JOURNAL OF SCIENCE
SAKARYA UNIVERSITY

Sakarya University Journal of Science

ISSN 1301-4048 | e-ISSN 2147-835X | Period Bimonthly | Founded: 1997 | Publisher Sakarya University | http://www.saujs.sakarya.edu.tr/

Title: Absolute Almost Weighted Summability Methods

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Recieved: 2019-01-31 16:17:44

Accepted: 2019-03-18 08:33:49

Article Type: Research Article
Volume: 23
Issue: 5
Month: October
Year: 2019
Pages: 763-766

How to cite
Mehmet Ali Sarıgöl; (2019), Absolute Almost Weighted Summability Methods. Sakarya University Journal of Science, 23(5), 763-766, DOI: 10.16984/saufenbilder.520449
Access link
http://www.saujs.sakarya.edu.tr/issue/44066/520449

JOURNAL of SCIENCE

Sakarya University Journal of Science

VOLUME: 23
ISSUE: 5
OCTOBER 2019

Sakarya University Journal of Science 23(5), 763-766, 2019

SAKARYA UNIVERSITY

Absolute almost weighted summability methods

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Abstract

In this paper, we introduce absolute almost weighted convergent series and treat with the classical results of Bor [3-4] and also study some relations between this method and the well known spaces.

Keywords: Absolute summability, almost summability, weighted mean, equivalence methods

1. INTRODUCTION

Let ℓ_∞ be the subspace of all bounded sequences of w, the set of all sequences of complex numbers. A sequence $(x_n) \in \ell_\infty$ said to be almost convergent to γ if all of its Banach limits (see, [1]) are equal to γ. Lorentz [8] proved that the sequence (x_n) is almost convergent to γ if and only if

$$\frac{1}{m+1}\sum_{v=0}^{m} x_{n+v} \to \gamma \text{ as } m \to \infty \text{ uniformly in } n \quad (1.1)$$

Let $\sum a_v$ be a given infinite series with s_n as its n-th partial sum and (p_n) be a sequence of positive real numbers such that, $P_{-1} = p_{-1}$,

$$P_n = p_0 + p_1 + \cdots + p_n \to \infty \text{ as } n \to \infty. \quad (1.2)$$

The series $\sum a_v$ is said to be summable $|\bar{N}, p_n|_k$, $k \geq 1$, if (see, [2])

$$\sum_{n=1}^{\infty}\left(\frac{P_n}{p_n}\right)^{k-1} |T_n - T_{n-1}|^k < \infty, \quad (1.3)$$

where

$$T_n = \frac{1}{P_n}\sum_{v=0}^{n} p_v s_v,$$

which reduces to the absolute Cesàro summability $|C,1|_k$ in Flett [5]' s notation in the special case $p_n = 1$ for $n \geq 0$. Many papers concerning the summability $|\bar{N}, p_n|_k$ was published by several authors (see, [2-3], [5-7], [10-15]). For example, it is well known that the classical results of Bor [2,3] give sufficient conditions for the equivalance of the summability methods $|C,1|_k$ and $|\bar{N}, p_n|_k$ as follows.

Theorem 1.1. Let (p_n) be a sequence of positive real numbers such that

(i) $np_n = O(P_n)$ and (ii) $P_n = O(np_n)$. (1.4)

Then, $\sum a_v$ is summable $|C,1|_k$ if and only if it is summable $|\bar{N}, p_n|_k$, $k \geq 1$.

2. MAIN RESULTS

The main purpose of this paper is to derive an absolute almost weighted summability from the absolute weighted summability just as absolute almost convergence emerges out of the concept of absolute convergence and to discuss Theorem 1.1 for new summability method. Also, we study some relations

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between this method and between some known spaces. Now for the sequence of partial sums (s_n) of the series $\sum a_v$ we define $T_m(n)$ by

$$T_{-1} = s_{n-1}, \qquad T_m(n) = \frac{1}{P_m}\sum_{v=0}^{m} p_v\, s_{n+v}\,, m \geq 0.$$

Then, it is easily seen that

$$T_m(n) - T_{m-1}(n) = \begin{cases} a_n\,, & m = 0 \\ \dfrac{p_m}{P_m P_{m-1}}\displaystyle\sum_{v=1}^{m} P_{v-1}\, a_{n+v}\,, & m \geq 1 \end{cases}$$

So we give the following definition.

Definition 2.1. Let $\sum a_v$ be an infinite series with partial sum s_n and (p_n) be a sequence of positive real numbers satisfying (1.2). The series $\sum a_v$ is said to be absolute almost weighted summable $|f(\bar{N}), p_m|_k$, $k \geq 1$, if

$$\sum_{m=0}^{\infty}\left(\frac{P_m}{p_m}\right)^{k-1} |\Delta T_{m-1}(n)|^k < \infty \tag{2.1}$$

uniformly in n, where $\Delta T_{-1}(n) = 0$, $\Delta T_{m-1}(n) = T_{m-1}(n) - T_m(n)$ for $m, n \geq 0$.

Note that, for $p_m = 1$ (resp. $k = 1$), it reduces to the absolute almost Cesàro summability $|f(C), 1|_k$ (resp. $\hat{\ell}$, given by Das et al [4]). Further, it is clear that every $|f(\bar{N}), p_m|_k$ summable series is also summable $|\bar{N}, p_n|_k$, but the converse is not true.

Before discussing a similar of Theorem 1.1 for the new method, we study a relation between this method and the space ℓ_k of all k-absolutely convergent series.

Theorem 2.2. Let (p_n) be a sequence of positive numbers satisfying the condition $P_n = O(p_n)$. If $\sum |a_v|^k < \infty$, then it is summable $|f(\bar{N}), p_m|_k$, $k \geq 1$. If $k = 1$, then the condition is omitted.

To prove this theorem, we require the following lemma of Maddox [9].

Lemma 2.3. If $\sum_m |b_m(n)| < \infty$ for each n and $\sum_m |b_m(n)| \to 0$ as $n \to \infty$, then $\sum_m |b_m(n)|$ is uniformly convergent in n.

Proof of Theorem 2.2. Since the proof is easy for $k = 1$, it is omitted. Now, for $k > 1$, it follows from Hölder's inequality that

$$\sum_{m=0}^{\infty}\left(\frac{P_m}{p_m}\right)^{k-1} |\Delta T_{m-1}(n)|^k$$

$$\leq |a_n|^k + \sum_{m=1}^{\infty}\frac{p_m}{P_m P_{m-1}^k}\left(\sum_{v=1}^{m}\left|\frac{P_{v-1}}{p_v} p_v a_{n+v}\right|\right)^k$$

$$\leq |a_n|^k$$

$$+ \sum_{m=1}^{\infty}\frac{p_m}{P_m P_{m-1}}\sum_{v=1}^{m}\left(\frac{P_{v-1}}{p_v}\right)^k p_v |a_{n+v}|^k \left(\frac{1}{P_{m-1}}\sum_{v=1}^{m} p_v\right)^{k-1}$$

$$= O(1)\left\{|a_n|^k + \sum_{v=1}^{\infty}\left(\frac{P_{v-1}}{p_v}\right)^k p_v |a_{n+v}|^k \sum_{m=v}^{\infty}\frac{p_m}{P_m P_{m-1}}\right\}$$

$$= O(1)\left\{|a_n|^k + \sum_{v=1}^{\infty}\left(\frac{P_{v-1}}{p_v}\right)^k \frac{p_v}{P_{v-1}} |a_{n+v}|^k\right\}$$

$$= O(1)\left\{|a_n|^k + \sum_{v=1}^{\infty}|a_{n+v}|^k\right\}$$

$$= O(1)\left\{\sum_{v=n}^{\infty}|a_v|^k\right\} \to 0 \; as \; n \to \infty.$$

Thus the proof is completed by Lemma 2.3.

Theorem 2.4. Let (p_n) satisfy the conditions of Theorem 1.1. If, $as\ m \to \infty$,

$$L_m(n) = \frac{1}{P_m}\sum_{v=1}^{m}(P_v - (v+1)p_v) \to 0$$

uniformly in n, then it is summable $|f(\bar{N}), p_m|_k$ whenever $\sum a_v$ is summable $|f(C), 1|_k$, $k \geq 1$.

Theorem 2.5. Let (p_n) satisfy the conditions of Theorem 1.1. If, $as\ m \to \infty$,,

$$R_m(n) = \frac{1}{m}\sum_{v=1}^{m}\left((v+1) - \frac{P_v}{p_v}\right) y_v(n) \to 0$$

uniformly in n, then, it is summable $|f(C), 1|_k$ whenever $\sum a_v$ is summable $|f(\bar{N}), p_m|_k$, $k \geq 1$.

Proof of Theorem 2.4. We define the sequences $(x_m(n))$ and $(y_m(n))$ by

$$x_0(n) = a_n,\; x_m(n) = \frac{1}{m(m+1)}\sum_{v=1}^{m} v a_{n+v} \tag{2.3}$$

and

$$y_0(n) = a_n, y_m(n) = \frac{p_m}{P_m P_{m-1}}\sum_{v=1}^{m} P_{v-1}\, a_{n+v} \tag{2.4}$$

Suppose that $\sum a_v$ is summable $|f(C), 1|_k$. Then,

$$\sum_{m=1}^{\infty} m^{k-1}|x_m(n)|^k < \infty$$

and the remaining term tends to zero uniformly in n, respectively. By using Abel's summations we write

$$y_m(n) = \frac{p_m}{P_m P_{m-1}} \sum_{v=1}^{m} \frac{P_{v-1}}{v} v a_{n+v}$$

$$= \frac{p_m}{P_m P_{m-1}} \left[\sum_{v=1}^{m-1} \Delta\left(\frac{P_{v-1}}{v}\right) v\,(v+1)\, x_v(n) + \frac{P_{m-1}}{m} m(m+1)\, x_m(n) \right]$$

$$= \frac{p_m}{P_m P_{m-1}} \sum_{v=1}^{m-1} (P_v - (v+1)p_v)\ x_v(n) + \frac{(m+1)p_m}{P_m}\ x_m(n)$$

$$= y_m^{(1)}(n) + y_m^{(2)}(n), say.$$

Now, by Minkowski's inequality, it is sufficient to show that the remaining term, $as\ j \to \infty$,

$$\sum_{m=j}^{\infty} \left(\frac{P_m}{p_m}\right)^{k-1} \left|y_m^{(r)}(n)\right|^k \to 0 \text{ uniformly in } n,$$

for $r = 1,2.$ By applying Hölder inequality for $k > 1$ (clearly for $k = 1$) we have, from the hypotheses of the theorem

$$\sum_{m=j}^{\infty} \left(\frac{P_m}{p_m}\right)^{k-1} \left|y_m^{(1)}(n)\right|^k$$

$$= O(1) \sum_{m=j}^{\infty} \frac{p_m}{P_m P_{m-1}^k} \left(\sum_{v=1}^{j-1} \quad + \sum_{v=j}^{m-1} \quad \right) |(P_v(v\ -\ \ (v+1)p_v)\, x_v(n)|^k$$

$$= O(1) \sum_{m=j}^{\infty} \frac{p_m}{P_m P_{m-1}^k} \left\{ \left|P_{j-1} L_{j-1}(n)\right|^k + \left| \sum_{v=j}^{m-1} (P_v - (v+1)p_v)\ x_v(n) \right|^k \right\}$$

$$= O(1) \left\{ \left|P_{j-1} L_{j-1}(n)\right|^k \sum_{m=j}^{\infty} \frac{p_m}{P_m P_{m-1}^k} + \sum_{m=j+1}^{\infty} \frac{p_m}{P_m P_{m-1}^k} \left(\sum_{v=T}^{m-1} v p_v \mid x_v(n) | \right)^k \right\}$$

$$= O(1) \left\{ \left|L_{j-1}(n)\right|^k + \sum_{m=j+1}^{\infty} \frac{p_m}{P_m P_{m-1}} \sum_{v=j}^{m-1} v^k p_v \mid x_v(n)|^k \left(\sum_{v=0}^{m-1} \frac{p_v}{P_{m-1}} \right)^{k-1} \right\}$$

$$= O(1) \left\{ \left|L_{j-1}(n)\right|^k + \sum_{v=j}^{\infty} v^k p_v \mid x_v(n)|^k \sum_{m=v+1}^{\infty} \frac{p_m}{P_m P_{m-1}} \right\}$$

$$= O(1) \left\{ \left|L_{j-1}(n)\right|^k + \sum_{v=j}^{\infty} \frac{v p_v}{P_v} v^{k-1} \mid x_v(n)|^k \right\}$$

$$= O(1) \left\{ \left|L_{j-1}(n)\right|^k \sum_{v=j}^{\infty} v^{k-1} \mid x_v(n)|^k \right\} \to 0 \text{ as } j \to \infty,$$

uniformly in n.

Also, by using $m p_m = O(P_m)$ we get

$$\sum_{m=j}^{\infty} \left(\frac{P_m}{p_m}\right)^{k-1} \left|y_m^{(2)}(n)\right|^k = \sum_{m=j}^{\infty} \left| \frac{(m+1)p_m}{P_m}\ x_m(n) \right|^k = O(1) \sum_{m=j}^{\infty} (m)^{k-1} |\ x_m(n)|^k \to 0$$

uniformly in n, which completes the proof of part (i).

The proof of Theorem 2.5 is proved by changing the roles of "p_n" and "1".

Also the following results are directly obtained by Lemma 2.3.

Theorem 2.6. Let (p_n) be a sequence of positive numbers satisfying the conditions of Theorem 1.1. If

$$\sum_{m=0}^{\infty}(m+1)^{k-1}\mid x_m(n)|^k \to 0 \ as\ n\to\infty$$

or

$$\sum_{m=0}^{\infty}\left(\frac{P_m}{p_m}\right)^{k-1}\mid y_m(n)|^k \to 0 \ as\ n\to\infty$$

holds, then a series $\sum a_v$ is summable $|f(C),1|_k$ and $|f(\bar{N}),p_m|_k$, $k\geq 1$, where the sequences $\left(x_m(n)\right)$ and $\left(y_m(n)\right)$ are as in (2.3) and (2.4).

Proof. By following the lines of the proof of Theorem 2.4, we have

$$\sum_{m=1}^{\infty}\left(\frac{P_m}{p_m}\right)^{k-1}\mid y_m(n)|^k = O(1)\left\{\sum_{v=1}^{\infty} v^{k-1}\mid x_v(n)|^k\right\}$$

and also

$$\sum_{m=1}^{\infty} m^{k-1}\mid x_m(n)|^k = O(1)\left\{\sum_{v=1}^{\infty}\left(\frac{P_v}{p_v}\right)^{k-1}\mid y_v(n)|^k\right\}$$

which completes the proof together with Lemma 2.3.

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