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Research Article

Finite Difference Solution to the Time-Fractional Differential-Difference Burgers Equation

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Abstract

This paper deals with the time-fractional differential-difference Burgers equation $\frac{d^{\alpha}u_n}{dt^{\alpha}} = (1+u_n)(u_{n+1}-u_n)$, $\alpha \in (0,1)$. The compact finite differences method (CFD-method) is used for numerical solution of this problem. According to the method, we approximate the unknown values u_n of the desired function by compact finite differences approximation. As an application, we demonstrate the capabilities of this method for identification of various values of order of fractional derivative with distinct two fractional (Riemann–Liouville and Caputo) derivatives. Numerical results show that the proposed version of CFD-method allows to obtain all data from the initial condition with sufficient high accuracy. AMS (MOS) subject classifications. 35R30, 47A52, 35L20

Keywords: Compact finite differences method, Time-fractional Burger equation, Time-fractional differential-difference equations.

Zamansal-Kesirli Diferansiyel Fark Burger Denkleminin Sonlu Farklar Yöntemiyle Çözümü

Öz

Bu makalede zamansal-kesirli diferansiyel fark Burger Denklemi $\frac{d^{\alpha}u_n}{dt^{\alpha}} = (1+u_n)(u_{n+1}-u_n)$, $\alpha \in (0,1)$ üzerinde durulmuştur. Bu denklemin sayısal çözümü için kompakt sonlu farklar metodu (CFD) kullanılmıştır. Bu metoda göre, kompakt sonlu fark yaklaşımı ile ilgili fonksiyonun bilinmeyen bir u_n değerine yaklaşılmıştır. Bir uygulama olarak, farklı iki kesir türevi (Riemann-Liouville ve Caputo) incelenmiştir. Bu iki kesir türev tipi için farklı mertebelerde bulunan değerler karşılaştırılmıştır. Sayısal sonuçlar, CFD yönteminin önerilen versiyonunun, başlangıç koşulundan tüm verilerin yeterli yüksek doğrulukta elde edilmesini sağladığını göstermektedir.

Anahtar Kelimeler: Kompakt sonlu farklar metodu, Zamansal-kesirli burger denklemi, Zamansal-kesirli diferansiyel-fark denklemi.

1. Introduction

Fractional calculus, formerly, has been used mainly by mathematicians as an abstract area which covering only pure mathematical manipulations. But, recently, the paradigm began to change from pure to applied mathematics with various applications. Due to applications of fractional calculus several kinds of numerical methods has appeared in the literature. Among various definitions of fractional derivatives including the most frequently used ones, the Riemann–Liouville derivative, the Caputo derivative, and some other fractional derivatives may be found state-of-the-art studies [Podlubny, 1999; Li and Zeng, 2015; Kilbas et al., 2006; Miller and Ross, 1993; Duarte, 2011; Podlubny,1988]. Hence, due to the rapid development of fractional numerical methods, more and more publications are emerging. [Mohan and Deekshitulu, 2012; Cui, 2009; Hodzic-Zivanovic and Jovanovic, 2017; Yokus and Kaya, 2017;Rawashdeh, 2017; Al-luhaibi, 2015]

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Here and now, we set up notations, basic definitions and main properties of Riemann- Liouville derivative and the relation between Riemann-Liouville derivative and Caputo fractional derivative. Hence, by definition, Riemann-Liouville fractional derivative with fractional order $\alpha \in (0,1)$ of the function $u = u(t)$, may be given [Podlubny, 1999; Cui, 2009] , i.e.,

$$\left[\frac{d^{\alpha}u(t)}{dt^{\alpha}}\right]_{RL} := \frac{1}{\Gamma(1-\alpha)}\frac{\partial}{\partial t}\int_{0}^{t}\frac{u(\tau)}{(t-\tau)^{\alpha}}d\tau, \quad t>0 \quad (1.1)$$

where $\Gamma(x)$ is the Euler's Gamma Function, which has following properties:

$$\Gamma(x) = \int_{0}^{\infty} e^{-t}t^{x-1}dt, \quad x>0; \Gamma(x+1) = x\Gamma(x);$$

$$\Gamma(n+1) = n!, \; n\epsilon N.$$

Also, Caputo fractional derivative with fractional order $\alpha \in (0,1)$ of the function $u = u(t)$ is defined by [Podlubny, 1999; Cui, 2009] as follows:

$$\left[\frac{d^{\alpha}u(t)}{dt^{\alpha}}\right]_{C} := \frac{1}{\Gamma(1-\alpha)}\int_{0}^{t}\frac{u'(\tau)}{(t-\tau)^{\alpha}}d\tau, \quad t>0. \; (1.2)$$

From (1.1) and (1.2), it is clear that definitions of Riemann–Liouville derivative and Caputo derivative are not equivalent. But, there is a fact that, almost all the numerical methods for the Riemann–Liouville derivative can be theoretically extended to the Caputo derivative if the function $u(t)$ satisfies suitable smooth conditions [Li and Zeng, 2015]. Following equality shows the relation between the Riemann–Liouville and Caputo derivatives for $0<\alpha<1$;

$$\left[\frac{d^{\alpha}u(t)}{dt^{\alpha}}\right]_{RL} = \left[\frac{d^{\alpha}u(t)}{dt^{\alpha}}\right]_{C} + \frac{t^{-\alpha}u(0)}{\Gamma(1-\alpha)}, \quad t>0, \quad (1.3)$$

where $\left[\frac{d^{\alpha}u(t)}{dt^{\alpha}}\right]_{RL}$ and $\left[\frac{d^{\alpha}u(t)}{dt^{\alpha}}\right]_{C}$ are Riemann–Liouville and Caputo fractional derivatives of the function $u = u(t)$, respectively [Podlubny, 1999].

In this study, in special, we will solve and make comparisons of the time-fractional differential-difference Burgers equation

$$\frac{d^{\alpha}u_n}{dt^{\alpha}} = (1+u_n)(u_{n+1}-u_n), \quad \alpha \in (0,1) \qquad (1.4)$$

with initial condition

$$u(t_1) = \phi, \; t_1 > 0, \; \phi\epsilon R \qquad (1.5)$$

where $\frac{d^{\alpha}u_n}{dt^{\alpha}}$ denotes the fractional derivative with fractional order α of the function $u = u(t)$ by applying Riemann–Liouville derivative and the Caputo derivative on fractional derivatives respectively and we will compare the numerical solutions with individual fractional derivatives numerically.

2. Numerical Implementation

Burger Equation for Riemann-Liouville fractional derivative: For the numerical solution to the considered problem (1.4) we consider Riemann-Liouville fractional derivative at left-hand side of (1.4):

$$\begin{cases}\left[\frac{d^{\alpha}u_n}{dt^{\alpha}}\right]_{RL} = (1+u_n)(u_{n+1}-u_n), \quad \alpha \in (0,1), \\ u(t_1) = \phi, \quad t_1 > 0.\end{cases} \qquad (2.1)$$

We construct a uniform grid of mesh points t_n with $t_n = n\tau$, $n = 0,1,2,\ldots,N$, and $\tau = T/N$. We denote the exact solution by $u_n = u(t_n)$ with $u(t_0) = 0$ and approximate solution by U_n at the same point t_n ($U_0 = 0$). We can approximate the Riemann-Liouville fractional derivative (1.3) by

$$\left[\frac{d^{\alpha}v(t)}{dt^{\alpha}}\right]_{RL}=\frac{1}{\tau^{\alpha}}\sum_{k=0}^{t/\tau}w_k^{\alpha}v(t-k\tau)+O(\tau^{p}),\quad t>0 \qquad (2.2)$$

where w_k^{α} are the coefficients of the generating function, that is $w_0^{\alpha}=1$, $w_k^{\alpha}=(1-\frac{\alpha+1}{k})w_{k-1}^{\alpha}$, $k\geq 1$ and $p=1$ [Li and Zeng, 2015; Cui, 2009]. Then the compact finite difference approximation of (2.1) is given as follows:

$$\frac{1}{\tau^{\alpha}}\sum_{k=0}^{n}w_k^{\alpha}U_{n-k}=(1+U_n)(U_{n+1}-U_n),\quad n\geq 1 \qquad (2.3)$$

So (2.3) gives the approximate solution for all points t_n as follows:

$$\begin{cases} U_{n+1}=U_n+\dfrac{w_0^{\alpha}U_n+w_1^{\alpha}U_{n-1}+\cdots+w_n^{\alpha}U_0}{\tau^{\alpha}(1+U_n)}, & n\geq 1 \\ U_1=\phi \end{cases} \qquad (2.4)$$

Numerical calculation: We consider here $\phi=0.5$ as initial data and $\alpha=0.8$ as fractional order of derivative. In this example the time step size is $\tau=0.5$ and number of time interval is $N=10$. The left Figure 1 shows numerical solution $U(t)$ for $t\in(0,T]$, $T=5$. The numerical solutions corresponding to the distinct values of time step size τ for $\alpha=0.8$ are plotted in the Figure 1, below.

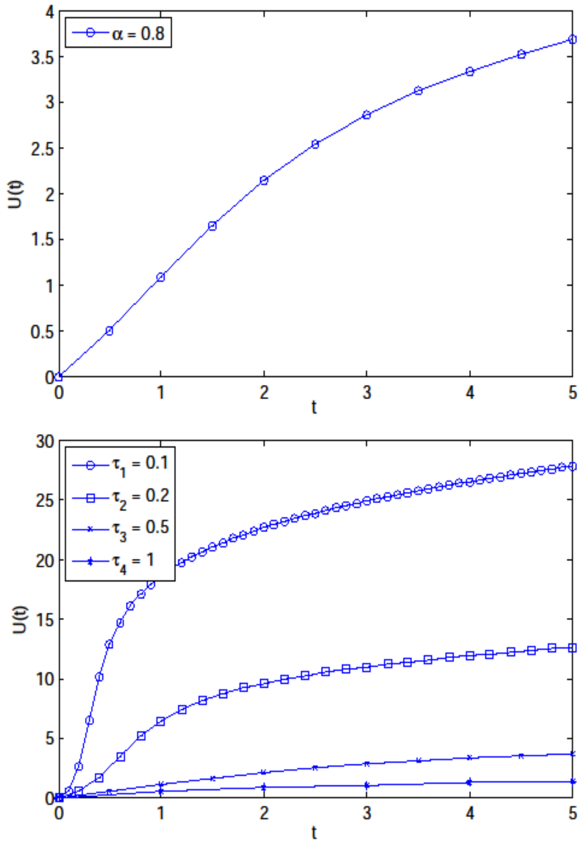

Figure 1. Solution for Riemann–Liouville fractional derivative ($\alpha=0.8$)

We think that Figure 1 verifies that solution $u(t)$ for distinct times t values preserves the characteristics of the problem, since no analytical solution is known. In order to get in to the numerical solutions, we run the problem with distinct $\alpha=0.2, 0.4, 0.6, 0.8$ values to see the consequences of fractional orders. Yet again, the time step size is $\tau=0.01$ and number of time interval is $N=20$. Figure 2 shows the numerical solutions corresponding to the distinct values of fractional order α.

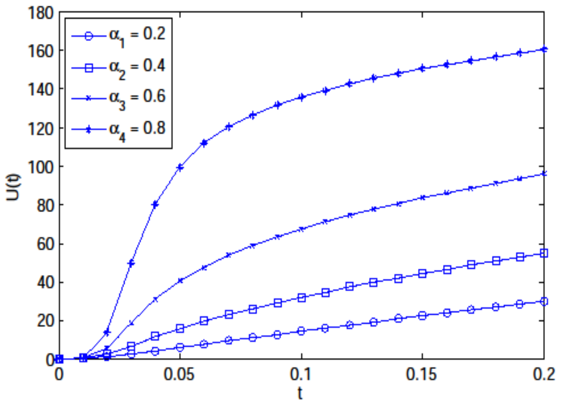

Figure 2. The solutions for distinct fractional orders ($\alpha = 0.2, 0.4, 0.6, 0.8$)

Since, the solutions with distinct fractional orders behave like, we conclude that obtained solution are good approximations for the problem in hand.

Burger Equation for Caputo fractional derivative: In case of Caputo fractional derivative in left-hand side of (1.4) we get

$$\begin{cases} \left[\dfrac{d^\alpha u_n}{dt^\alpha}\right]_C = (1+u_n)(u_{n+1}-u_n), \quad \alpha \in (0,1), \\ u(t_1) = \phi, \quad t_1 > 0. \end{cases} \tag{2.5}$$

We consider L1 approximation of the Caputo fractional derivative for (2.5). According to the L1 method, the Caputo fractional derivative is approximated as follows [Podlubny, 1999]:

$$\left[\frac{d^\alpha u_n}{dt^\alpha}\right]_C = \sum_{k=0}^{n-1} b_{n-k-1}\big(u(k+1)-u(k)\big) + O(\tau^{2-\alpha}), \quad 0<\alpha<1, \tag{2.6}$$

where $b_k = \dfrac{\tau^{-\alpha}}{\Gamma(2-\alpha)}\left((k+1)^{1-\alpha} - k^{1-\alpha}\right)$.

Taking into account (2.6) in (2.5), we get the approximate solution of (2.5) as follows:

$$\begin{cases} U_{n+1} = U_n + \dfrac{b_{n-1}(U_1-U_0)+b_{n-2}(U_2-U_1)+\cdots+b_0(U_n-U_{n-1})}{1+U_n}, \quad n \ge 1. \\ U_1 = \phi. \end{cases} \tag{2.7}$$

Numerical computation: We, again, consider here $\phi = 0.5$ as initial data and $\alpha = 0.8$ as fractional order of derivative. In this example the time step size is $\tau = 0.5$ and number of time interval is $N = 10$. The left Figure 3 shows numerical solutions $U(t)$ of (1.4) and (2.5) for $t \in (0,T]$, $T = 5$. The numerical solutions of (2.5) corresponding to the distinct values of time step size τ for $\alpha = 0.8$ are plotted in the right Figure 3.

In order to compare the numerical solutions of (2.5), we consider here $\phi = 0.5$ as initial data and $\alpha = 0.2, 0.4, 0.6, 0.8$ as fractional order of derivative. In this example the time step size is $\tau = 0.01$ and number of time interval is $N = 20$. Figure 4 shows the numerical solutions corresponding to the distinct values of fractional order α.

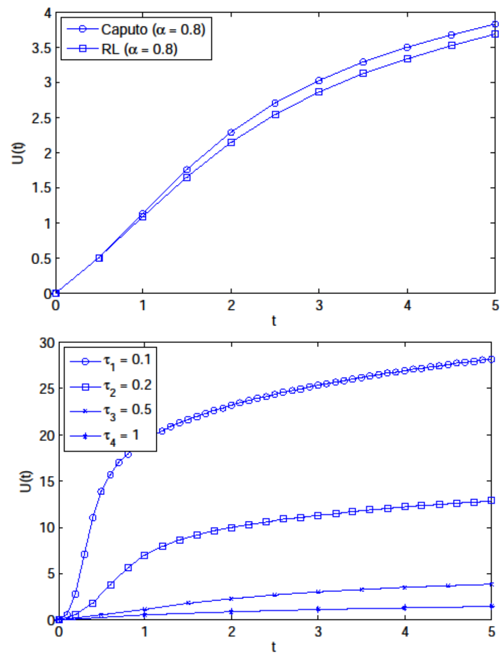

Figure 3. Solutions for Caputo fractional derivatives ($\alpha = 0.8$)

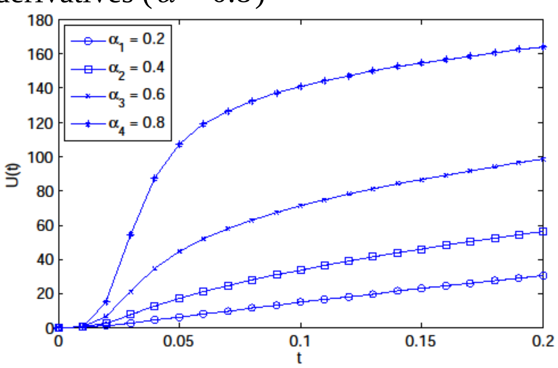

Figure 4. Solutions with respect to the α values

It is noticeable that the left figure in Figure 3 shows slight difference on the solutions with Riemann–Liouville fractional derivative and Caputo fractional derivatives for $\alpha = 0.8$. We think that this slight difference is due to the second term on r.h.s of equation (1.3) which states the relation between the Riemann–Liouville and Caputo derivatives for $0 < \alpha < 1$ Hence, numerical experiments clearly show that the outcome of this term in equation (1.3) is insignificant, and any of the derivative may be used for future problems.

3. Conclusions

In this study the time-fractional differential-difference Burgers equation is experimented numerically with Riemann–Liouville and Caputo derivatives. We use the compact finite differences method for numerical solution of the problem and present computational results for the case of distinct values of fractional order α as well as initial data. Numerical experiments show that any of the fractional (Riemann–Liouville and Caputo) derivatives may be used for any physical problem without any reluctance and the choice of the fractional derivative is negligible at least the problem considered in this study.

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