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# A Practical Method for the Dynamic Analysis of Non-Uniform Piezoelectric Rod

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Laplace transform,
Pseudospectral Chebyshev Method

**Abstract:** In this paper, a unified approach for the dynamic analysis of non-uniform piezoelectric rod is presented. It is assumed that the cross sectional area of the rod is varying along the longitudinal axis, arbitrarily. Therefore, the partial differential equations that govern the non-uniform piezoelectric isotropic rod in a forced vibration analysis are obtained with a variable coefficient taking into account mechanical and electrostatic equations. Analytical solutions of these equations are only possible for simple cross-section areas. First, the governing equations are transformed to the Laplace space and then solved numerically by pseudospectral Chebyshev approach for arbitrary cross-section area under four different load functions. The final results are transformed to the time domain using modified Durbin's procedure. The technique is validated for simple cross-section area results that can also be solved analytically.

# Düzgün Olmayan Piezoelektrik Çubuğun Dinamik Analizi için Pratik Bir Yöntem

**Anahtar Kelimeler**
Zorlanmış titreşim,
Piezoelektrik çubuk,
Laplace dönüşümü,
Pseudospectral Chebyshev yöntemi

**Özet:** Bu çalışmada, düzgün olmayan piezoelektrik çubuğun dinamik analizi için birleşik bir yaklaşım sunulmaktadır. Çubuğun enine kesit alanının rastgele olarak uzunlamasına eksen boyunca değiştiği varsayılmaktadır. Bu nedenle, zorlanmış titreşim analizinde düzgün olmayan piezoelektrik izotropik çubuğu idare eden kısmi diferansiyel denklemler, mekanik ve elektrostatik denklemler dikkate alınarak değişken bir katsayılı olarak elde edilirler. Bu denklemlerin analitik çözümleri sadece basit kesit alanları için mümkündür. İlk olarak, sistemi idare eden denklemler Laplace uzayına dönüştürülür ve daha sonra dört farklı yük fonksiyonu altında rastgele kesit alanı için pseudospektral Chebyshev yaklaşımı ile sayısal olarak çözülür. Nihai sonuçlar, modifiye edilmiş Durbin prosedürü kullanılarak zaman uzayına dönüştürülür. Yöntem, analitik olarak da çözülebilen basit kesit alanına sahip piezoelektrik çubuk sonuçları ile doğrulanmıştır.

## 1. Introduction

Smart materials are new generation materials that have ability to change their properties in a particular manner in response to specific stimulus input. Piezoelectric materials are very common example of smart materials that are able to deform under an electric field or produce an electric sigmal as a result of any mechanical effect. Piezoelectric materials, considered to be active smart materials and therefore, they can be used as force transducers and actuators. These materials find application in the field of structural engineering such as monitoring the civil engineering structures to evaluate their stability. Some materials, such as rods and beams, are designed with variable cross-section due to technical advantages in specific piezoelectric applications. Vibration behavior of non-uniform rods with variable cross section were investigated extensively in the literature [1–6]. Studies on the piezoelectricity are usually related to finite and infinite dimensional structures in different geometries such as thin rods, solid or hollow cylinders, plates, discs, cylindrical shells. In the case of complicated geometrical shapes and material properties, applying numerical methods is often inevitable. Piezoelectric transducers can be treated as rods in terms of mathematical modelling. Thus, it is important to confirm the vibration behavior of piezoelectric rods.

Chen and Zhang [7] obtained analytical equations of a non-uniform cross section rod. For piezoelectromagnetic beam, one dimensional equations derived by Zhang et al.

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[8] for the flexure and extension with shear deformation from three dimensional equations. Nadal and Pigache [9] established accurate electromechanical model of a piezoelectric transformer by using Hamilton's principle to obtain the equations of motion for free vibrations. Li and Zhifie [10] furthered the field based on the elasticity theory and piezoelectricity with state space based differential quadrature method to the free vibrations of a functionally graded piezoelectric beam under different boundary conditions. By the help of piezoelectric patch actuator, active vibration controlling in structural elements like beam and plate etc. is also promosing research area. The optimal time dependent vibration control of the Timoshenko beam is examined by Yildirim [11] and with damping effect by Kucuk [12]. The optimal control is determined by using maximum principle in their paper. The vibrations activated by extarnal force in a rectangular plate with Kelvin-Voigt damping are controlled by piezoelectric patch actuators bonded two sides of the plate in Yildirim's study [13]. He analytically solved the problem by using Galerkin expansions. In our previous studies, free [14, 15] and forced longitudinal vibrations [16] of non-uniform piezoelectric rod is solved numerically by complementary function method (CFM), pseudospectral Chebyshev method and analytically by Bessel functions for an arbitrary cross-section area. Differentiation matrices that signify the approximations at grid points play an important role in the implementation of spectral collocation methods [17]. Constructing procedure of Chebyshev differentiation matrices with the help of Chebyshev points (grid point) is found in [18] and the implementation for numerical solution of convection-diffusion problem in [19]. In this study, forced longitudinal vibration of non-uniform piezoelectric (PZT-4) fixed-free supported rod is solved in Laplace domain numerically by Chebyshev pseudospectral approach for arbitrary cross-section area under four different load functions. Durbin's Laplace inversion procedure is used to get the results in the time domain [20]. Then, the numerical and analytical consequences are compared.

## 2. Material and Method

Rosen type transducers provide the most efficient tools in piezoelectric applications. The transducer consists of two parts including a driving portion and receiving portion that is operated by exploiting extensional vibrations of these parts. Each of the driving portion and receiving portion with different coordinates and under different polarization conditions are to be considered as a piezoelectric rod. The non-uniform piezoelectric rods polarized along the longitudinal axis are considered for three different cross-sectional areas: power, exponential and cosinusoidal form [16].

Constitutive equations of piezoelectric materials that exhibit linear behavior define electromechanical properties and can be derived in variety ways to tailor for desired properties [7]. Under the assumption of material properties do not change along the x-axis, and with the consideration of mechanical and electrostatic equations together, the governing equation of the system can be written in the following form [7, 14],

$$\frac{d^2u}{dx^2}+\frac{1}{A(x)}\frac{dA(x)}{dx}\frac{du}{dx}=\frac{\rho}{\bar{c}_{11}}\frac{\partial^2 u}{\partial t^2} \tag{1}$$

where $u$ is the extensional displacement, $A(x)$ is the cross sectional area varying along the longitudinal axis, $\bar{c}_{11}=\tilde{c}_{11}+\frac{\tilde{e}_{11}^2}{\tilde{\varepsilon}_{11}}$ and $\tilde{c}_{11},\tilde{e}_{11}$ and $\tilde{\varepsilon}_{11}$ elastic, piezoelectric and dielectric constants for longitudinal motions respectively. Clamped-free supported rod which is electrically open on two ends is considered, therefore the initial and boundary conditions become,

$$u(x,0)=0, \qquad \frac{\partial u}{\partial t}(x,0)=0 \tag{2a}$$

$$u(0,t)=0, \qquad \frac{\partial u}{\partial x}(l,t)=\frac{P(t)}{\bar{c}_{11}A(l)} \tag{2b}$$

where $P(t)$ represents load functions applied to the free end of the rod. Four different types of loading functions have been applied to the system: step, sinusoidal impulsive, arbitrary and ramped [16, 21]. If Laplace transformation is applied to Equation (1) by using initial conditions (2a), the following ordinary differential equation is obtained

$$\bar{U}''+P(x,s)\bar{U}'+Q(s)\bar{U}=0 \tag{3}$$

with

$$P(x,s)=\frac{1}{A(x)}\frac{dA(x)}{dx}, \quad Q(s)=-\frac{s^2\rho}{\bar{c}_{11}}$$

where $()'$ denotes the derivative with respect to $x$, $\bar{U}=U(x,s)$ is the Laplace transformation of $u$, and $s$ is the Laplace transform parameter. And, the transformed boundary conditions (2b) written in the following way:

$$U(0,s)=0, \qquad \frac{\partial U}{\partial x}(l,s)=\frac{p(s)}{\bar{c}_{11}A(l)} \tag{4}$$

where $p(s)$ is the Laplace transform of the load $P(t)$ [16].

## 3. Pseudospectral Chebyshev Model

Pseudospectral Chebyshev Model used to perform forced vibration analysis of non-uniform piezoelectric rod by referring to the study of Gottlieb [22] and Trefethen [17] that depends on discretization the governing equations (3) with respect to the spatial variable using pseudospectral Chebyshev method. With regard to collacation points, the first order $(n+1)\times(n+1)$ Chebyshev differentiation matrix

$$0=x_0<x_1\cdots<x_n,\ with\ x_j=\frac{1}{2}[1-\cos(j\pi/n)]$$

$(j=0,1,\cdots,n)$ will be denoted by $D$. First-order Chebyshev differentiation matrix D provides highly precise approxtimation to $\bar{U}'(x_j),\bar{U}''(x_j),...,$ simply by multiplication differential matrix with vector data $\bar{U}'(x_j)=(D\,\bar{U})_j$, $\bar{U}''(x_j)=(D^2\bar{U})_j$, suchlike where $\bar{U}=[\bar{U}_0,\ldots,\bar{U}_n]^T$ discrete vector data at positions $x_j$. The computation procedure of the Chebyshev differentiation matrix and codes as a m-file can be found in notable references, see e.g., Trefethen [17], where the collocation points $x_i$ are numbered from right to left and defined in $[-1,1]$. With a small revision, the method can be implemented to any interval.

### 3.1. Formulation

Three different cross-section forms depend on the longitudinal axis are considered in this research. The cross-sectional areas are given in the following forms:

$$\text{Power form:}\quad A_p(x) = A_0\left(a+b\frac{x}{l}\right)^n \tag{5a}$$

$$\text{Exponential form:}\quad A_e(x) = A_0 e^{\frac{-nx}{l}} \tag{5b}$$

$$\text{Cosinusoidal form:}\quad A_c(x) = A_0\, Cos^2\left[a\frac{n\,x}{l}+b\right] \tag{5c}$$

The linear ordinary differential equation in Laplace domain (3) is basically converted to a linear system by using the differential matrix operator. First and second-order derivatives in Equation (3) can be discretized with differential matrix in the following way:

$$\begin{bmatrix} \frac{dU}{dx}(x_0,s) \\ \frac{dU}{dx}(x_1,s) \\ \vdots \\ \frac{dU}{dx}(x_n,s) \end{bmatrix} \approx D \begin{bmatrix} U(x_0,s) \\ U(x_1,s) \\ \vdots \\ U(x_n,s) \end{bmatrix} \tag{6}$$

and

$$\begin{bmatrix} \frac{dU^2}{dx^2}(x_0,s) \\ \frac{dU^2}{dx^2}(x_1,s) \\ \vdots \\ \frac{dU^2}{dx^2}(x_n,s) \end{bmatrix} \approx D^2 \begin{bmatrix} U(x_0,s) \\ U(x_1,s) \\ \vdots \\ U(x_n,s) \end{bmatrix} \tag{7}$$

Rearranging the Equation (3) by using differential matrix approximation,

$$M\bar{U} = 0 \tag{8}$$

linear equation system is obtained, where

$$M = D^2 + P(x,s)D + Q(s) \tag{9}$$

Here, $P(x,s)$ is a variable coefficient and depending on the cross-sectional areas. This coefficient takes the following forms for three different cross-section:

$$\text{Power form:}\quad P(x,s) = \frac{nb}{(al+bx)},$$

$$\text{Exponential form:}\quad P(x,s) = \frac{n}{l},$$

$$\text{Cosinusoidal form:}\quad P(x,s) = -\frac{2n\,a}{l} Tan\left[a\frac{n\,x}{l}+b\right].$$

When boundary conditions (4) are applied to this linear system (8), nontrivial solution is obtained in Laplace space. Then, the results are transformed into the time domain by using inverse Laplace transformation (modified Durbin's) procedure.

## 4. Results

This section will present some numerical example showing the capability of the presented method and also to confirm the simulated results with analytical examinations for constant cross-section case. For this purpose, a general objective computer program is coded in Matlab to analyze the forced vibration of non-uniform piezoelectric rods. Material constants for PZT-4 was taken from [23]. The geometrical parameters $a$, $b$ are taken as 0.8, $-0.2$ and inhomogeneity parameter $n$ is taken as 1, 1.5, 1.8 for the entirety of this study. The results for $n = 0$ are corresponding to uniform cross-section with constant material properties. Comparison of analytical results are presented for uniform rod to ensure the efficacy and adequacy of the present method and monitored in Table 1 [16]. It can be noted from Table 1, the pseudospectral Chebyshev method results match quite well (six-digit accuracy for $N = 10$) with the analytical results. Figure 3(a-l) show the displacement on the end of the rod ($x = l$) for different geometrical models subjected to dynamic loads. As shown in Figure 3(a-l), the inhomogeneity parameter is a useful parameter for controlling the displacement amplitude.

## 5. Discussion and Conclusion

In this study, a unified numerical approach for the dynamic analysis of non-uniform piezoelectric rod is presented. It is assumed that the cross sectional area of the rod is varying along the longitudinal axis, arbitrarily. The numerical models of the forced vibration of non-uniform piezoelectric (PZT-4) rods are obtained in the Laplace space and then solved numerically by pseudospectral Chebyshev method. Inverse transformation into the time domain is achieved by modified Durbin's method. It can be concluded from this research that:

- This unified procedure can be easily applied to any dynamical problems.
- The solution procedure has sufficient accuracy, small computational costs and ease of application.
- In terms of design perspective, the inhomogeneity parameter constitutes an adjustment variable for particular applications. This further enables control on the displacement amplitude.

**Table 1.** Comparison of displacement $(u)$ at the end of the rod $(x = L)$ for constant cross-section.$(u \times 10^9)$

| Step Load | | Sinusoidal Load | | Arbitrary Load | | Periodic Load | |
|---|---|---|---|---|---|---|---|
| Numeric | Analytic[16] | Numeric | Analytic[16] | Numeric | Analytic[16] | Numeric | Analytic[16] |
| 0.004884 | 0.004884 | 0.000350 | 0.000350 | -0.014654 | -0.014654 | 0.000046 | 0.000046 |
| 0.070423 | 0.070423 | 0.074244 | 0.074244 | -0.096752 | -0.096752 | 0.009464 | 0.009464 |
| 0.077362 | 0.077362 | -0.017375 | -0.017375 | -0.167576 | -0.167576 | 0.014953 | 0.014953 |
| 0.010689 | 0.010689 | -0.053253 | -0.053253 | 0.243588 | 0.243588 | 0.024008 | 0.024008 |
| 0.062570 | 0.062570 | 0.064356 | 0.064356 | -0.016346 | -0.016346 | 0.033869 | 0.033869 |
| 0.083473 | 0.083473 | -0.003522 | -0.003522 | -0.212244 | -0.212244 | 0.039452 | 0.039452 |
| 0.013151 | 0.013151 | -0.061249 | -0.061249 | 0.220716 | 0.220716 | 0.048004 | 0.048004 |
| 0.054286 | 0.054286 | 0.057556 | 0.057556 | 0.041104 | 0.041104 | 0.054218 | 0.054218 |
| 0.088319 | 0.088319 | 0.010472 | 0.010472 | -0.256964 | -0.256964 | 0.051812 | 0.051812 |
| 0.017159 | 0.017159 | -0.066794 | -0.066794 | 0.185620 | 0.185620 | 0.049435 | 0.049435 |

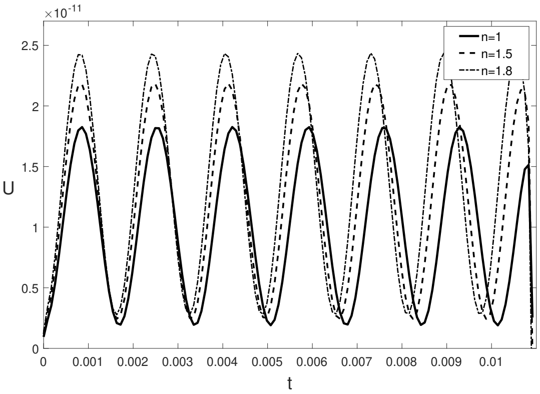


**Figure 1 (a):**Step load with power form cross section

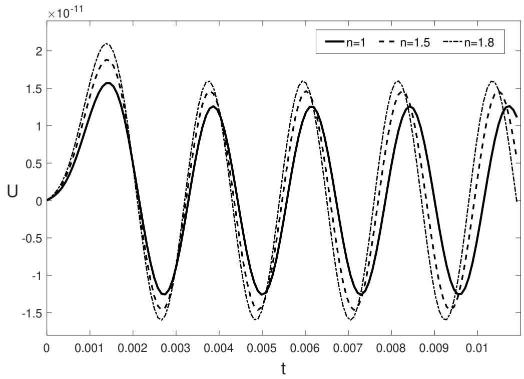


**Figure 1 (d):** Sinusoidal load with power form cross sec.

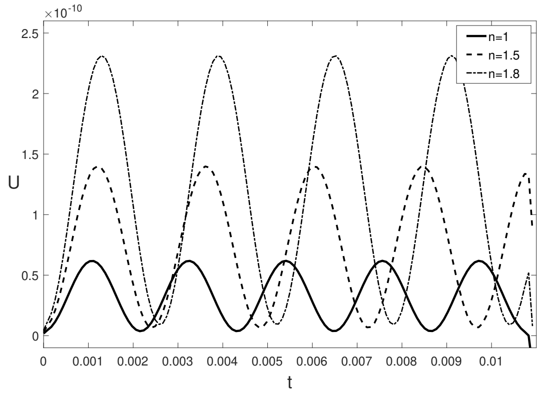


**Figure 1 (b):**Step load with exponential form cross sec.

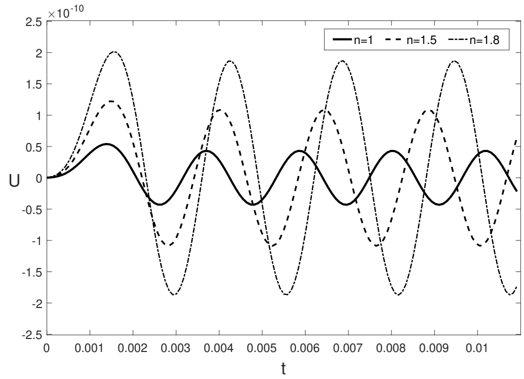


**Figure 1 (e):**Sinusoidal load with exponential form cross section

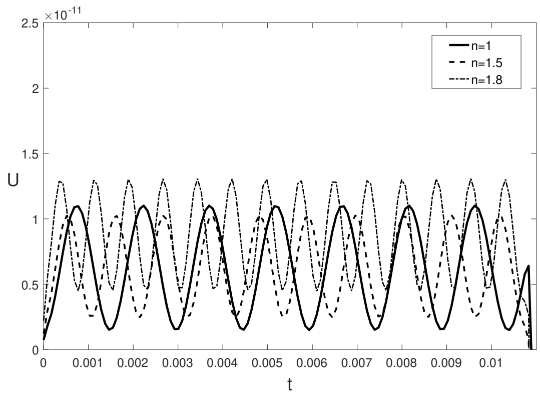


**Figure 1 (c):**Step load with cosinusoidal form cross sec.

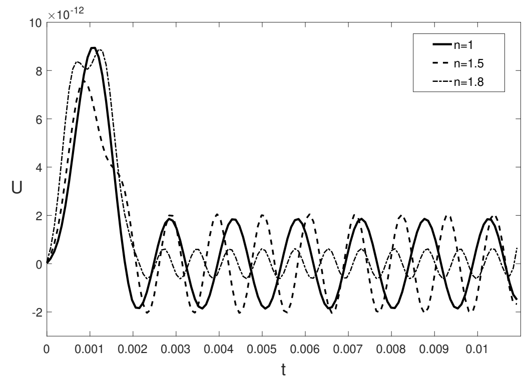


**Figure 1 (f):**Sinusoidal load with cosinusoidal form cross section

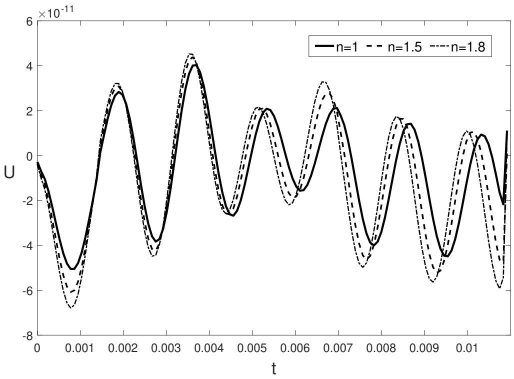


**Figure 1 (g):**Arbitrary load with power form cross sec.

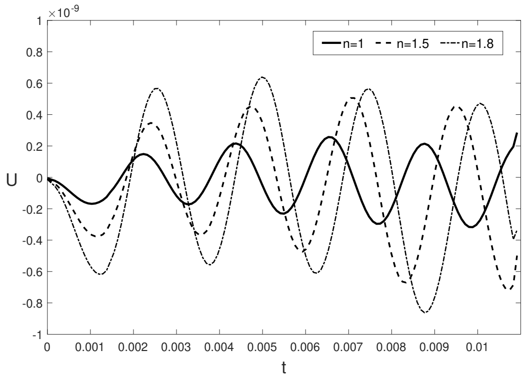


**Figure 1 (h):**Arbitrary load with exponential form cross section

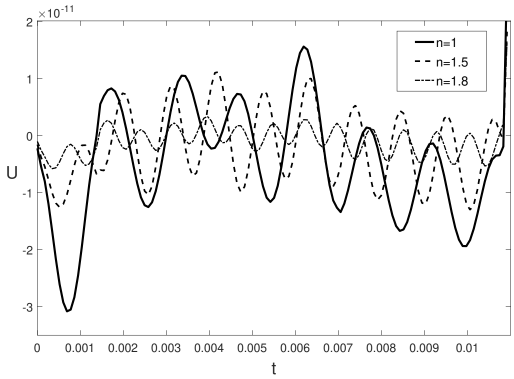


**Figure 1 (i):**Arbitrary load with cosinusoidal form cross section

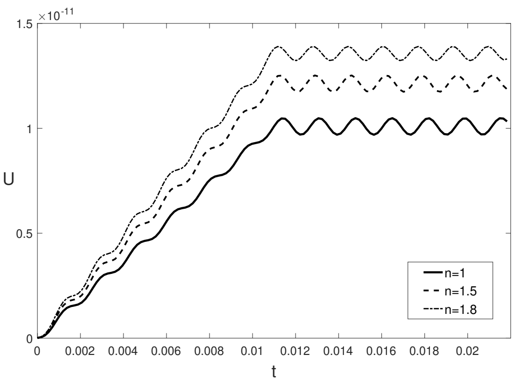


**Figure 1 (j):**Ramped load with power form cross sec.

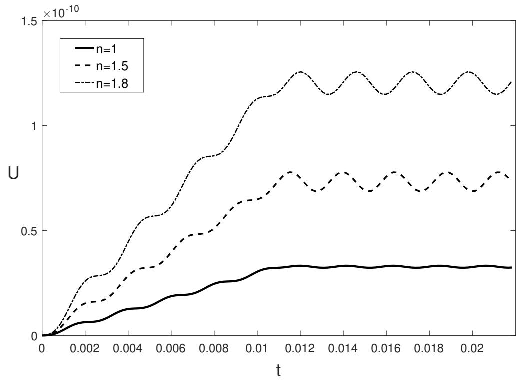


**Figure 1 (k):**Ramped load with exponential form cross section

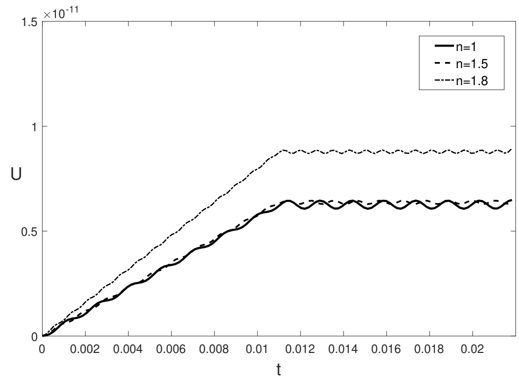


**Figure 1 (l):**Ramped load with cosinusoidal form cross section

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