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New Approach to Slant Helix

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ABSTRACT

A slant helix is a curve for which the principal normal vector field makes a constant angle with a fixed direction. In this study, we solve a system of linear ordinary differential equations involving an alternative moving frame, then determine the position vectors of slant helices through integration in Minkowski 3-space.

Keywords: *Slant helix; Alternative moving frame.*

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1. Introduction

In the local differential geometry, we think of curves as a geometric set of points. Intuitively, we are thinking of a curve as the path traced out by a particle moving in $\mathbb{E}^3$. So, the investigating position vectors of the curves in a classical aim to determine behavior of the curve. A curve of constant slope or general helix in Euclidean $3-$space is well-known curve in the classical differential geometry of space curves and is defined by the property that the tangent vector field makes a constant angle with a fixed straight line (the axis of the general helix). Helix is one of the most fascinating curves in science and nature. We can see the helical structures in nano-springs, carbon nano-tubes, DNA double, vines, screws, springs, and sea shells and also in fractal geometry [2, 10, 13]. A classical result stated by M. A. Lancret in 1802 and first proved by B. de Saint Venant in 1845 is: A necessary and sufficient condition that a curve be a general helix is that the ratio of curvature to torsion be constant [8, 12]. Later, Izumiya and Takeuchi have introduced the concept of slant helix in Euclidean $3-$space saying that the normal lines makes a constant angle with a fixed direction. They characterize a slant helix if and only if the geodesic curvature

$$\sigma = \frac{\kappa^2}{(\kappa^2+\tau^2)^{3/2}}\left(\frac{\tau}{\kappa}\right)'$$

of the principal image of the principal normal indicatrix is a constant function [5].

On the studies of general helices in Lorentzian space forms, Lorentz-Minkowski spaces, semi-Riemannian manifolds, we refer to the papers [1, 3, 4]. In [7], Kula and Yayli have studied spherical images of tangent indicatrix and binormal indicatrix of a slant helix and they showed that the spherical images are spherical helix. Recently, Kula et al. investigated the relations between a general helix and a slant helix [6].

In this study, we construct a slant helices using the system of linear ordinary differential equations. We determine the position vectors of slant helices through integration according to alternative moving frame in Minkowski $3-$space.

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2. Preliminaries

Let $\mathbb{E}_1^3$ be Minkowski $3-$space which endowed with the standard flat metric given by

$$\begin{aligned} \langle , \rangle &= \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R} \\ (u,v) &\to \langle u,v \rangle = u_1v_1 + u_2v_2 - u_3v_3 \end{aligned}$$

where $u = (u_1, u_2, u_3)$ and $v = (v_1, v_2, v_3)$ are the usual coordinate system in $\mathbb{E}_1^3$. Due to semi-Riemannian metric, an arbitrary vector $u \in \mathbb{E}_1^3$ said spacelike if $\langle u,u \rangle > 0$ or $u = 0$, timelike if $\langle u,u \rangle < 0$ and null (lightlike) if $\langle u,u \rangle = 0$ but $u \neq 0$. This classification can be generalized for a given regular curve $\alpha : I \subset \mathbb{R} \to \mathbb{E}_1^3$ depending on the casual character of their tangent vectors, that is, the curve α is called a spacelike (resp. timelike and lightlike) if its velocity vector $\alpha'(t)$ is spacelike (resp. timelike and lightlike) for every $t \in I$. The norm of a vector u is given by $\|u\| = \sqrt{|\langle u,u \rangle|}$. Two non-null vectors u and v are orthogonal, if $\langle u,v \rangle = 0$.

Assume that $\{T(s), N(s), B(s)\}$ is the moving positive directed Frenet frame along the unit speed curve α in Minkowski $3-$space. Here $T(s) = \alpha'(s)$ is a tangent vector, $N(s) = \dfrac{\alpha''(s)}{\|\alpha''(s)\|}$ is a principal normal vector and $B(s) = T(s) \times N(s)$ is a binormal vector field along the curve α.

If $\alpha(s)$ is spatial curve in Minkowski $3-$space, then Frenet formulas can be given as follows, see [9]:

$$\begin{pmatrix} T'(s) \\ N'(s) \\ B'(s) \end{pmatrix} = \begin{pmatrix} 0 & \varepsilon_2\kappa(s) & 0 \\ -\varepsilon_1\kappa(s) & 0 & \varepsilon_3\tau(s) \\ 0 & -\varepsilon_2\tau(s) & 0 \end{pmatrix} \begin{pmatrix} T(s) \\ N(s) \\ B(s) \end{pmatrix} \tag{2.1}$$

where $\kappa(s) = \left\langle T'(s), N(s) \right\rangle$ and $\tau(s) = \left\langle N'(s), B(s) \right\rangle$ are curvature and torsion of $\alpha(s)$, respectively. Moreover, the Frenet vectors satisfy

$$\begin{aligned} \langle T(s), T(s) \rangle &= \varepsilon_1,\ \langle N(s), N(s) \rangle = \varepsilon_2,\ \langle B(s), B(s) \rangle = \varepsilon_3, \\ \langle T(s), N(s) \rangle &= \langle T(s), B(s) \rangle = \langle N(s), B(s) \rangle = 0 \end{aligned}$$

Introduce a new alternative moving frame $\{N(s), C(s), W(s)\}$ defined by Uzunoglu et al.

$$\begin{pmatrix} N(s) \\ C(s) \\ W(s) \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ -\dfrac{\varepsilon_1\kappa}{\sqrt{\kappa^2+\tau^2}} & 0 & \dfrac{\varepsilon_3\tau}{\sqrt{\kappa^2+\tau^2}} \\ -\dfrac{\varepsilon_3\tau}{\sqrt{\kappa^2+\tau^2}} & 0 & -\dfrac{\varepsilon_1\kappa}{\sqrt{\kappa^2+\tau^2}} \end{pmatrix} \begin{pmatrix} T(s) \\ N(s) \\ B(s) \end{pmatrix} \tag{2.2}$$

where $W(s)$ is the Darboux vector field [14]. Note that $\{N(s), C(s), W(s)\}$ is still orthonormal basis. For the triple $\{N(s), C(s), W(s)\}$ we have the following formulas

$$\begin{pmatrix} N'(s) \\ C'(s) \\ W'(s) \end{pmatrix} = \begin{pmatrix} 0 & \varepsilon_2 f(s) & 0 \\ -\varepsilon_1 f(s) & 0 & \varepsilon_3 g(s) \\ 0 & -\varepsilon_2 g(s) & 0 \end{pmatrix} \begin{pmatrix} N(s) \\ C(s) \\ W(s) \end{pmatrix} \tag{2.3}$$

where

$$f(s) = \kappa\sqrt{1+H^2},\ \ g(s) = \frac{H'}{1+H^2},\ \ H = \frac{\tau}{\kappa}.$$

If the curve $\alpha(s)$ is a slant helix, then

$$\frac{g(s)}{f(s)} = \frac{H'}{\kappa(1+H^2)^{3/2}} = \sigma = \text{constant} \tag{2.4}$$

or equivalently

$$\frac{H'}{1+H^2} = \sigma f(s). \tag{2.5}$$

3. New Approach to Slant Helix

In this section, we discuss the construction of slant helices. Under the assumption (2.4), firstly we solve the system (2.3), then obtain the position vector of slant helices through integration.

Introduce new parameter t by

$$t(s) = \int f(s)ds. \tag{3.1}$$

If we take derivatives based on t

$$\begin{aligned} \frac{dN}{dt} &= \varepsilon_2 C \\ \frac{dC}{dt} &= -\varepsilon_1 N + \varepsilon_3 \sigma W \\ \frac{dW}{dt} &= -\varepsilon_2 \sigma C \end{aligned}$$

then with respect to t, (2.3) becomes

$$\frac{d}{dt}\begin{pmatrix} N \\ C \\ W \end{pmatrix} = \begin{pmatrix} 0 & \varepsilon_2 & 0 \\ -\varepsilon_1 & 0 & \varepsilon_3\sigma \\ 0 & -\varepsilon_2\sigma & 0 \end{pmatrix}\begin{pmatrix} N \\ C \\ W \end{pmatrix} \tag{3.2}$$

Therefore C satisfies the following second-order linear ordinary partial differential equation

$$\frac{d^2C}{dt^2} = -\varepsilon_1\varepsilon_2 C - \varepsilon_2\varepsilon_3\sigma^2 C$$

$$\frac{d^2C}{dt^2} + z^2 C = 0$$

where $z = \sqrt{\varepsilon_1\varepsilon_2 + \varepsilon_2\varepsilon_3\sigma^2}$. So

$$C = \cos(zt)\mathbf{a} + \sin(zt)\mathbf{b} \tag{3.3}$$

where $\mathbf{a}, \mathbf{b}$ are two constant of integration. Substituting (3.3) into the first equation of (3.2) gives

$$\frac{dN}{dt} = \varepsilon_2 \cos(zt)\mathbf{a} + \varepsilon_2 \sin(zt)\mathbf{b}$$

$$N = \frac{\varepsilon_2}{z}[\sin(zt)\mathbf{a} - \cos(zt)\mathbf{b} + \frac{\sigma}{\varepsilon_2}\mathbf{d}] \tag{3.4}$$

where $\mathbf{d}$ is another constant of integration. Substituting (3.4) into the second equation of (3.2) yields

$$\frac{dW}{dt} = -\varepsilon_2\sigma C$$

$$W = -\frac{\varepsilon_2\sigma}{z}[\sin(zt)\mathbf{a} - \cos(zt)\mathbf{b}] + \frac{\varepsilon_2}{z}\mathbf{d} \tag{3.5}$$

Therefore we get a general solution $\{N, C, W\}$ of the system (3.2). At $t = 0$ (or equivalently $s = 0$) in order the triple $\{N, C, W\}$ to be an orthonormal basis we need to put some restrictions on the constants of integration $\mathbf{a}, \mathbf{b}, \mathbf{d}$. Since

$$\begin{pmatrix} N(0) \\ C(0) \\ W(0) \end{pmatrix} = \begin{pmatrix} 0 & -\frac{\varepsilon_2}{z} & \frac{\varepsilon_2\sigma}{z} \\ 1 & 0 & 0 \\ 0 & \frac{\varepsilon_2\sigma}{z} & \frac{\varepsilon_2}{z} \end{pmatrix}\begin{pmatrix} \mathbf{a} \\ \mathbf{b} \\ \mathbf{d} \end{pmatrix}$$

where the matrix of transformation is orthogonal with determinant 1, we only need to choose the constant of integration $\mathbf{a}, \mathbf{b}, \mathbf{d}$ such that they form an orthogonal basis.

We are now ready to determine the position vector of slant helices. From (2.5) and (3.1) we see

$$H = -\cot(\sigma t) \tag{3.6}$$

From (2.2) , we have

$$\begin{aligned} \frac{d\alpha}{ds} &= T = -\frac{\varepsilon_1 \kappa}{\sqrt{\kappa^2+\tau^2}} C - \frac{\varepsilon_3 \tau}{\sqrt{\kappa^2+\tau^2}} W \\ T &= -\frac{\varepsilon_1}{\sqrt{1+H^2}} C - \frac{\varepsilon_3 H}{\sqrt{1+H^2}} W \end{aligned}$$

and using equation (3.6) , we obtain that

$$T = -\varepsilon_1 \sin(\sigma t) C + \varepsilon_3 \cos(\sigma t) W \tag{3.7}$$

Substituting (3.3) and (3.5) into (3.7)

$$\begin{aligned} \frac{d\alpha}{ds} &= \left(-\varepsilon_1 \sin(\sigma t)\cos(zt) - \varepsilon_2\varepsilon_3 \frac{\sigma}{z}\cos(\sigma t)\sin(zt)\right)\mathbf{a} \\ &+ \left(-\varepsilon_1 \sin(\sigma t)\sin(zt) + \varepsilon_2\varepsilon_3 \frac{\sigma}{z}\cos(\sigma t)\cos(zt)\right)\mathbf{b} + \varepsilon_2\varepsilon_3 \frac{1}{z}\cos(\sigma t)\mathbf{d} \end{aligned} \tag{3.8}$$

So the coordinates of a slant helix with respect to the orthonormal basis $\{\mathbf{a}, \mathbf{b}, \mathbf{d}\}$ are

$$\alpha = (\alpha_1, \alpha_2, \alpha_3) \tag{3.9}$$

where

$$\alpha_1 = -\frac{1}{z}\int (\varepsilon_1 z \sin(\sigma t)\cos(zt) + \varepsilon_2\varepsilon_3 \sigma \cos(\sigma t)\sin(zt))\, ds \tag{3.10}$$

$$\alpha_2 = -\frac{1}{z}\int (\varepsilon_1 z \sin(\sigma t)\sin(zt) - \varepsilon_2\varepsilon_3 \sigma \cos(\sigma t)\cos(zt))\, ds \tag{3.11}$$

$$\alpha_3 = \frac{1}{z}\int \varepsilon_2\varepsilon_3 \cos(\sigma t) ds \tag{3.12}$$

We summarize the above argument in the following theorem.

Theorem 3.1. *A slant helix with curvature κ and torsion τ in Minkowski 3−space is given by (3.9) – (3.12) where $t(s) = \int \sqrt{\kappa^2+\tau^2} ds$, σ is an arbitrary constant and $z = \sqrt{\varepsilon_1\varepsilon_2 + \varepsilon_2\varepsilon_3\sigma^2}$.*

Example 3.1. Let $\alpha(s)$ be slant helix with curvatures $\kappa(s) = -\sqrt{3}\sin\sqrt{2}s$ and $\tau(s) = \sqrt{3}\sin\sqrt{2}s$ as follows [11];

$$\alpha(s) = (\alpha_1(s), \alpha_2(s), \alpha_3(s))$$

where

$$\begin{aligned} \alpha_1(s) &= \frac{7+2\sqrt{10}}{6\sqrt{5}}\sin(\sqrt{5}-\sqrt{2})s - \frac{7+2\sqrt{10}}{6\sqrt{5}}\sin(\sqrt{5}+\sqrt{2})s \\ \alpha_2(s) &= -\frac{7+2\sqrt{10}}{6\sqrt{5}}\cos(\sqrt{5}-\sqrt{2})s + \frac{7+2\sqrt{10}}{6\sqrt{5}}\cos(\sqrt{5}+\sqrt{2})s \\ \alpha_3(s) &= \frac{\sqrt{3}}{\sqrt{10}}\sin\sqrt{2}s \end{aligned}$$

So, the figure of this slant helix is given in Figure 1.

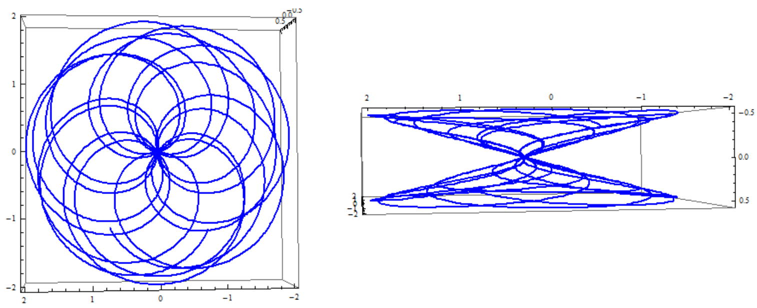

Figure 1. The curve $\alpha(s)$

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