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Certain inversion and representation formulas for q-sumudu transforms

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Abstract

In the present paper, we explore the formal properties of the q-Sumudu transforms to derive a number of inversion and representation formulas. Some interesting applications of the main results are also presented in the concluding section.

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1. Introduction and Preliminaries

In the classical analysis, integral transforms are the most widely used to solve differential equations and integral equations. Thus a lot of work has been done on the theory and applications of integral transforms (for instance, see [15] and [16]). Most popular integral transforms are due to Laplace, Fourier, Mellin and Hankel. In 1993 the Sumudu transform was proposed originally by Watugala [17] as follow:

$$S\{f(t);s\} = \frac{1}{s}\int_0^\infty e^{-t/s} f\,(t)\,dt, \qquad s \in (-\tau_1, \tau_2)\,,$$

over the set of functions

$$A = \left\{ f(t) | \exists M, \tau_1, \tau_2 > 0,\ |f(t)| < Me^{|t|/\tau_j},\ t \in (-1)^j \times [0, \infty) \right\}$$

and he applied it to the solution of ordinary differential equations in control engineering problems. Subsequently, Weerakoon [18] gave the Sumudu transform of partial derivatives and the complex inversion transform and applied it to the solution of partial differential equations. The Sumudu transform is not a new integral transform, but simply s-multiplied Laplace transform, providing the relation between them (see [12] and [5]).

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The Sumudu transform which is itself linear, preserves linear functions, and hence in particular does not change units (Belgacem and Karaballı [6]). One may, for instance, refer to such type of works in the recent papers [9] and [10].

The theory of q-analysis, in recent past, have been applied in the many areas of mathematics and physics like ordinary fractional calculus, optimal control problems, q-transform analysis, geometric function theory and in finding solutions of the q-difference and q-integral equations (for instance, see [2], [11], [13] and [14]). Albayrak, Purohit and Uçar [3] introduced the q-analogues of the Sumudu transform and established several theorems related to q-Sumudu transforms of some functions. They also introduced the convolution theorem for q-Sumudu transform.

Abdi [1] gave certain representation and inversion formulae for the two basic analogues of the Laplace transform. Here, it is worth to note that, the results given in this paper, can be derived from the corresponding results from q-analogue of Laplace transform which is given by Theorem 2 and Theorem 4 in [3]:

Theorem 2: Let $F_1(s) = L_q\{f(t);s\}$ and $G_1(s) = S_q\{f(t);s\}$. Then we have

$$G_1(s) = \frac{1}{s}F_1\left(\frac{1}{s}\right).$$

Theorem 4: Let $F_2(s) = \mathcal{L}_q\{f(t);s\}$ and $G_2(s) = \mathbb{S}_q\{f(t);s\}$. Then we have

$$G_2(s) = \frac{1}{s}F_2\left(\frac{1}{s}\right).$$

The aim of this paper is to give certain inversion and representation formulas for q-Sumudu transforms. Throughout this paper we will assume that functions are analytic at infinity. This paper is organized in following manner. In Section 2, we will give some theorems, which exhibit the inversion formulae for q-Sumudu transforms. In section 3, we give some examples for theorems.

As for prerequisites, the reader is expected to be familiar with notations of q-calculus. We start with basic definitions and facts from the q-calculus necessary for understanding of this study. Throughout this paper, we will assume that q satisfies the condition $0 < |q| < 1$. The q-derivative $D_q f$ of an arbitrary function f is given by

$$(D_q f)(x) = \frac{f(x) - f(qx)}{(1-q)x},$$

where $x \neq 0$. Clearly, if f is differentiable, then

$$\lim_{q\to 1^-} (D_q f)(x) = \frac{df(x)}{dx}.$$

For any real number α,

$$[\alpha] := \frac{q^\alpha - 1}{q-1}.$$

In particular, if $n \in \mathbb{Z}^+$, we denote

$$[n] = \frac{q^n - 1}{q-1} = q^{n-1} + \cdots + q + 1,$$

and the q-binomial coefficients is defined by

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{[n]!}{[k]!\,[n-k]!},$$

where $[n]! = [n]\,[n-1]\cdots[2]\,[1]$.
Following usual notations are very useful in the theory of q-calculus:

$$(a;q)_n = \prod_{k=0}^{n-1}\left(1-aq^k\right),$$

$$(a;q)_\infty = \prod_{k=0}^{\infty}\left(1-aq^k\right),$$

$$(a;q)_t = \frac{(a;q)_\infty}{(aq^t;q)_\infty}, \qquad (\, t\in\mathbb{R}), \tag{1.1}$$

$$(a;q)_{-n} = \frac{(-q/a)^n\, q^{n(n-1)/2}}{(q/a;q)_n}. \tag{1.2}$$

The q-analogues of the classical exponential functions are defined by

$$e_q(t) = \sum_{n=0}^{\infty}\frac{t^n}{(q;q)_n} = \frac{1}{(t;q)_\infty}, \qquad |t|<1, \tag{1.3}$$

$$E_q(t) = \sum_{n=0}^{\infty}\frac{(-1)^n q^{n(n-1)/2} t^n}{(q;q)_n} = (t;q)_\infty, \qquad (t\in\mathbb{C})\,. \tag{1.4}$$

q-exponential functions have the following properties

$$\lim_{q\to 1^-} e_q\left((1-q)\,x\right) = e^x,$$

$$\lim_{q\to 1^-} E_q\left((q-1)\,x\right) = e^x.$$

The q-integrals are defined as (see [7])

$$\int_0^x f(t)d_qt = x(1-q)\sum_{k=0}^{\infty} q^k f(xq^k), \tag{1.5}$$

$$\int_0^{\infty/A} f(t)d_qt = (1-q)\sum_{k\in\mathbb{Z}}\frac{q^k}{A}f(\frac{q^k}{A}). \tag{1.6}$$

The following definition is due to Albayrak, Purohit and Uçar [3]:

1.1. Definition. The q-analogues of Sumudu transform are defined as follow:

$$S_q\{f(t);s\} = \frac{1}{(1-q)\,s}\int_0^s E_q\left(\frac{q}{s}t\right) f\,(t)\,d_qt, \qquad s\in(-\tau_1,\tau_2)\,, \tag{1.7}$$

over the set of functions

$$A = \left\{ f(t)|\exists M,\tau_1,\tau_2>0,\ |f(t)| < ME_q\left(|t|\,/\tau_j\right),\ t\in(-1)^j\times[0,\infty)\right\}.$$

and

$$\mathbb{S}_q\{f(t);s\} = \frac{1}{(1-q)\,s}\int_0^\infty e_q\left(-\frac{1}{s}t\right) f\,(t)\,d_qt, \qquad s\in(-\tau_1,\tau_2)\,, \tag{1.8}$$

over the set of functions

$$B = \left\{ f(t)|\exists M,\tau_1,\tau_2>0, |f(t)| < Me_q\left(|t|\,/\tau_j\right), t\in(-1)^j\times[0,\infty)\right\}.$$

By virtue of (1.5) and (1.6), q-Sumudu transforms can be expressed as

$$S_q\left\{f\,(t)\,;s\right\} = (q;q)_\infty\sum_{k=0}^{\infty}\frac{q^k f\left(sq^k\right)}{(q;q)_k}, \tag{1.9}$$

and

$$\mathbb{S}_q\left\{f\left(t\right);s\right\}=\frac{s^{-1}}{\left(-\frac{1}{s};q\right)_\infty}\sum_{k\in\mathbb{Z}}q^k f(q^k)\left(-\frac{1}{s};q\right)_k. \tag{1.10}$$

Furthermore, the q-hypergeometric functions are defined by [11]:

$$_r\Phi_s\left[\begin{matrix}a_1 & a_2 & \cdots & a_r\\ b_1 & b_2 & \cdots & b_s\end{matrix};q,z\right]=\sum_{n=0}^{\infty}\frac{(a_1,a_2,\ldots,a_r;q)_n}{(b_1,b_2,\ldots,b_s;q)_n}\left[(-1)^n q^{\binom{n}{2}}\right]^{s+1-r}\frac{z^n}{(q;q)_n},$$

where

$$(a_1,a_2,\ldots,a_r;q)_n=\prod_{i=1}^{r}(a_i;q)_n.$$

The q-Taylor's series is defined by (see also [13])

$$f\left(x\right)=\sum_{n=0}^{\infty}\frac{a_n}{[n]!}\left(x-a\right)_q^n, \tag{1.11}$$

where

$$a_n=\lim_{x\to a}D_q^{(n)}f\left(x\right).$$

The q-Bessel functions were introduced by Jackson [8] and are referred to as Jackson's q-Bessel functions. Some q-analogues of the Bessel functions are given by

$$\begin{aligned}J_\nu^{(1)}\left(z;q\right)&=\frac{\left(q^{\nu+1};q\right)_\infty}{(q;q)_\infty}\left(\frac{z}{2}\right)^\nu {}_2\Phi_1\left[\begin{matrix}0 & 0\\ q^{\nu+1} & \end{matrix};q,-\frac{z^2}{4}\right],\qquad |z|<2,\\ &=\left(\frac{z}{2}\right)^\nu\sum_{n=0}^{\infty}\frac{\left(-z^2/4\right)^n}{(q;q)_{\nu+n}(q;q)_n},\end{aligned} \tag{1.12}$$

and

$$\begin{aligned}J_\nu^{(2)}\left(z;q\right)&=\frac{\left(q^{\nu+1};q\right)_\infty}{(q;q)_\infty}\left(\frac{z}{2}\right)^\nu {}_0\Phi_1\left[\begin{matrix}-\\ q^{\nu+1}\end{matrix};q,-\frac{q^{\nu+1}z^2}{4}\right],\qquad |z|<2,\\ &=\left(\frac{z}{2}\right)^\nu\sum_{n=0}^{\infty}\frac{q^{n(n+\nu)}\left(-z^2/4\right)^n}{(q;q)_{\nu+n}(q;q)_n}.\end{aligned} \tag{1.13}$$

The relation between these two q-Bessel functions is

$$J_\nu^{(2)}\left(z;q\right)=\left(-\frac{z^2}{4};q\right)_\infty J_\nu^{(1)}\left(z;q\right),\qquad |z|<2.$$

q-Bessel functions are q-extensions of the Bessel function of the first kind since

$$\lim_{q\to1^-}J_\nu^{(k)}\left((1-q)\,z;q\right)=J_\nu\left(z\right),\qquad k=1,2.$$

The third kind q-analogue of the Bessel function is given by following formula

$$\begin{aligned}J_\nu^{(3)}\left(z;q\right)&=\frac{\left(q^{\nu+1};q\right)_\infty}{(q;q)_\infty}z^\nu {}_1\Phi_1\left[\begin{matrix}0\\ q^{\nu+1}\end{matrix};q,qz^2\right],\\ &=z^\nu\sum_{n=0}^{\infty}\frac{(-1)^n q^{n(n-1)/2}\left(qz^2\right)^n}{(q;q)_{\nu+n}(q;q)_n}.\end{aligned} \tag{1.14}$$

This third kind q-Bessel function is also known as the Hahn-Exton q-Bessel function. This is also q-extension of the Bessel function of the first kind since

$$\lim_{q\to1^-}J_\nu^{(3)}\left((1-q)\,z;q\right)=J_\nu\left(2z\right).$$

2. Main Theorems

In the sequel, we need the following results:
Recently, Albayrak, Purohit and Uçar [4] proved the following theorem.

Theorem: *Corresponding to the bounded sequence A_n, let $f(x)$ be given by*

$$f(x) = \sum_{n=0}^{\infty} A_n x^n,$$

then for $\alpha > 0$, the following results hold:

$$S_q\left\{x^{\alpha-1} f(x); s\right\} = s^{\alpha-1} \sum_{n=0}^{\infty} A_n (q;q)_{\alpha+n-1} s^n, \tag{2.1}$$

$$\mathbb{S}_q\left\{x^{\alpha-1} f(x); s\right\} = s^{\alpha-1} \sum_{n=0}^{\infty} A_n \frac{(q;q)_{\alpha+n-1}}{K(s;\alpha+n)} s^n. \tag{2.2}$$

If we write $\alpha = j+1$ and $f(x) = 1$ in the above theorem, we get

$$S_q\left\{x^j; s\right\} = s^j (q;q)_j, \tag{2.3}$$

$$\mathbb{S}_q\left\{x^j; s\right\} = q^{-j(j+1)} s^j (q;q)_j. \tag{2.4}$$

If we write $\alpha = j/2+1$ and choose $f(x) = a^{-j/2} J_j^{(1)}(2\sqrt{ax};q)$ or $f(x) = \left(\frac{q}{a}\right)^{j/2} J_j^{(3)}\left(\sqrt{q^{-1}ax};q\right)$ in (2.1), we have

$$S_q\left\{\left(\frac{x}{a}\right)^{j/2} J_j^{(1)}(2\sqrt{ax};q); s\right\} = s^j e_q(-as), \tag{2.5}$$

$$S_q\left\{\left(\frac{qx}{a}\right)^{j/2} J_j^{(3)}\left(\sqrt{q^{-1}ax};q\right); s\right\} = s^j E_q(as), \tag{2.6}$$

and similarly, if we write $\alpha = j/2+1$ and choose $f(x) = q^{j(j+1)/2} a^{-j/2} J_j^{(2)}(2\sqrt{ax};q)$ or $f(x) = \left(\frac{q}{a}\right)^{j/2} J_j^{(3)}\left(\sqrt{q^j ax};q\right)$ in (2.2), we have

$$\mathbb{S}_q\left\{q^{j(j+1)/2}\left(\frac{x}{a}\right)^{j/2} J_j^{(2)}(2\sqrt{ax};q); s\right\} = s^j E_q(as), \tag{2.7}$$

$$\mathbb{S}_q\left\{\left(\frac{qx}{a}\right)^{j/2} J_j^{(3)}\left(\sqrt{q^j ax};q\right); s\right\} = s^j e_q(-as), \tag{2.8}$$

where $\mathrm{Re}(s) > 0$ and $\mathrm{Re}(\upsilon+1) > 0$. The following theorem is due to Belgacem, Karaballı and Kalla [5].

Theorem: *The inverse discrete Sumudu transform, $f(t)$, of the power series,*

$$G(u) = \sum_{n=0}^{\infty} b_n u^n,$$

is given by

$$S^{-1}G(u) = f(t) = \sum_{n=0}^{\infty} \left(\frac{1}{n!}\right) b_n t^n.$$

In this section, we will give some theorems, which exhibit the inversion formulae for q-Sumudu transforms. The usefulness of the theorems are also exhibited by considering some examples.

2.1. Theorem. *Let* $S_q\{f(x);s\} = G_1(s)$ *and* $\mathbb{S}_q\{g(x);s\} = G_2(s)$*. If* $\frac{1}{s}G_1\left(\frac{1}{s}\right)$ *and* $\frac{1}{s}G_2\left(\frac{1}{s}\right)$ *are analytic outside a given circle* $|s| = r$*,* $r > 0$ *and their values are zero at* ∞*, then we have*

$$f(x) = \sum_{j=0}^{\infty} a_j (1-q)^j \frac{x^j}{\left\{(q;q)_j\right\}^2}, \tag{2.9}$$

where

$$\lim_{s\to 0} D_q^{(j)} G_1(s) = a_j,$$

and

$$g(x) = \sum_{j=0}^{\infty} b_j q^{j(j+1)/2} (1-q)^j \frac{x^j}{\left\{(q;q)_j\right\}^2}, \tag{2.10}$$

where

$$\lim_{s\to 0} D_q^{(j)} G_2(s) = b_j.$$

Proof. We only give the proof of the identity (2.9). Because, the proof of the identity (2.10) is similar. We suppose that $\frac{1}{s}G_1\left(\frac{1}{s}\right)$ is analytic in $|s| > r$, for some $r > 0$ and vanishes at ∞. Then, it can be represented as an absolutely convergent power series as follows:

$$\frac{1}{s}G_1\left(\frac{1}{s}\right) = \sum_{j=0}^{\infty} c_j s^{-j-1}.$$

Therefore, we can write

$$G_1(s) = \sum_{j=0}^{\infty} c_j s^j.$$

Making use of the q-Taylor formula (1.11), we have

$$c_j = \frac{(1-q)^j}{(q;q)_j} \lim_{s\to 0} D_q^{(j)} G_1(s).$$

Thus, we have

$$G_1(s) = \sum_{j=0}^{\infty} a_j (1-q)^j \frac{s^j}{(q;q)_j},$$

where

$$a_j = \lim_{s\to 0} D_q^{(j)} G_1(s).$$

Taking into account (2.3), we obtain

$$f(x) = \sum_{j=0}^{\infty} a_j (1-q)^j \frac{x^j}{\left\{(q;q)_j\right\}^2}.$$

2.2. Theorem. *Let* $S_q\{f(x);s\} = G_1(s)$ *and* $\mathbb{S}_q\{f(x);s\} = G_2(s)$*. If* $\frac{1}{s}G_1\left(\frac{1}{s}\right)$ *and* $\frac{1}{s}G_2\left(\frac{1}{s}\right)$ *are analytic outside a given circle* $|s| = r$*,* $r > 0$ *and their values are zero at* ∞*, then*

$$f(x) = \sum_{j=0}^{\infty} a_j \frac{(1-q)^j}{(q;q)_j} \left(\frac{qx}{a}\right)^{j/2} J_j^{(3)}\left(\sqrt{\frac{a}{q}x};q\right), \tag{2.11}$$

where

$$a_j = \lim_{s\to 0} D_q^{(j)} \left\{ e_q(as)\, G_1(s) \right\}.$$

$$\text{(2.12)} \quad f(x) = \sum_{j=0}^{\infty} b_j \frac{(1-q)^j}{(q;q)_j} \left(\frac{qx}{a}\right)^{j/2} J_j^{(3)}\left(\sqrt{q^j ax};q\right),$$

where

$$b_j = \lim_{s\to 0} D_q^{(j)} \left\{ E_q(-as)\, G_2(s) \right\}.$$

$$\text{(2.13)} \quad f(x) = \sum_{j=0}^{\infty} b_j \frac{(1-q)^j}{(q;q)_j} \left(\frac{x}{a}\right)^{j/2} J_j^{(1)}\left(2\sqrt{at};q\right),$$

where

$$b_j = \lim_{s\to 0} D_q^{(j)} \left\{ E_q(-as)\, G_2(s) \right\}.$$

$$\text{(2.14)} \quad f(x) = \sum_{j=0}^{\infty} a_j \frac{(1-q)^j}{(q;q)_j} q^{j(j+1)/2} \left(\frac{x}{a}\right)^{j/2} J_j^{(2)}\left(2\sqrt{ax};q\right),$$

where

$$a_j = \lim_{s\to 0} D_q^{(j)} \left\{ e_q(as)\, G_1(s) \right\}.$$

Proof. We only give the proof of the identity (2.11). Because, by using the formula (2.8), (2.5), (2.7), respectively, the proof of the identities (2.12)-(2.14) are similar. Let

$$\frac{1}{s} F\left(\frac{1}{s}\right) = \frac{1}{s} e_q(a/s)\, G_1\left(\frac{1}{s}\right).$$

Suppose $\frac{1}{s} G_1\left(\frac{1}{s}\right)$ is analytic outside the circle $|s| = r$ and its value is zero at ∞. Therefore $\frac{1}{s} F\left(\frac{1}{s}\right)$ is analytic outside the circle $|s| = p$ and its value is zero at ∞. Thus, Theorem 2.1 can be applied for $F(s)$ and we can write

$$F(s) = \sum_{j=0}^{\infty} a_j \ (1-q)^j \ \frac{s^j}{(q;q)_j},$$

where

$$a_j = \lim_{s\to 0} D_q^{(j)} \left\{ F(s) \right\}.$$

Following the definition of $F(s)$, we have

$$e_q(as)\, G_1(s) = \sum_{j=0}^{\infty} a_j \ (1-q)^j \ \frac{s^j}{(q;q)_j},$$

$$G_1(s) = \sum_{j=0}^{\infty} a_j \ (1-q)^j \ \frac{s^j E_q(as)}{(q;q)_j},$$

where

$$a_j = \lim_{s\to 0} D_q^{(j)} \left\{ e_q(as)\, G_1(s) \right\}.$$

Now, on using the formula (2.6), we get

$$f(x) = \sum_{j=0}^{\infty} a_j \ (1-q)^j \ \left(\frac{qx}{a}\right)^{j/2} \frac{J_j^{(3)}\left(\sqrt{\frac{a}{q}x};q\right)}{(q;q)_j},$$

where

$$a_j = \lim_{s\to 0} D_q^{(j)} \left\{ e_q\left(as\right) G_1\left(s\right)\right\}.$$

2.3. Theorem. *Let* $S_q\left\{f\left(x\right);s\right\} = G_1(s)$, $\mathbb{S}_q\left\{f\left(x\right);s\right\} = G_2(s)$. *We suppose that* $\frac{1}{s}G_1\left(\frac{1}{s}\right)$ *and* $\frac{1}{s}G_2\left(\frac{1}{s}\right)$ *are analytic outside a given circle* $|s| = r$, $r > 0$ *and their values are zero at* ∞. *If* $G_1\left(s\right)$ *and* $G_2\left(s\right)$ *have a series expansion as follow*

$$G_i\left(s\right) = \sum_{j=0}^{\infty} c_j \left(as;q\right)_j, \ \ (i = 1, 2),$$

where $\sum\limits_{j=0}^{\infty}|c_j|$ *convergent, then* f *has a series expansion as follow*

$$f\left(x\right) = \sum_{j=0}^{\infty} a_j \ q^{-j(j-1)/2}\left(q-1\right)^j a^{-j} \ \frac{{}_2\Phi_1\left(q^{-j},0;q;q,q^j ax\right)}{\left(q;q\right)_j}, \tag{2.15}$$

where

$$a_j = \lim_{s\to a^{-1}q^{-j}} D_q^{(j)} G_1\left(s\right),$$

$$f\left(x\right) = \sum_{j=0}^{\infty} b_j \ q^{-j(j-1)/2} a^{-j}\left(q-1\right)^j \ \frac{{}_1\Phi_1\left(q^{-j};q;q,-q^{j+1}ax\right)}{\left(q;q\right)_j}, \tag{2.16}$$

where

$$b_j = \lim_{s\to a^{-1}q^{-j}} D_q^{(j)} G_2\left(s\right).$$

Proof. We only give the proof of the identity (2.15). The proof of the identity (2.16), can be easily seen by making use of following known identity [4, p.419, (2.11)],

$$\mathbb{S}_q\left\{{}_1\Phi_1\left(q^{-j};q;q,-q^{j+1}ax\right);s\right\} = \left(as;q\right)_j.$$

Suppose $\frac{1}{s}G_1\left(\frac{1}{s}\right)$ is analytic in $|s| > r$, for some $r > 0$ and its value is zero at ∞. Then, $G_1\left(s\right)$ has a series expansion as follow

$$G_1\left(s\right) = \sum_{j=0}^{\infty} c_j\left(as;q\right)_j,$$

where $\sum\limits_{j=0}^{\infty}|c_j|$ is convergent. Using the q-Taylor's formula, we get

$$c_j = \lim_{s\to a^{-1}q^{-j}} D_q^{(j)}\left\{G_1\left(s\right)\right\}.q^{-j(j-1)/2}\left(q-1\right)^j a^{-j}/\left(q;q\right)_j.$$

Thus, we can write

$$G_1\left(s\right) = \sum_{j=0}^{\infty} a_j \ q^{-j(j-1)/2}\left(q-1\right)^j a^{-j} \ \frac{\left(as;q\right)_j}{\left(q;q\right)_j},$$

where

$$a_j = \lim_{s\to a^{-1}q^{-j}} D_q^{(j)} G_1\left(s\right).$$

Furthermore, making use of formula [4, p.419, (2.10)],

$$S_q\left\{{}_2\Phi_1\left(q^{-j},0;q;q,q^j ax\right);s\right\} = \left(as;q\right)_j,$$

we obtain

$$f\left(x\right)=\sum_{j=0}^{\infty}a_{j}\ q^{-j(j-1)/2}\left(q-1\right)^{j}a^{-j}\ \frac{{}_{2}\Phi_{1}\left(q^{-j},0;q;q,q^{j}ax\right)}{\left(q;q\right)_{j}},$$

where

$$a_{j}=\lim_{s\to a^{-1}q^{-j}}D_{q}^{(j)}G_{1}\left(s\right).$$

2.4. Theorem. *Let $S_q\left\{f\left(x\right);s\right\}=G(s)$. We suppose that $\frac{1}{s}G\left(\frac{1}{s}\right)$ is analytic outside the circle $|s|=r$, $r>0$ and its value is zero at ∞. Then, $G\left(s\right)$ has the form*

$$G\left(s\right)=s^{-1}\sum_{j=1}^{\infty}c_{j}\left(q\left(as\right)^{-1};q\right)_{-j},$$

where $\sum_{j=0}^{\infty}|c_j|$ convergent, if and only if, f has a series expansion as follow

$$f\left(x\right)=e_{q}\left(ax\right)\sum_{j=1}^{\infty}c_{j}\left(-1\right)^{j}a^{j}q^{j(j-1)/2}\frac{x^{j-1}}{\left(q;q\right)_{j-1}},$$

where

$$\sum_{j=1}^{\infty}\left|q^{j(j-1)/2}c_{j}\right|<\infty.$$

Proof. Suppose f has a series expansion as follow

$$f\left(x\right)=e_{q}\left(ax\right)\sum_{j=1}^{\infty}c_{j}\left(-1\right)^{j}a^{j}q^{j(j-1)/2}\frac{x^{j-1}}{\left(q;q\right)_{j-1}}.$$

Using the series representation (1.9) of S_q-transform, we have

$$G\left(s\right)=\left(q;q\right)_{\infty}\sum_{k=0}^{\infty}\frac{q^{k}}{\left(q;q\right)_{k}}\sum_{j=1}^{\infty}c_{j}\left(-1\right)^{j}a^{j}q^{j(j-1)/2}\frac{\left(sq^{k}\right)^{j-1}e_{q}\left(asq^{k}\right)}{\left(q;q\right)_{j-1}}.$$

Clearly, $G\left(s\right)$ has simple poles at $s=a^{-1}q^{-k}$, $(k=0,1,2,...)$ which obviously all on lie or in the exterior of the circle $|s|=1/a$. So, for $|s|<1/a$, $G\left(s\right)$ is absolutely convergent. Hence interchanging the order of summations in the right side of $G\left(s\right)$, we obtain

$$G\left(s\right)=\left(q;q\right)_{\infty}\sum_{j=1}^{\infty}c_{j}\left(-1\right)^{j}a^{j}q^{j(j-1)/2}\frac{s^{j-1}}{\left(q;q\right)_{j-1}}\sum_{k=0}^{\infty}\frac{\left(q^{j}\right)^{k}}{\left(q;q\right)_{k}}e_{q}\left(asq^{k}\right).$$

Making use of the definitions (1.3) of e_q and (1.1) of $\left(a;q\right)_t$, we have

$$\begin{aligned}G\left(s\right)&=\left(q;q\right)_{\infty}\sum_{j=1}^{\infty}c_{j}\left(-1\right)^{j}a^{j}q^{j(j-1)/2}\frac{s^{j-1}}{\left(q;q\right)_{j-1}}\sum_{k=0}^{\infty}\frac{\left(q^{j}\right)^{k}}{\left(q;q\right)_{k}}\frac{1}{\left(asq^{k};q\right)_{\infty}}\\&=\left(q;q\right)_{\infty}\sum_{j=1}^{\infty}c_{j}\left(-1\right)^{j}a^{j}\frac{q^{j(j-1)/2}}{\left(q;q\right)_{j-1}}\frac{s^{j-1}}{\left(as;q\right)_{\infty}}\sum_{k=0}^{\infty}\frac{\left(as;q\right)_{k}}{\left(q;q\right)_{k}}\left(q^{j}\right)^{k}.\end{aligned}$$

Now on using the formula

$$\sum_{k=0}^{\infty}\frac{\left(a;q\right)_{k}}{\left(q;q\right)_{k}}z^{k}=\frac{\left(az;q\right)_{\infty}}{\left(z;q\right)_{\infty}},$$

we get

$$G(s)=\sum_{j=1}^{\infty}c_j(-1)^j a^j q^{j(j-1)/2}\frac{s^{j-1}}{(as;q)_j}.$$

Then using (1.2), we obtain

$$\begin{aligned}G(s)&=\sum_{j=1}^{\infty}c_j(-1)^j a^j q^{j(j-1)/2}\frac{s^{j-1}}{(-as)^j q^{j(j-1)/2}}(q/as;q)_{-j}\\&=\frac{1}{s}\sum_{j=1}^{\infty}c_j(q/as;q)_{-j}.\end{aligned}$$

On the other hand, let $G(s)$ has a series expansion as follow

$$G(s)=s^{-1}\sum_{j=1}^{\infty}c_j\left(q(as)^{-1};q\right)_{-j},$$

where

$$\sum_{j=1}^{\infty}\left|q^{j(j-1)/2}c_j\right|<\infty.$$

By the property (1.2) of $(a;q)_{-j}$, we can write

$$G(s)=\sum_{j=1}^{\infty}c_j(-1)^j a^j q^{j(j-1)/2}\frac{s^{j-1}}{(as;q)_j}.$$

For some $s=s_0$, $|s_0|<\frac{1}{a}$,

$$|G(s_0)|=\sum_{j=1}^{\infty}\left|c_j(-1)^j a^j q^{j(j-1)/2}\frac{s_0^{j-1}}{(as_0;q)_j}\right|.$$

Since

$$|as_0|<1 \text{ and } \left|\frac{s_0^{-1}}{(as_0;q)_j}\right|<\left|s_0^{-1}e_q(as_0)\right|,$$

we have

$$|G(s_0)|=\left|s_0^{-1}e_q(as_0)\right|\sum_{j=1}^{\infty}\left|q^{j(j-1)/2}c_j\right|.$$

Hence, for the absolute convergence of $G(s_0)$, we need

$$\sum_{j=1}^{\infty}\left|q^{j(j-1)/2}c_j\right|<\infty.$$

Now on using the formula

$$S_q\left\{(-a)^j q^{j(j-1)/2}x^{j-1}e_q(ax)/(q;q)_{j-1};s\right\}=\frac{(q/as;q)_{-j}}{s},$$

we obtain

$$f(x)=e_q(ax)\sum_{j=1}^{\infty}c_j(-1)^j a^j q^{j(j-1)/2}\frac{x^{j-1}}{(q;q)_{j-1}}.$$

This completes the proof.

2.5. Theorem. *Let $\mathbb{S}_q\{f(x);s\} = G(s)$ and the function $\frac{1}{s}G\left(\frac{1}{s}\right)$ is analytic outside a given circle $|s| = r$, $r > 0$ and its value is zero at ∞. Then, $G(s)$ has the form*

$$G(s) = s^{-1}\sum_{j=1}^{\infty} c_j \left(-q(as)^{-1};q\right)_{-j},$$

where $\sum_{j=0}^{\infty}\left|q^{j(j-1)/2}c_j\right|$ convergent, if and only if, f has a series expansion as follow

$$f(x) = \sum_{j=1}^{\infty} c_j a^j q^{j(j-1)} \frac{x^{j-1}}{(q;q)_{j-1}} E_q\left(aq^j x\right).$$

Proof. Suppose f has a series expansion as follow

$$f(x) = \sum_{j=1}^{\infty} c_j a^j q^{j(j-1)} \frac{x^{j-1}}{(q;q)_{j-1}} E_q\left(q^j ax\right).$$

Making use of series representation (1.10) of $\mathbb{S}_q$-transform, we have

$$G(s) = \frac{s^{-1}}{\left(-\frac{1}{s};q\right)_\infty}\sum_{k\in Z} q^k\left(-\frac{1}{s};q\right)_k \sum_{j=1}^{\infty} c_j a^j q^{j(j-1)} \frac{\left(q^k\right)^{j-1}}{(q;q)_{j-1}} E_q\left(aq^{k+j}\right).$$

Clearly, $G(s)$ has simple poles at $s = a^{-1}q^{-k}$, $(k = 0, 1, 2, ...)$ which obviously all on lie or in the exterior of the circle $|s| = 1/a$. So, for $|s| < 1/a$, $G(s)$ is absolutely convergent. Hence, interchanging the order of summations in the right hand side of $G(s)$, we obtain

$$G(s) = \sum_{j=1}^{\infty} c_j a^j \frac{q^{j(j-1)}}{(q;q)_{j-1}} \frac{s^{-1}}{\left(-\frac{1}{s};q\right)_\infty}\sum_{k\in Z}\left(-\frac{1}{s};q\right)_k \left(q^j\right)^k E_q\left(aq^{k+j}\right).$$

Making use of definitions (1.4) of E_q and (1.1) of $(a;q)_t$, we have

$$\begin{aligned} G(s) &= \sum_{j=1}^{\infty} c_j a^j \frac{q^{j(j-1)}}{(q;q)_{j-1}} \frac{\left(aq^j;q\right)_\infty s^{-1}}{\left(-\frac{1}{s};q\right)_\infty}\sum_{k\in Z}\frac{(-1/s;q)_k}{(aq^j;q)_k}\left(q^j\right)^k \\ &= \sum_{j=1}^{\infty} c_j a^j \frac{q^{j(j-1)}}{(q;q)_{j-1}} \frac{\left(aq^j;q\right)_\infty s^{-1}}{\left(-\frac{1}{s};q\right)_\infty}\ {}_1\Psi_1\left(-1/s;aq^j;q,q^j\right). \end{aligned}$$

Now on using the formula

$${}_1\Psi_1(a;b;q,z) = \frac{(q, b/a, az, q/az;q)_\infty}{(b, q/a, z, b/az;q)_\infty},$$

we get

$$\begin{aligned} G(s) &= \sum_{j=1}^{\infty} c_j a^j \frac{q^{j(j-1)}}{(q;q)_j} \frac{\left(aq^j;q\right)_\infty s^{-1}}{\left(-\frac{1}{s};q\right)_\infty} \frac{\left(q, -asq^j, -q^j/s, -q^{1-j}s;q\right)_\infty}{\left(aq^j, -qs, q^j, -as;q\right)_\infty} \\ &= \sum_{j=1}^{\infty} c_j a^j q^{j(j-1)} \frac{1}{s}\frac{1}{(-as;q)_j(-1/s;q)_j(-qs;q)_{-j}}. \end{aligned}$$

By the property (1.2) of $(a;q)_{-j}$, we obtain

$$\begin{aligned} G(s) &= \sum_{j=1}^{\infty} c_j a^j q^{j(j-1)} \frac{1}{s}\frac{(-q/as;q)_{-j}(-qs;q)_{-j}}{(as)^j q^{j(j-1)/2}(1/s)^j q^{j(j-1)/2}(-qs;q)_{-j}} \\ &= \frac{1}{s}\sum_{j=1}^{\infty} c_j(-q/as;q)_{-j}. \end{aligned}$$

On the other hand, let $G(s)$ has a series expansion as follow

$$G(s) = s^{-1}\sum_{j=1}^{\infty} c_j \left(-q\,(as)^{-1};q\right)_{-j},$$

where

$$\sum_{j=1}^{\infty}\left|q^{j(j-1)/2}c_j\right| < \infty.$$

By the property (1.2) of $(a;q)_{-j}$, we can write

$$G(s) = \sum_{j=1}^{\infty} c_j a^j q^{j(j-1)/2}\frac{s^{j-1}}{(-as;q)_j}.$$

For some $s = s_0$, $|s_0| < \frac{1}{a}$

$$|G(s_0)| = \sum_{j=1}^{\infty}\left|c_j a^j q^{j(j-1)/2}\frac{s_0^{j-1}}{(-as_0;q)_j}\right|.$$

Since

$$|as_0| < 1 \text{ and } \left|\frac{s_0^{-1}}{(-as_0;q)_j}\right| < \left|s_0^{-1}e_q(-as_0)\right|,$$

we have

$$|G(s_0)| = \left|s_0^{-1}e_q(-as_0)\right|\sum_{j=1}^{\infty}\left|c_j q^{j(j-1)/2}\right|.$$

Therefore for absolute convergence

$$\sum_{j=1}^{\infty}\left|q^{j(j-1)/2}c_j\right| < \infty.$$

Now on using the formula

$$\mathbb{S}_q\left\{a^j q^{j(j-1)}x^{j-1}E_q\left(aq^jx\right)/(q;q)_{j-1};s\right\} = \frac{(-q/as;q)_{-j}}{s},$$

we obtain

$$f(x) = \sum_{j=1}^{\infty} c_j a^j q^{j(j-1)}\frac{x^{j-1}}{(q;q)_{j-1}}E_q\left(aq^jx\right).$$

This completes the proof.

3. Applications

As illustrations of the above theorems, we shall now give some examples. The following result was obtained previously in [4]:

3.1. Example. Let

$$G(s) = e_q(-as) = \sum_{n=0}^{\infty}\frac{(-1)^n a^n}{(q;q)_n}s^n.$$

Then, $\frac{1}{s}G\left(\frac{1}{s}\right)$ is analytic outside the circle $|s| = |a|$ and its value is zero at ∞. Also, it is easy to see that

$$D_q^{(n)}\left(e_q(-as)\right) = \frac{a^n}{(q-1)^n}e_q(-as).$$

If we set $G(s)=G_1(s)=G_2(s)$ in Theorem 2.1, then we have

$$\begin{aligned}
f(x) &= \sum_{n=0}^{\infty} a_n (1-q)^n \frac{x^n}{\left\{(q;q)_n\right\}^2} \\
&= \sum_{n=0}^{\infty} \frac{a^n}{(q-1)^n} (1-q)^n \frac{x^n}{\left\{(q;q)_n\right\}^2} \\
&= {}_2\Phi_1 \left[\begin{matrix} 0 \quad 0 \\ q \end{matrix} ; q, -ax \right] \\
&= J_0^{(1)}\left(2\sqrt{ax};q\right).
\end{aligned}$$

and

$$\begin{aligned}
f(x) &= \sum_{n=0}^{\infty} b_n q^{n(n+1)/2} (1-q)^n \frac{x^n}{\left\{(q;q)_n\right\}^2} \\
&= \sum_{n=0}^{\infty} \frac{a^n}{(q-1)^n} q^{n(n+1)/2} (1-q)^n \frac{x^n}{\left\{(q;q)_n\right\}^2} \\
&= {}_1\Phi_1 \left[\begin{matrix} 0 \\ q \end{matrix} ; q, -ax \right] \\
&= J_0^{(3)}\left(\sqrt{ax};q\right)
\end{aligned}$$

where $J_0^{(1)}(x;q)$ is the first kind q-Bessel function and $J_0^{(3)}(x;q)$ is the third kind q-Bessel function. Thus we have

$$S_q\left\{J_0^{(1)}\left(2\sqrt{ax};q\right);s\right\} = \mathbb{S}_q\left\{J_0^{(3)}\left(\sqrt{ax};q\right);s\right\} = e_q(-as).$$

3.2. Example. Let

$$G_1(s) = E_q(bs)\, E_q(cs),$$

and

$$G_2(s) = e_q(-bs)\, e_q(-cs).$$

Then, $\frac{1}{s}G_1\left(\frac{1}{s}\right)$ and $\frac{1}{s}G_2\left(\frac{1}{s}\right)$ are analytic outside the circle $|s|=|a|$ and their value are zero at ∞. If we write $G_1(s)$ and $G_2(s)$ in Theorem 2.2, respectively, then we have

$$f(x) = \begin{cases} \sum\limits_{j=0}^{\infty} (-b)^j q^{j(j-1)/2} \left(\frac{qx}{c}\right)^{j/2} \frac{J_j^{(3)}\left(\sqrt{q^{-1}cx};q\right)}{(q;q)_j}, & \text{if } a=c, \\ \sum\limits_{j=0}^{\infty} (-c)\, q^{j(j-1)/2} \left(\frac{qx}{b}\right)^{j/2} \frac{J_j^{(3)}\left(\sqrt{q^{-1}bx};q\right)}{(q;q)_j}, & \text{if } a=b. \end{cases}$$

where

$$a_j = \lim_{s\to 0} D_q^{(j)}\left\{e_q(as)\, G_1(s)\right\} = \begin{cases} (q-1)^{-j} q^{j(j-1)/2} b^j, & \text{if } a=c, \\ (q-1)^{-j} q^{j(j-1)/2} c^j, & \text{if } a=b. \end{cases}$$

$$f(x) = \begin{cases} \sum\limits_{j=0}^{\infty} (-b)^j \left(\frac{qx}{c}\right)^{j/2} \frac{J_j^{(3)}\left(\sqrt{q^{j}cx};q\right)}{(q;q)_j}, & \text{if } a=c, \\ \sum\limits_{j=0}^{\infty} (-c)^j \left(\frac{qx}{b}\right)^{j/2} \frac{J_j^{(3)}\left(\sqrt{q^{j}bx};q\right)}{(q;q)_j}, & \text{if } a=b. \end{cases}$$

where

$$b_j = \lim_{s\to 0} D_q^{(j)}\left\{E_q(-as)\, G_2(s)\right\} = \begin{cases} (q-1)^{-j} b^j, & \text{if } a=c, \\ (q-1)^{-j} c^j, & \text{if } a=b. \end{cases}$$

$$f(x)=\begin{cases}\sum\limits_{j=0}^{\infty}(-b)^j\left(\frac{x}{c}\right)^{j/2}\frac{J_j^{(1)}\left(2\sqrt{cx};q\right)}{(q;q)_j}, & \text{if } a=c,\\ \sum\limits_{j=0}^{\infty}(-c)^j\left(\frac{x}{b}\right)^{j/2}\frac{J_j^{(1)}\left(\sqrt{2bx};q\right)}{(q;q)_j}, & \text{if } a=b.\end{cases}$$

where

$$b_j=\lim_{s\to 0}D_q^{(j)}\left\{E_q(-as)\,G_2(s)\right\}=\begin{cases}(q-1)^{-j}\,b^j, & \text{if } a=c,\\ (q-1)^{-j}\,c^j, & \text{if } a=b.\end{cases}$$

$$f(x)=\begin{cases}\sum\limits_{j=0}^{\infty}(-qb)^j\,q^{j(j-1)}\left(\frac{x}{c}\right)^{j/2}\frac{J_j^{(2)}\left(2\sqrt{cx};q\right)}{(q;q)_j}, & \text{if } a=c,\\ \sum\limits_{j=0}^{\infty}(-qc)^j\,q^{j(j-1)}\left(\frac{x}{b}\right)^{j/2}\frac{J_j^{(2)}\left(\sqrt{bx};q\right)}{(q;q)_j}, & \text{if } a=b.\end{cases}$$

where

$$a_j=\lim_{s\to 0}D_q^{(j)}\left\{e_q(as)\,G_1(s)\right\}=\begin{cases}(q-1)^{-j}\,q^{j(j-1)/2}b^j, & \text{if } a=c,\\ (q-1)^{-j}\,q^{j(j-1)/2}c^j, & \text{if } a=b.\end{cases}$$

3.3. Example. Let

$$\begin{aligned}G(s)&=e_q(y)/e_q(s)=(s;q)_\infty/(y;q)_\infty\\&=\sum_{j=0}^{\infty}\frac{(s/y;q)_j}{(q;q)_j}y^j.\end{aligned}$$

Then, $\frac{1}{s}G\left(\frac{1}{s}\right)$ is analytic outside the circle $|s|=1$ and its value is zero at ∞. Furthermore, it can be easily seen that

$$D_q^{(j)}G(s)=(-1)^j(1-q)^{-j}\,q^{j(j-1)/2}e_q(y)/e_q\left(q^js\right).$$

If we write $G(s)=G_1(s)=G_2(s)$ in Theorem 2.3, respectively, then we have

$$\begin{aligned}f(x)&=\sum_{j=0}^{\infty}a_j\;q^{-j(j-1)/2}(q-1)^j\,y^j\;\frac{{}_2\Phi_1\left(q^{-j},0;q;q,q^jx/y\right)}{(q;q)_j}\\&=\sum_{j=0}^{\infty}y^j\;\frac{{}_2\Phi_1\left(q^{-j},0;q;q,q^jx/y\right)}{(q;q)_j},\end{aligned}$$

and

$$\begin{aligned}f(x)&=\sum_{j=0}^{\infty}b_j\;q^{-j(j-1)/2}(q-1)^j\,y^j\;\frac{{}_1\Phi_1\left(q^{-j};q;q,-q^{j+1}x/y\right)}{(q;q)_j}\\&=\sum_{j=0}^{\infty}y^j\;\frac{{}_1\Phi_1\left(q^{-j};q;q,-q^{j+1}x/y\right)}{(q;q)_j},\end{aligned}$$

where

$$a_j=b_j=\lim_{s\to yq^j}D_q^{(j)}G(s)=(-1)^j(1-q)^{-j}\,q^{j(j-1)/2}.$$

Thus, we obtain

$$S_q\left\{\sum_{j=0}^{\infty}y^j\;\frac{{}_2\Phi_1\left(q^{-j},0;q;q,q^jx/y\right)}{(q;q)_j};s\right\}=\mathbb{S}_q\left\{\sum_{j=0}^{\infty}y^j\;\frac{{}_1\Phi_1\left(q^{-j};q;q,-q^{j+1}x/y\right)}{(q;q)_j};s\right\}=e_q(y)/e_q(s).$$

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References

[1] Abdi, W.H. *Certain inversion and representation formulae for q-Laplace transforms*, Math. Zeitschr., **83**, 238–249, 1964.

[2] Abu-Risha, M., Annaby, M., Ismail, M.E.H. and Mansour, Z. *Linear q-difference equations*, Z. Anal. Anwend., **26**, 481–494, 2007.

[3] Albayrak, D., Purohit, S.D. and Uçar, F. *On q-analogues of Sumudu transforms*, An. Ştiinţ. Univ. "Ovidius" Constanţa, Ser. Mat., **21**(1), 239–260, 2013.

[4] Albayrak, D., Purohit, S.D. and Uçar, F. *On q-Sumudu transforms of certain q-polynomials*, Filomat, **27**(2), 413–429, 2013.

[5] Belgacem, F.B.M., Karaballı, A.A. and Kalla, S.L. *Analytical investigations of the Sumudu transform and applications to integral production equations*, Math. Probl. Eng., **2003**, 103–118, 2003.

[6] Belgacem, F.B.M., Karaballı, A.A. *Sumudu transform fundamental properties investigations and applications*, Journal of Applied Mathematics and Stochastic Analysis, Article ID 91083, 1–23, 2006.

[7] Jackson, F. H. *On q-definite Integrals*, Q. J. Pure Appl. Math., **41**, 193–203, 1910.

[8] Jackson, F. H. *Applications of basic numbers to Bessel's and Legendre's functions*, Proc. London Math. Soc., **2**, 192–220, 1904.

[9] Jarad, F. and Taş, K. *On Sumudu transform method in discrete fractional calculus*, Abstr. Appl. Anal., **2012**, doi:10.1155/2012/270106.

[10] Kilicman, A., Atan, K. A. M. and Eltayeb, H. *A note on the comparison between Laplace and Sumudu transforms*, Bulletin of the Iranian Mathematical Society, **37**(1), pp. 131–141, 2011.

[11] Koelink, E. Quantum groups and q-special functions, Report 96–10 (1996), Universiteit van Amsterdam.

[12] McLachlan, N. W. Modern Operational Calculus with Applications in Technical Mathematics, Macmillan, London, 1948.

[13] Purohit, S.D. and Raina, R.K. *Generalized q-Taylor's series and applications*, Gen. Math., **18**(3), 19–28, 2010.

[14] Purohit, S.D. and Raina, R.K. *Certain subclass of analytic functions associated with fractional q-calculus operators*, Math. Scand., **109** (1), 55–70, 2011.

[15] Purohit, S.D. and Kalla, S.L. *On fractional partial differential equations related to quantum mechanics*, J. Phys. A: Math. Theor., **44**(4) (8pp.), 2011.

[16] Saxena, R.K., Saxena, R. and Kalla, S.L. *Solution of space-time fractional Schrodinger equation occuring in quantum mechanics*, Fract. Calc. Appl. Anal., **13**(2), 177–190, 2010.

[17] Watugala, G. *Sumudu transform: A new integral transform to solve differential equations and control engineering problems*, Int. J. Math. Educ. Sci. Technol., **24**(1), 35–43, 1993.

[18] Weerakoon, S. *Application of Sumudu transform to partial differential equations*, Int. J. Math. Educ. Sci. Technol., **25**, 277–283, 1994.