Haliç Üniversitesi Fen Bilimleri Dergisi 2019, 2/1: 17-34
DOI:

On Coincidence Degree Theory Some Corrections and Explanations

Ali SIRMA*

*Haliç University, Department of Industrial Engineering, Istanbul, Turkey

Received Date: 18.12.2018
***Corresponding author e mail:** alisirma@halic.edu.tr
Accepted Date: 04.03.2019

Atıf/Citation: Sırma, A. "On Coincidence Degree Theory Some Corrections and Explanations", Haliç Üniversitesi Fen Bilimleri Dergisi 2019, 2/1: 17-34

Abstract

Coincidence degree theory, also known Mawhin's coincidence theory is very powerful technique especially in existence of solutions problems in nonlinear equations. It has especially so broad applications in the existence of periodic solutions of nonlinear differential equations so that many researchers have used it for their investigations. In coincidence degree, mainly existence of solutions of the operator equation in the form $Lx = Nx$ in an open and bounded set Ω in some Banach space was researched. Here, L is a linear operator and N is a nonlinear operator satisfying some special properties. In this study mainly the studies of Gaines and Mahwin are followed, the statement of continuation theory in a coincidence degree theory was corrected and the reason is expressed. A continuation theorem was expressed in different manner. In order to help the researchers with their studies on this subject, the proof that was provided by Gaines and Mawhin has now been presented with more detailed explanation.

Keywords: Coincidence Degree Theory, L-compact operator, Homotopy Theory.

Örtüşen Derece Teorisi Üzerine Bazı Düzeltme ve İzahlar

Özet

Gaines-Mawhin örtüşen derece teorisi olarak da bilinen örtüşen derece teorisi, özellikle doğrusal olmayan denklemlerdeki çözümün varlığı probleminde güçlü bir tekniktir. Özellikle doğrusal olmayan diferansiyel denklemlerin periyodik çözümlerinin varlığının gösterilmesinde çok geniş bir uygulaması olduğundan pek çok araştırmacı çalışmalarında bu metodu kullanmışlardır. Örtüşen derece teorisinde, bir Banach uzayındaki Ω açık ve sınırlı kümesinde tanımlı $Lx = Nx$ formundaki bir operatör denkleminin çözümlerinin varlığı araştırılır. Burada L bir doğrusal operatör ve N doğrusal olmayan bir operatör olmak üzere L ve N bazı özel koşulları sağlayan operatörlerdir. Bu çalışmada esas olarak Gaines ve Mawhin'in çalışmaları takip edilmiş, örtüşen teorinin sürdürülebilirlik teoreminin ifadesindeki ikinci sonuç düzeltilmiş ve gerekçesi belirtilmiş. Her ne kadar uygulamalarda birinci sonuç kullanılsa da bu ikincisinin düzeltilmesi de önemli bir çalışmadır. Bu sürdürülebilirlik teoremi farklı bir şekilde ifade edilmiştir. Gaines ve Mawhin'in çok az izahla verdiği ispat ilerideki çalışmalara yardımcı olmak amacıyla yeterince detaylı bir şekilde izah edilmeye çalışılmıştır.

Anahtar Kelimeler: Örtüşen Derece Teorisi, L-kompakt operatör, Homotopi Teorisi.

1. Introduction

It is known that in a finite dimensional case, for an open and bounded set $\Omega \subset \mathbb{R}^n$, for a continuous function $f \in C(\overline{\Omega})$ and for a point $p \in \mathbb{R}^n \backslash f(\partial\Omega)$, the degree of f on Ω with respect to p, $d(f, \Omega, p)$, is well defined. But unfortunately this is not the case in infinite dimension for a continuous function $f \in C(\overline{\Omega})$. Luckily, in an arbitrary Banach space X, Leray and Schauder proved that for an open and bounded set $\Omega \subset X$, for a compact operator $M: \overline{\Omega} \to X$ and for a point $p \in X \backslash (I - M)(\partial\Omega)$, the degree of compact perturpation of identity, $I - M$, in $\overline{\Omega}$ with respect to point p denoted by $d(I - M, \Omega, p)$ is well defined. One of the useful properties of degree theory is that if the degree $d(I - M, \Omega, p) \neq 0$ then the equation $(I - M)x = p$ has at least one solution in Ω. In particular if it is taken $p = 0 \in X \backslash (I - M)(\partial\Omega)$, and if $d(I - M, \Omega, 0) \neq 0$ then the compact operator M has at least one fixed point in Ω.

Gaines and Mawhin in [1] studied existence of a solution of an operator equation

$$Lx = Nx \tag{1}$$

defined on a Banach space X in an open and bounded set Ω using the Leray-Schauder degree theory. But since the operator $I - (L - N)$ is not compact in general the need to define a compact operator M such that its set of fixed points in Ω would be equal to a solution set of operator equation (1) in Ω aroused. In [1], the compact operator M is given and the coincidence degree for the couple (L,N) in Ω is defined by $d[(L,N),\Omega] = d[I-M,\Omega,0]$. Coincidence degree theory has especially so broad applications in the existence of periodic solutions of nonlinear differential equations so that many researchers have used it for their investigations (see [2-20] and references therein).

The aim of this paper is to make an effort to understand, to explain and to correct generalized continuation theorem for coincidence degree given and proven densely in [1]. In this study, the theory that was given in [1] tried to explain. Besides proofs of some results that their proofs not given [1] is given. Namely, proofs of Lemma 1 and 2 and proofs of Lemmas 4-6 are given. In the proof of generalized continuation theorem, Theorem 7, important contributions are made to make the proof much more understandable and so that it can improved by interested researchers. Also an important contribution it is done in this study is that one expresion of result of generalized continuation theorem is corrected and given in the statement of Theorem 7 and the reason is explained after Theorem 8 of Gaines and Mawhin. The last contribution is that first two assumption of generalized continuation theorem is unified and given in Theorem 9.

2. Material and Method

Let X and Z be two normed space, $\mathrm{Dom}L$ is the domain of the operator L, $L:\mathrm{Dom}L \to Z$ be a linear operator. Assume that the operators $P: X \to X$ and $Q: Z \to Z$ are linear projection operators such that the chain

$P: X \to X$, $L:\mathrm{Dom}L \subset X \to Z$, $Q: Z \to Z$

is exact. That is the conditions $\mathrm{Im}P = \mathrm{Ker}L$ and $\mathrm{Im}L = \mathrm{Ker}Q$ are satisfied. Here $\mathrm{Im}P$ and $\mathrm{Im}L$ respectively indicate image of the operators P and L, $\mathrm{Ker}L$ and $\mathrm{Ker}Q$ respectively indicate kernels of the operators P and L. Beside this, restriction of the linear operator

L on $\mathrm{Dom}L \cap \mathrm{Ker}P$, the operator $L_p: \mathrm{Dom}L \cap \mathrm{Ker}P \to \mathrm{Im}L$ is an algebraic isomorfizm [2]. Furthermore, the operators $K_p := L_p^{-1}$ and $K_{P,Q}: Z \to Z,\ K_{P,Q} = K_p(I - Q)$ are defined.

Definition 1: Let X and Z be two normed space, $\Omega \subset X$ is an open and bounded subset of X, $\overline{\Omega}$ is a closure of the Ω. Assume that the operators L:DomL $\subset X \to Z$ and $N: \overline{\Omega} \subset X \to Z$ satisfy the following condtions:

i) L is linear and $\mathrm{Im}L$ is a closed subset of Z,
ii) The vector spaces $\mathrm{Ker}L$ and $\mathrm{Coker}L = Z/\mathrm{Im}L$ are finite dimensional vector spaces and $\dim\mathrm{Ker}L = \dim\mathrm{Coker}L$,
iii) The operator $N: \overline{\Omega} \subset X \to Z$ is a continuous operator and $\Pi: Z \to \mathrm{Coker}L$ such that $\Pi(z) = z + \mathrm{Im}L$, the operator $\Pi N(\overline{\Omega})$ is a bounded operator.
iv) The operator $K_{P,Q}N: \overline{\Omega} \subset X \to Z$ is a compact operator on $\overline{\Omega}$.
   a) If the operator L satisfes the conditions (i) and (ii) then L is called a Fredholm operator of index zero.
   b) If the operator N satisfes the conditions (iii) and (iv) then N is called $L -$ compact operator.

Lemma 1: Let L:$\mathrm{Dom}L \subset X \to Z$ be a Fredholm operator of index zero and the operator

$N^*: \overline{\Omega} \times [0,1] \to Z$
$(x,\lambda) \to N^*(x,\lambda)$

be a $L -$compact operator on $\overline{\Omega} \times [0,1]$ with $N^*(.,1) = N$.
Let $y \in \mathrm{Im}L$ be any element, for $\lambda \in [0,1]$ let us consider the family of equations

$$Lx = \lambda N^*(x,\lambda) + y, \qquad \lambda \in [0,1]. \tag{2}$$

Therefore for any $\lambda \in (0,1]$ the set of solutions of equation (2) is equal the set of solutions of equation

$$Lx = QN^*(x,\lambda) + \lambda(I-Q)N^*(x,\lambda) + y. \tag{3}$$

And for $\lambda = 0$, any solution of the eqation (3) is also a solution of the equation (2).

Proof: Let $\lambda \in]0,1]$.
Since any $z \in Z$ can be written in the form $z = Qz + (I-Q)z$ and since $N^*(x,\lambda) \in Z$ then the equality $N^*(x,\lambda) = QN^*(x,\lambda) + (I-Q)N^*(x,\lambda)$ holds. Therefore if we write the last equality in the equation (2) then it is obtained the equation

$$Lx = \lambda\, QN^*(x,\lambda) + \lambda(I-Q)N^*(x,\lambda) + y, \tag{4}$$

that is the equation

$$Lx + 0 = \lambda QN^*(x,\lambda) + \lambda(I-Q)N^*(x,\lambda) + y \tag{5}$$

is obtained. Since Z can be written by a direct sum as $Z = \mathrm{Ker}Q \oplus \mathrm{Im}Q = \mathrm{Im}L \oplus \mathrm{Im}Q$ and $Lx \in \mathrm{Im}L = \mathrm{Ker}Q$ and $y \in \mathrm{Im}L$, $\lambda(I-Q)N^*(x,\lambda) \in \mathrm{Ker}Q$, $QN^*(x,\lambda) \in \mathrm{Im}Q$ and because of the uniqueness of indicating with direct sum, the equation (5) is equivalent to the system of equations

$$0 = \lambda QN^*(x,\lambda),\ Lx = \lambda(I-Q)N^*(x,\lambda) + y. \tag{6}$$

Since $\lambda \neq 0$, this system of equations (6) is equivalent to equation (3). If λ=0 then the equation (3) turns into the form $Lx = QN^*(x,0) + y$. Again because of uniqueness of indicating with direct sum, this turns into the system of equation $0 = QN^*(x,0)$ and $Lx = y$. From here the result is trivial. Therefore the proof is completed.

Lemma 2: Let $y \in \mathrm{Im}L$ be an arbitrary but fixed point and $L^{\leftarrow}$ be an inverse image of the operator L then the linear space $L^{\leftarrow}\{y\}$ is a finite dimensional space.

Proof: It is known that $\mathrm{dimCoker}L = \mathrm{dimKer}L = n < \infty$.
If $y = 0$ since $\dim L^{\leftarrow}\{y\} = \dim L^{\leftarrow}\{0\}$=$\mathrm{dimKer}L = n < \infty$ then the space $L^{\leftarrow}\{y\}$ is a finite dimensional space.
For the case $y \neq 0$ consider any points $x, x_0 \in L^{\leftarrow}\{y\}$. Using linearity of L, $L(x - x_0) = L(x) - L(x_0) = y - y = 0$ is satisfied. Thus $x - x_0 \in \mathrm{Ker}L$. That is any element $x \in L^{\leftarrow}\{y\}$ is also an element $x \in x_0 + \mathrm{Ker}L$, hence $L^{\leftarrow}\{y\} = x_0 + \mathrm{Ker}L$. Here the space $x_0 + \mathrm{Ker}L$ is an affin space of $\mathrm{Ker}L$. Since $\dim L^{\leftarrow}\{y\} = \dim\{x_0 + \mathrm{Ker}L\} = \dim \mathrm{Ker}L = n < \infty$, hence the proof is completed.

Theorem 3: (see [1]) (The property of invariance of coincidence degree under homotopy) Let L be a Fredholm operator of index zero,
$\tilde{N}: \overline{\Omega} \times [0,1] \to Z$
$(x,\lambda) \to \tilde{N}(x,\lambda)$
be an L-compact operator on $\overline{\Omega} \times [0,1]$ and $0 \notin \left[L - \tilde{N}(.,\lambda)\right](\mathrm{dom}L \cap \partial\Omega)$.

Therefore the value of coincidence degree $d\left[\left(L,\tilde{N}(.,\lambda)\right),\Omega\right]$ is independent of the value of $\lambda \in [0,1]$.

In order to show continuation theorem for the coincidence theory, first the following lemmas will be expressed and proven.

Lemma 4: Assume that the operators L and N^* as in above. Therefore the homotopy $\tilde{N}(x,\lambda)$ defined by

$$\tilde{N}(x,\lambda) = QN^*(x,\lambda) + \lambda(I-Q)N^*(x,\lambda) + y \qquad (7)$$

for $\lambda \in [0,1]$ is L-compact.

Proof: In order to show that the homotopy $\tilde{N}(x,\lambda)$ is L-compact for any $\lambda \in [0,1]$, by the definition of L-compactness given in Definition 1, it should be shown that for any $\lambda \in [0,1]$ the operator $\tilde{N}(.,\lambda): \overline{\Omega} \subset X \to Z$ is continuous and with $\Pi: Z \to \mathrm{Coker}L$, $\Pi(z) = z + \mathrm{Im}L$ the operator $\Pi\tilde{N}(\overline{\Omega},\lambda)$ is bounded and the operator $K_{P,Q}\tilde{N}(.,\lambda): \overline{\Omega} \subset X \to Z$ is a compact operator on $\overline{\Omega}$. In the right hand side of equation (7), the operators Q and $N^*(x,\lambda)$ appear. It is given that the projection operator Q is continuous and for $\lambda \in [0,1]$ the operator $N^*(x,\lambda)$ is given that L-compact therefore continuous. Hence for any $\lambda \in [0,1]$ the homotopy $\tilde{N}(x,\lambda)$ is continuous.

Now let us show that the operator $\Pi\tilde{N}(x,\lambda)$ is bounded on the set $\overline{\Omega}$. From the equation (7) the equation

$$\Pi\tilde{N}(x,\lambda) = \Pi QN^*(x,\lambda) + \lambda\Pi(I-Q)N^*(x,\lambda) + \Pi y \qquad (8)$$

is obtained. Since $y \in \mathrm{Im}L$ then $\Pi y = 0$ and since $\mathrm{Im}(I-Q) = \mathrm{Ker}Q$ and $\mathrm{Ker}Q = \mathrm{Im}L$ then $\mathrm{Im}(I-Q) = \mathrm{Im}L$ therefore the equality $\lambda\Pi(I-Q)N^*(x,\lambda) \equiv 0$ is obtained. Hence in order to show that the operator $\Pi\tilde{N}(x,\lambda)$ is bounded on the set $\overline{\Omega}$ it is enough to show that the operator $\Pi QN^*(x,\lambda)$ is bounded on the set $\overline{\Omega}$. Namely:

Since the operator $N^*(x,\lambda)$ is L-compact, by the definition of L-compactness the operator $\Pi N^*(x,\lambda)$ is bounded on the set $\overline{\Omega}$. Since the linear vector space Z can be written direct sum of $\mathrm{Im}L$ and $\mathrm{Im}Q$ as $Z = \mathrm{Ker}Q \oplus \mathrm{Im}Q = \mathrm{Im}L \oplus \mathrm{Im}Q$ then the set $N^*(\overline{\Omega},\lambda) \subseteq Z$ is also can be written as $N^*(\overline{\Omega},\lambda) = \left(\left(N^*(\overline{\Omega},\lambda) \cap \mathrm{Im}L\right) \oplus \left(N^*(\overline{\Omega},\lambda) \cap \mathrm{Im}Q\right)\right)$. Thus the set $\Pi N^*(\overline{\Omega},\lambda) = \Pi\left(N^*(\overline{\Omega},\lambda) \cap \mathrm{Im}Q\right)$ is a bounded set. Because the equalities

$$QN^*(\overline{\Omega},\lambda) = Q\left(\left(N^*(\overline{\Omega},\lambda) \cap \mathrm{Im}L\right) \oplus \left(N^*(\overline{\Omega},\lambda) \cap \mathrm{Im}Q\right)\right)$$

$$= Q\left(\left(N^*(\overline{\Omega},\lambda) \cap \mathrm{Ker}Q\right) \oplus \left(N^*(\overline{\Omega},\lambda) \cap \mathrm{Im}Q\right)\right) = Q\left(N^*(\overline{\Omega},\lambda) \cap \mathrm{Im}Q\right) = N^*(\overline{\Omega},\lambda) \cap \mathrm{Im}Q \qquad (9)$$

are satisfed, the set $\Pi QN^*(\overline{\Omega},\lambda)$ is a bounded set. Hence boundedness of the operator $\Pi\tilde{N}(x,\lambda)$ on the set $\overline{\Omega}$ has been shown.

Now in order to show that the homotopy operator $\tilde{N}(x,\lambda)$ for any $\lambda \in [0,1]$ is L-compact, lastly let us show that for any $\lambda \in [0,1]$ the operator $K_{P,Q}\tilde{N}(x,\lambda): \overline{\Omega} \subset X \to Z$ is compact on $\overline{\Omega}$. Because $K_{P,Q} = K_P(I-Q)$ then

$$K_{P,Q}\tilde{N}(x,\lambda) = K_P(I-Q)QN^*(x,\lambda) + \lambda K_P(I-Q)(I-Q)N^*(x,\lambda) + K_P(I-Q)y \qquad (10)$$

can be written. Since here $y \in \mathrm{Im}L = \mathrm{Ker}Q$ then $K_P(I-Q)y = K_P y$. Since constant operator is compact thus the operator $K_P(I-Q)y$ is compact. Since the operator Q is a projection operator hence $I-Q$ is also projection operator. So that $(I-Q)(I-Q) = (I-Q)^2 = (I-Q)$ holds, since $K_P(I-Q)(I-Q)N^*(x,\lambda) = K_P(I-Q)N^*(x,\lambda) = K_{P,Q}N^*(x,\lambda)$ and by our assumption the operator $K_{P,Q}N^*(x,\lambda)$ is compact on $\overline{\Omega}$ then the part $\lambda K_P(I-Q)(I-Q)N^*(x,\lambda)$ is also compact operator on $\overline{\Omega}$. Beside this, since $(I-Q)Q = Q - Q^2 = Q - Q = 0$ then we have $K_P(I-Q)QN^*(x,\lambda) \equiv 0$, so that in this case also the operator $K_P(I-Q)QN^*(x,\lambda)$ is compact on $\overline{\Omega}$. Since sum of the compact operators is also compact, we have that for any $\lambda \in [0,1]$ the operator $K_{P,Q}\tilde{N}(x,\lambda): \overline{\Omega} \subset X \to Z$ is compact on the set $\overline{\Omega}$. As a result the proof of Lemma 4 is completed.

Lemma 5: Assume that the operators L and N^* as in above, $y \in \mathrm{Im}L$ is any number and Λ:Coker$L \to$ Ker$L$ is any isomorphism. Hence the following equalities hold:

a) $d\left[\{-\Lambda\Pi N^*(.+K_p y,0)\}_{|KerL}, (-K_p y+ \ \Omega) \cap \mathrm{Ker}L, 0\right]$
$= \pm d\left[\{\Pi N^*(.+K_p y,0)\}_{|\mathrm{Ker}L}, (-K_p y+ \ \Omega) \cap \mathrm{Ker}L, 0\right]$ (11)

b) $d\left[\{\Pi N^*(.+K_p y,0)\}_{|\mathrm{Ker}L}, (-K_p y+ \ \Omega) \cap \mathrm{Ker}L, 0\right]$
$= d\left[\Pi N^*(x,0)_{|L^{\leftarrow}\{y\}}, \Omega \cap L^{\leftarrow}\{y\}, 0\right]$ (12)

Proof:

a) Now to an expression

$d\left[\{-\Lambda\Pi N^*(.+K_p y,0)\}_{|KerL}, (-K_p y+ \ \Omega) \cap \mathrm{Ker}L, 0\right]$ (13)

product rule in Brower degree is applied. Namely; here since the mapping $-\Lambda$:Coker$L \to$ Ker$L$ is any isomorphism then $(-\Lambda)^{-1}\{0\} = \{\bar{0}\}$, so that if we take the set $V \subset$ Coker$L$, any open and bounded set which contains $\bar{0}$, then by the product rule the expression (13) is equal to expression

$= d\left[-\Lambda, V, \bar{0}\right] d\left[\{\Pi N^*(.+K_p y,0)\}_{|KerL}, (-K_p y+ \ \Omega) \cap KerL, 0\right]$. (14)

Because of degree of linear isomorphism ± 1 then we have $d\left[-\Lambda, V, \bar{0}\right] = \pm 1$, hence in this case the expression (14) is equal to expression

$= \pm d\left[\{\Pi N^*(.+K_p y,0)\}_{|\mathrm{Ker}L}, (-K_p y+ \ \Omega) \cap \mathrm{Ker}L, 0\right]$ (15)

which is same with (11).

b) In the expression of $d\left[\{\Pi N^*(.+K_p y,0)\}_{|\mathrm{Ker}L}, (-K_p y+ \ \Omega) \cap \mathrm{Ker}L, 0\right]$ since

$x \in (-K_p y+ \ \Omega) \cap \mathrm{Ker}L = (-L_p^{-1} y+ \ \Omega) \cap \mathrm{Ker}L$
$\Leftrightarrow x \in (-L_p^{-1} y+ \ \Omega)$ ve $x \in \mathrm{Ker}L$
$\Leftrightarrow x \in \Omega$ ve $x \in (-L_p^{-1} y+ \mathrm{Ker}L)$
$\Leftrightarrow x \in \Omega$ ve $x \in L^{\leftarrow}\{y\}$

are satisfied. Then the equality

$d\left[\Pi N^*(.+K_p y,0)_{|\mathrm{Ker}L}, (-K_p y+ \ \Omega) \cap \mathrm{Ker}L, 0\right]$
$= d\left[\Pi N^*(x,0)_{|L^{\leftarrow}\{y\}}, \Omega \cap L^{\leftarrow}\{y\}, 0\right]$

holds. In this way the proof of Lemma 5 is completed.

Lemma 6: The operator $M^*(.,0) = P + \left(\Lambda\Pi + K_{P,Q}\right)(QN^*(.,0) + y)$ can be written in the form $M^*(.,0) = P + \Lambda\Pi N^*(.,0) + K_P y$.

Proof: By definition the open expression of operator $M^*(.,0)$ is
$M^*(.,0) = P + \left(\Lambda\Pi + K_{P,Q}\right)(QN^*(.,0) + y)$
$= P + \Lambda\Pi QN^*(.,0) + \Lambda\Pi y + K_{P,Q}QN^*(.,0) + K_{P,Q}y$.
Here since $y \in \mathrm{Im}L y \in \mathrm{Im}L$ then $\Pi y = \overline{0}\, \Pi y = \overline{0}$ and $\Lambda\overline{0} = 0$ $\Lambda\overline{0} = 0$ and beside this, since
$y \in \mathrm{Im}L = \mathrm{Ker}Q$ thus $K_{P,Q}y = K_P(I - Q)y{=}K_P y - K_P Qy = K_P y$.
Therefore the equalites
$M^*(.,0) = P + \Lambda\Pi QN^*(.,0) + 0 + K_P(I - Q)QN^*(.,0) + K_P(I - Q)y$
$= P + \Lambda\Pi QN^*(.,0) + K_P(Q - Q^2)N^*(.,0) + K_P y - K_P Qy$
$= P + \Lambda\Pi QN^*(.,0) + K_P(Q - Q)N^*(.,0) + K_P y - 0$
$= P + \Lambda\Pi QN^*(.,0) + K_P y$
are obtained.

But since $\Lambda\Pi N^*(.,0)$ can be written in the form $\Lambda\Pi N^*(.,0) = \Lambda\Pi QN^*(.,0) + \Lambda\Pi(I - Q)N^*(.,0)$ and $\mathrm{Ker}Q = \mathrm{Im}L$ hence $\Pi(I - Q) = \overline{0}$ and this also follows that we have $\Lambda\Pi N^*(.,0) = \Lambda\Pi QN^*(.,0)$. In this way the expression $M^*(.,0) = P + \Lambda\Pi N^*(.,0) + K_P y$ is obtained. Thus, the proof of Lemma 6 is completed.

Theorem 7: (Generalized Continuation Theorem) Assume that the operators L and N^* as in above and $y \in \mathrm{Im}L$ is any number. Assume also that the following conditions are satisfied:

1) For any $x \in \mathrm{DomL} \cap \partial\Omega$ and for any $\lambda \in (0,1)$
   $Lx \neq \lambda N^*(x,\lambda) + y$
2) For any $x \in L^{\leftarrow}\{y\} \cap \partial\Omega$ için
   $\Pi N^*(x,0) \neq 0$
3) $d\left[\Pi N^*(.,0)_{|L^{\leftarrow}\{y\}},\ \Omega \cap L^{\leftarrow}\{y\}, 0\right] \neq 0$.

   Therefore the equation
   $Lx = Nx + y$ (16)

has at least one solution on $\overline{\Omega}$. Beside this, whenever the equation (16) does not have any solution on $\mathrm{DomL} \cap \partial\Omega$ then for any $\lambda \in [0,1]$ the equation

$$Lx = \lambda N^*(x,\lambda) + y \tag{17}$$

has at least one solution in Ω.

Proof: In Lemma 2 we showed that space $L^{\leftarrow}\{y\}$ has a finite n dimension, same with the dimension of $\mathrm{Coker}L$. If basis for the spaces $L^{\leftarrow}\{y\}$ and $\mathrm{Coker}L$ are chosen and if a sign of the degree is taken with respect to this orientation, then degree in the assumption 3) can be considered as a clasical Brouwer degree from R^n to R^n. For the detail see Mawhin [3]. In Lemma 4 it is shown that the operator $\tilde{N}(x,\lambda)$ defined by

$$\tilde{N}(x,\lambda) = QN^*(x,\lambda) + \lambda(I-Q)N^*(x,\lambda) + y \tag{18}$$

is L-compact for any $\lambda \in [0,1]$. Now to an homotopy operator $L - \tilde{N}(x,\lambda)$ the theorem of invariance of coincidence degree under homotopy given in Theorem 3 will be applied. In order to apply the theorem of invariance of coincidence degree under homotopy, it is needed to show that the operator $L - \tilde{N}(x,\lambda)$ does not have any root on $\mathrm{dom}L \cap \partial\Omega$ for any $\lambda \in [0,1]$. Namely:

Assume that $\lambda \in (0,1)$. By the assumption 1), for any $x \in \mathrm{dom}L \cap \partial\Omega$ the inequality

$$Lx \neq \lambda N^*(x,\lambda) + y \tag{19}$$

is satisfied. Therefore by the consequence of Lemma 1, for any $x \in \mathrm{dom}L \cap \partial\Omega$ and for any $\lambda \in (0,1)$ the inequality

$$Lx \neq QN^*(x,\lambda) + \lambda(I-Q)N^*(x,\lambda) + y \tag{20}$$

is obtained.

The equation (20) for the case $\lambda = 0$ is equivalent the system of equation

$$Lx = y \qquad \text{ve} \qquad QN^*(x,0) = 0.$$

$$Lx = y \Leftrightarrow x \in L^{\leftarrow}\{y\} \qquad \text{and}$$

$$QN^*(x,0) = 0 \Leftrightarrow N^*(x,0) \in \mathrm{Ker}Q = \mathrm{Im}L \Leftrightarrow \Pi N^*(x,0) \in \Pi(\mathrm{Im}L) = \overline{0}$$

hold. Here $\overline{0}$, equivalence class of 0 in $\mathrm{Coker}L$, hence for $\lambda = 0$ the equation (20) is equivalent to equation

$\Pi N^*(x,0) = \bar{0}$ ve $x \in L^{\leftarrow}\{y\}$. (21)

But by the assumption 2) in this theorem the equation (21) does not have any solution on $\partial\Omega$ and therefore for any $x \in \mathrm{domL} \cap \partial\Omega$ and for any $\lambda \in [0,1[$

$Lx \neq QN^*(x,\lambda) + \lambda(I-Q)N^*(x,\lambda) + y$ (22)

and

$Lx \neq \lambda N^*(x,\lambda) + y$ (23)

are satisfied.

Now the case $\lambda = 1$ is considered. There are two cases related the equation

$Lx = N^*(x,1) + y = N(x) + y$. (the case λ=1) (24)

It does not have any solution on the boundary set $\mathrm{Dom}L \cap \partial\Omega$ or it has at least one solution on $\mathrm{Dom}L \cap \partial\Omega$.

Assume that the equation (24) that is the equation (16) does not have any solution on $\mathrm{Dom}L \cap \partial\Omega$. Therefore since by assumptions 1) and 2) the inequality (25) is satisfied for any $\lambda \in [0,1[$ by the above arguments for any $\lambda \in [0,1]$ and $x \in \mathrm{Dom}L \cap \partial\Omega$ the inequality

$Lx \neq \lambda N^*(x,\lambda) + y$ (25)

holds. Thus by the Theorem 3 the value of coincidence degree $d\left[\left(L,\tilde{N}(.,\lambda)\right),\Omega\right]$ is independent from the value of $\lambda \in [0,1]$ and hence it is equal to its value at $\lambda = 0$.

Now let us try to find the value of coincidence degree $d\left[\left(L,\tilde{N}(.,\lambda)\right),\Omega\right]$ at $\lambda = 0$, that is let us try to find the value of degree

$d[(L,\tilde{N}(.,\lambda)),\Omega] = [(L,\tilde{N}(.,0)),\Omega] = d[(L,QN^*(.,0)+y),\Omega] = d[I - M^*(.,0),\Omega,0]$. (26)

By the definition

$$M = P + (\Lambda\Pi + K_{P,Q})N$$

hence

$M^*(.,0) = P + \left(\Lambda\Pi + K_{P,Q}\right)(QN^*(.,0) + y)$. (27)

But by the Lemma 6 the equality (27) can be written as

$M^*(.,0) = P + \Lambda\Pi N^*(.,0) + K_P y$. (28)

Therefore we have

$d\left[\left(L,\tilde{N}(.,\lambda)\right),\Omega\right] = d[I - M^*(.,0),\Omega,0]$

$= d[I - P - \Lambda\Pi N^*(.,0) - K_p y, \Omega, 0]$.

If $\mathrm{Ker}L = \{0\}$ then because $\mathrm{Ker}L = \mathrm{Im}P$, $P = 0$ is obtained. But because of $\mathrm{dimKer}L = \mathrm{dimCoker}L = \dim(Z/\mathrm{Im}L)$ and $\dim \mathrm{Ker}L = 0 \dim \mathrm{Ker}L = 0$ then we have $\dim(Z/\mathrm{Im}L) = 0$ thus we have $\mathrm{Im}L = Z$ hence $\Pi = 0$ is obtained. Since $Z = \mathrm{Im}L = \mathrm{Ker}Q$ then $\mathrm{Ker}Q = Z$ hence this implies $Q = 0$.

$Q = 0$, and $K_p = L^{-1}$ implies that

$d\left[\left(L, \tilde{N}(.,0), \Omega\right)\right] = d[I - L^{-1}y, \Omega, 0]$. (29)

Assumption 2) puts the condition for any $x \in L^{\leftarrow}\{y\} \cap \partial\Omega$, $\Pi N^*(x,0) \neq 0$.Butinthiscasesince$\Pi = 0$thenforany$x \in L^{\leftarrow}\{y\} \cap \partial\Omega$ we have $\Pi N^*(x,0) = 0$. Thus in this case the assumption 2) can be satisfied if and only if the condition $L^{\leftarrow}\{y\} \cap \partial\Omega = \emptyset$ is satisfied.

Now let's write the assumption 3) again:

$d\left[\Pi N^*(.,0)_{|L^{\leftarrow}\{y\}}, \Omega \cap L^{\leftarrow}\{y\}, 0\right] \neq 0$ (30)

Since $\Pi N^*(.,0) = 0$ then the necessary and suffcient condition that the inequality (30) holds is that $\Omega \cap L^{\leftarrow}\{y\} \neq \emptyset$. That is why,

if $\Omega \cap L^{\leftarrow}\{y\} = \emptyset$ then $d[0, \emptyset, 0] = 0$.

If $\Omega \cap L^{\leftarrow}\{y\} \neq \emptyset$ then $d[0, \Omega \cap L^{\leftarrow}\{y\}, 0] = 1 \neq 0$.

Since now we investigate the case $\mathrm{Ker}L = \{0\}$, because in this case the linear operator L one-to-one and $y \in \mathrm{Im}L$ therefore we have $L^{\leftarrow}\{y\} = \{x_0\}$.

$\Omega \cap L^{\leftarrow}\{y\} \neq \emptyset$

That is

$$\Omega \cap L^{\leftarrow}\{y\} = \{x_0\}$$

this means that $L^{-1}y = x_0 \in \Omega$.

Since the set Ω is a open set then $x_0 \in \Omega$ but $x_0 \notin \partial\Omega$. That is

$$L^{\leftarrow}\{y\} \cap \partial\Omega = \emptyset.$$

That is assumption 2) and assumption 3) occur if and only if under the condition $L^{\leftarrow}\{y\} \cap \partial\Omega = \emptyset$. Therefore since $0 \notin (I - L^{\leftarrow}\{y\})(\partial\Omega)$ then the triple $(I - L^{-1}y, \Omega, 0)$ is an admissiable triple.

Now again let us return to calculate the degree

$d\left[\left(L, \tilde{N}(.,0), \Omega\right)\right] = d[I - L^{-1}y, \Omega, 0]$. (31)

Because the equalities

$$d[I - L^{-1}y, \Omega, 0] = d[I, \Omega, L^{-1}y] = d[I, \Omega, x_0] \tag{32}$$

and $x_0 \in \Omega$ then $d[I, \Omega, x_0] = 1$.

As a result because of existence theorem and Theorem 3, for the case $\mathrm{Ker}L = \{0\}$, for $y \in \mathrm{Im}L$ the equation

$$Lx = \lambda N^*(x, \lambda) + y$$

for any $\lambda \in [0,1]$ has at least one solution in Ω.

Now let us consider the case $\mathrm{Ker}L \neq \{0\}$. Using the rule of invariance of Leray-Schauder degree under translation and using its definiton the equality

$$d[I - P - \Lambda\Pi N^*(.,0) - K_p y, \Omega, 0] = d[I - P - \Lambda\Pi N^*(.+K_p y, 0), K_p y + \Omega, 0] \tag{33}$$

is satisfied . Since here

$$\left(I - P - \Lambda\Pi N^*(.+K_p y, 0)\right)(x) = 0 \Leftrightarrow x - Px - \Lambda\Pi N^*(x + K_p y, 0) = 0 \tag{34}$$

$$\Leftrightarrow \; Px + \Lambda\Pi N^*(x + K_p y, 0) = x \tag{35}$$

and $Px \in \mathrm{Im}P = \mathrm{Ker}L$, $\Lambda\Pi N^*(x + K_p y, 0) \in \mathrm{Ker}L$ then $x \in \mathrm{Ker}L$. So that the domain of the expression $\left(I - P - \Lambda\Pi N^*(.+K_p y, 0)\right)$ becomes a subset of $\mathrm{Ker}L$. Hence we have

$$d[I - P - \Lambda\Pi N^*(.,0) - K_p y, \Omega, 0] = d[I - P - \Lambda\Pi N^*(.+K_p y, 0), -K_p y + \Omega, 0] \tag{36}$$

$$= d\left[\left(I - P - \Lambda\Pi N^*(.+K_p y, 0)\right)_{\mathrm{Ker}L}, (-K_p y + \Omega) \cap \mathrm{Ker}L, 0\right]. \tag{37}$$

But on the set $\mathrm{Ker}L = \mathrm{Im}P$ we have $(I - P) = 0$ then this implies

$$= d\left[\{-\Lambda\Pi N^*(.+K_p y, 0)\}_{|KerL}, (-K_p y + \Omega) \cap \mathrm{Ker}L, 0\right]. \tag{38}$$

Beside by the result of Lemma 5 (a) the expression (38) is equal to

$$= \pm d\left[\{\Pi N^*(.+K_p y, 0)\}_{|\mathrm{Ker}L}, (-K_p y + \Omega) \cap \mathrm{Ker}L, 0\right]. \tag{39}$$

Moreover by the result of Lemma 5 (b) the expression (39) is equal to

$$= \pm d\left[\Pi N^*(x, 0)_{|L^{\leftarrow}\{y\}}, \Omega \cap L^{\leftarrow}\{y\}, 0\right]. \tag{40}$$

Since by the assumption (3) the degree $d\left[\Pi N^*(x, 0)_{|L^{\leftarrow}\{y\}}, \; \Omega \cap L^{\leftarrow}\{y\}, 0\right] \neq 0$. So that, because of existence theorem and Theorem 3, for the case $\mathrm{Ker}L \neq \{0\}$, for $y \in \mathrm{Im}L$ the equation

$$Lx = \lambda N^*(x, \lambda) + y$$

for any $\lambda \in [0,1]$ has at least one solution in Ω.

Therefore beside the assumptions 1) and 2) if we assume that the equation (16) or (24) does not have any solution on $\mathrm{Dom}L \cap \partial\Omega$ then for $y \in \mathrm{Im}L$ the equation

$$Lx = \lambda N^*(x,\lambda) + y$$

for any $\lambda \in [0,1]$ has at least one solution in Ω. Hence the second part of Theorem 7 is proved. In particular for $\lambda = 1$, if we assume that the equation (16) does not have any solution on $\mathrm{Dom}L \cap \partial\Omega$ then for $y \in \mathrm{Im}L$ the equation

$$Lx = N^*(x,1) + y = N(x) + y$$

has at least one solution in Ω. So this indicates first part of Theorem 7 for the case the equation (16) does not have any solution on $\mathrm{Dom}L \cap \partial\Omega$. But if the equation (16) does have a solution on $\mathrm{Dom}L \cap \partial\Omega$, then the first part of Theorem 7 still clearly holds, however because in this case the necessary condition for the homotopy does not hold, we cannot apply Theorem 3, so that we cannot talk about the existence of solution of equation (17) for $\lambda \in [0,1[$. In this case equation (16) has a solution on $\mathrm{Dom}L \cap \partial\Omega$. This completes the proof of Theorem 7.

However, Gaines and Mawhin [1] expressed Therorem 7 in the following way and this expresion is partly false, namely in the second part of Thorem 8 there is an important absence of a condition. Namely:

Theorem 8: Assume that the operators L and N^* as in above and $y \in \mathrm{Im}L$ is any number. Assume also that assumptions 1), 2) and 3) of Theorem 7 holds. Therefore the equation

$$Lx = Nx + y \tag{41}$$

has at least one solution on the set $\overline{\Omega}$ and for any $\lambda \in [0,1[$ the equation

$$Lx = \lambda N^*(x,\lambda) + y \tag{42}$$

has at least on solution in Ω.

Theorem 8 as a consequence says that the equation (41) which is the equation (16) has at least one solution on $\overline{\Omega}$ and for any $\lambda \in [0,1[$ the equation (42) which is the equation (17) has at least one solution in Ω. But this two expression sometimes cannot hold at the same time. Namely; the equation (41) has at least one solution on $\partial\Omega$ or does not have any solution on $\partial\Omega$. Beside the assumptions 1) and 2) if the equation (41) does not have any solution on $\partial\Omega$, the rule of invariance of coincidence degree under homotopy can be applied like in the proof of Theorem 7. In this way it is shown that for any $\lambda \in [0,1[$ the equation (42) does have a solution in Ω. But, however, if the equation (41) has a solution on $\partial\Omega$, because of the rule of invariance of coincidence degree under homotopy can not be applied, then for any $\lambda \in [0,1[$ it can not be said anything about existence of solution of equation (42) in Ω with this method. That is this proof does not say anything about existence of solution of equation (42) in Ω for $\lambda \in [0,1[$ under the condition equation (41) has a solution on $\partial\Omega$. Because of this, the result of Theorem 8 should be like the result of Theorem 7.

In fact assumption 2) in Theorem 7 or Theorem 8 is a special case of assumption 1) for $\lambda = 0$, so that assumption 1) and assumption 2) can be unified as in the following theorem.

Theorem 9: Assume that the operators L and N^* as in above and $y \in \mathrm{Im}L$ is any number. If the conditions

1) For $x \in \mathrm{DomL} \cap \partial\Omega$ and $\lambda \in [0,1[$ için
   $Lx \neq \lambda N^*(x,\lambda) + y$
2) $d\left[\Pi N^*(.,0)_{|L^{\leftarrow}\{y\}},\ \Omega \cap L^{\leftarrow}\{y\}, 0\right] \neq 0$

are satisfied, then the equation (41) has at least one solution on $\overline{\Omega}$. And whenever the equation (41) does not have any solution on $\mathrm{DomL} \cap \partial\Omega$ then for any $\lambda \in [0,1]$ the equation (42) does have at least one solution on Ω.

3. Results and Discussion

As a result, in this study it is seen that for $y \in \mathrm{Im}L$ the linear space $L^{\leftarrow}\{y\}$ is a finite dimensional space; the homotopy $\widetilde{N}(x,\lambda)$ defined by $\widetilde{N}(x,\lambda) = QN^*(x,\lambda) + \lambda(I-Q)N^*(x,\lambda) + y$ for $\lambda \in [0,1]$ is L

-compact. The operator $M^*(.,0) = P + (\Lambda\Pi + K_{P,Q})(QN^*(.,0) + y)$ can be written in the form $M^*(.,0) = P + \Lambda\Pi N^*(.,0) + K_P y$. Also it is seen that in order the operator equation $Lx = \lambda N^*(x,\lambda) + y$ has at least one solution in Ω, the equation $Lx = Nx + y$ should not have any solution on $\mathrm{Dom}L \cap \partial\Omega$, this fact is not expressed in the study of Gaines and Mawhin in [1] which is expressed in Theorem 8.

4. Conclusions

Generalized continuation theorem for coincidence degree given and proven densely by Gaines and Mahwin in this paper explained in detail and corrected. Besides the proofs of some results that their proofs not given by them are given in this study. Namely, the proofs of Lemma 1 and 2 and proofs of Lemmas 4-6 are given. In the proof of generalized continuation theorem, Theorem 7, important contributions to make the proof much more understandable are made. Also it is explained that in result of generalized continuation theorem, for the existence of solution of the equation (42) it is necessary that the equation (41) should not have any solution on the boundary set $\mathrm{DomL} \cap \partial\Omega$ with respect to proof of generelized continuation theorem. So generalized continuation theorem given by Gaines and Mahwin, here is corrected, reason is explained and proof is given in detail. Lastly first two assumption of generalized continuation theorem are unified and given in Theorem 9.

References

[1] Gaines, R. E., and Mawhin, J. L., Coincidence Degree and Nonlinear Differential Equations, vol. 568 of Lecture Notes in Mathematics, Springer, 1977.

[2] Sırma, A., and Şevgin S., A Note on Coincidence Degree Theory, Abstract and Applied Analysis, (2012) 370946; 1-18. Doi:10.1155/2012/370946.

[3] Mawhin, J. L., Reduction and Continuation Theorems for Brouwer Degree and Applications to Nonlinear Difference Equations, Opuscula Math., (2008) 28(4); 541-560.

[4] Feltrin G., and Zanolin F., Existence of Positive Solutions in Superlinear Case via Coincidence Degree: The Neumann and the Periodic Boundary Value Problems, Adv. Differential Equations, 20(9-10), (2015) 937-982.

[5] Dong, X., Bai, Z. and Zhang, S., Positive Solutions to Boundary Value Problems of p-Laplacian with Fractional Derivative, Boundary Value Problems, 2017(5), (2017). Doi:10.1186/s13661-016-0735-Z

[6] Tian-Wei-Tian, Z., Multiplicity of Positive Almost Periodic Solutions in a Delayed Hassell-Varley-Type Predator-Prey Model with Harvesting on Prey, Math. Meth. Appl. Sci., 37(5), (2014) 686-697. Doi:10.1002/mma.2826

[7] Liu Z., Dynamics of Positive Solutions to SIR and SEIR Epidemic Models with Saturated Incidence Rates, Nonlinear Analysis: Real Word Applications, 14 (2013) 1286-1295. Doi:10/1016/j.nonrwa.2012.09.2016

[8] Muhammadhaji, A., and Teng, Z., Global Attractivity of a Periodic Delayed N-Species Model of Facultative Mutualism, Discrete Dynamics in Nature and Society, 2013, 580185, (2013) 1-11. Doi:10.1155/2013/580185

[9] Li Y., and Ye Y., Multiple Positive Almost Periodic Solutions to an Impulsive Non-autonomous Lotka-Violtera Predator-prey System with Harvesting Terms, Commun Nonlinear Sci Numer Simulat, 18(2013) 3190-3201. Doi:10.1016/j.cnsns.2013.03.014

[10] Hou X., Duan L., and Huang Z., Permanence and Periodic Solutions for a Class of Delay Nicholson's Blowfies Models, Applied Mathematical Modelling, 37(2013) 1537-1544. Doi:10.1016/j.apm.2012.04.2018

[11] Moussaoui A., Bassaid S., and Ait Dads El H., The Impact of Water Level Fluctuations on a Delayed Prey-Predator Model, Nonlinear Analysis: Real World Applications, 21(2015) 170-184. Doi:10.1016/j.nonrwa.2014.07.011

[12] Niyaz T., and Muhammadhaji A., Positive Periodic Solutions of Cooperative Systems with Delays and Feedback Controls, International Journal of Differential Equations, (2013) 502963 1-9. Doi:10.1155/2013/502963

[13] Zhang A., Qiu J., and She J., Existence and Global Stability of Periodic Solution for High-order Discrete-time BAM Neural Networks, Neural Networks, 50(2014) 98-109. Doi:10/1016/j.neuret.2013.11.005

[14] Barcaggin A., Feltrin G., and Zanolin F., Positive Solutions for Super-sublinear Indefinite Problems: High Multiplicity Results via Coincidence Degree, Trans. Amer. Math. Soc., 370(2018) 791-845. Doi:10.1090/tran/6992

[15] Xie D., and Jiang Y., Global Exponential Stability of Periodic Solution for Delayed Complex-valued Networks with Impulses, Neurocomputing 207(2016) 528-538. Doi:10.1016/j.neucom.2016.04.054

[16] Feltrin G., and Zanolin F., An Application of Coincidence Degree Theory to Cyclic Feedback Type Systems Associated with Nonlinear Differential

Operators, Topol. Methods Nonlinear Anal., 50(2), (2017) 683-726. Doi:10.12775/TMNA.2017.038

[17] Lv W., Solvability for a Discrete Fractional Three-Point Bondary Value Problem at Resonance, Abstract and Applied Analysis, (2014) 601092, 1-7. Doi:10.1155/2014/601092

[18] Li Y., and Qin J., Existence and Global Stability of Periodic Solutions for Quaternion-Valued Cellular Neural Networks with Time-Varying Delays, Neurocomputing, 292(2018) 91-103. Doi:10.1016/j.neucom2018.02.077

[19] Mohammed M.J., Ibrahim R.W., and Ahmad M.Z., Periodicity Computation of Genralized Mathematical Biology Problems Involving Delay Differential Equations, Saudi Journal of Biological Sciences, 24(2017) 737-740. Doi:10.1016/j.sjbs.2017.01.050

[20] Chen F., Coincidence Degree and Fractional Boundary Value Problems with Impulses, Computers and Mathematics with Applications, 64(2012) 3444-3455. Doi:10.1016/j.camwa.2012.02.022