CONSTRUCTIVE MATHEMATICAL ANALYSIS
2 (2019), No. 2, pp. 64-80
http://dergipark.gov.tr/cma
ISSN 2651-2939

General Multivariate Iyengar Type Inequalities

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ABSTRACT. Here we give a variety of general multivariate Iyengar type inequalities for not necessarily radial functions defined on the shell and ball. Our approach is based on the polar coordinates in $\mathbb{R}^N$, $N \geq 2$, and the related multivariate polar integration formula. Via this method we transfer well-known univariate Iyengar type inequalities and univariate author's related results into general multivariate Iyengar inequalities.

Keywords: Iyengar inequality, Polar coordinates, Not necessarily radial function, Shell, Ball.

2010 Mathematics Subject Classification: 26D10, 26D15.

1. BACKGROUND

In the year 1938, Iyengar [5] proved the following interesting inequality:

Theorem 1.1. *Let f be a differentiable function on $[a,b]$ and $|f'(x)| \leq M_1$. Then*

$$\left|\int_a^b f(x)\,dx - \frac{1}{2}(b-a)(f(a)+f(b))\right| \leq \frac{M_1(b-a)^2}{4} - \frac{(f(b)-f(a))^2}{4M_1}. \tag{1.1}$$

In 2001, X.-L. Cheng [4] proved that

Theorem 1.2. *Let $f \in C^2([a,b])$ and $|f''(x)| \leq M_2$. Then*

$$\left|\int_a^b f(x)\,dx - \frac{1}{2}(b-a)(f(a)+f(b)) + \frac{1}{8}(b-a)^2(f'(b)-f'(a))\right| \tag{1.2}$$

$$\leq \frac{M_2}{24}(b-a)^3 - \frac{(b-a)}{16M_2}\Delta_1^2,$$

where

$$\Delta_1 = f'(a) - \frac{2(f(b)-f(a))}{(b-a)} + f'(b).$$

In 1996, Agarwal and Dragomir [1] obtained a generalization of (1.1):

Theorem 1.3. *Let $f : [a,b] \to \mathbb{R}$ be a differentiable function such that for all $x \in [a,b]$ with $M > m$ we have $m \leq f'(x) \leq M$. Then*

$$\left|\int_a^b f(x)\,dx - \frac{1}{2}(b-a)(f(a)+f(b))\right|$$

$$\leq \frac{(f(b)-f(a)-m(b-a))(M(b-a)-f(b)+f(a))}{2(M-m)}.$$

In [7], Qi proved the following:

Received: 22 January 2019; Accepted: 27 March 2019; Published Online: 29 March 2019
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DOI: 10.33205/cma.543560

Theorem 1.4. *Let $f:[a,b]\to\mathbb{R}$ be a twice differentiable function such that for all $x\in[a,b]$ with $M>0$ we have $|f''(x)|\le M$. Then*

$$\left|\int_a^b f(x)\,dx-\frac{(f(a)+f(b))}{2}(b-a)+\frac{(1+Q^2)}{8}(f'(b)-f'(a))(b-a)^2\right|\le\frac{M(b-a)^3}{24}\left(1-3Q^2\right),$$

where

$$Q^2=\frac{\left(f'(a)+f'(b)-2\left(\frac{f(b)-f(a)}{b-a}\right)\right)^2}{M^2(b-a)^2-(f'(b)-f'(a))^2}.$$

In 2005, Zheng Liu, [6], proved the following:

Theorem 1.5. *Let $f:[a,b]\to\mathbb{R}$ be a differentiable function such that f' is integrable on $[a,b]$ and for all $x\in[a,b]$ with $M>m$ we have*

$$m\le\frac{f'(x)-f'(a)}{x-a}\le M\ \text{ and } m\le\frac{f'(b)-f'(x)}{b-x}\le M.$$

Then

$$\left|\int_a^b f(x)\,dx-\frac{(f(a)+f(b))}{2}(b-a)+\left(\frac{1+P^2}{8}\right)(f'(b)-f'(a))(b-a)^2-\left(\frac{1+3P^2}{48}\right)(m+M)(b-a)^3\right|\le\frac{(M-m)(b-a)^3}{48}\left(1-3P^2\right),$$

where

$$P^2=\frac{\left(f'(a)+f'(b)-2\left(\frac{f(b)-f(a)}{b-a}\right)\right)^2}{\left(\frac{M-m}{2}\right)^2(b-a)^2-\left(f'(b)-f'(a)-\left(\frac{m+M}{2}\right)(b-a)\right)^2}.$$

Next we list some author's related results, (here $L_\infty([a,b])$ is the normed space of essentially bounded functions over $[a,b]$):

Theorem 1.6. ([3]) *Let $n\in\mathbb{N}$, $f\in AC^n([a,b])$ (i.e. $f^{(n-1)}\in AC([a,b])$, absolutely continuous functions). We assume that $f^{(n)}\in L_\infty([a,b])$. Then*

(i)

$$\left|\int_a^b f(x)\,dx-\sum_{k=0}^{n-1}\frac{1}{(k+1)!}\left[f^{(k)}(a)(t-a)^{k+1}+(-1)^k f^{(k)}(b)(b-t)^{k+1}\right]\right|\le\frac{\left\|f^{(n)}\right\|_{L_\infty([a,b])}}{(n+1)!}\left[(t-a)^{n+1}+(b-t)^{n+1}\right],\tag{1.3}$$

for all $t\in[a,b]$,

(ii) at $t=\frac{a+b}{2}$, the right hand side of (1.3) is minimized, and we get:

$$\left|\int_a^b f(x)\,dx-\sum_{k=0}^{n-1}\frac{1}{(k+1)!}\frac{(b-a)^{k+1}}{2^{k+1}}\left[f^{(k)}(a)+(-1)^k f^{(k)}(b)\right]\right|\le\frac{\left\|f^{(n)}\right\|_{L_\infty([a,b])}}{(n+1)!}\frac{(b-a)^{n+1}}{2^n},$$

(iii) *if* $f^{(k)}(a) = f^{(k)}(b) = 0$ *for all* $k = 0, 1, ..., n-1$, *then we obtain*

$$\left|\int_a^b f(x)\,dx\right| \leq \frac{\|f^{(n)}\|_{L_\infty([a,b])}}{(n+1)!}\frac{(b-a)^{n+1}}{2^n}$$

which is a sharp inequality,

(iv) *more generally, for* $j = 0, 1, 2, ..., N \in \mathbb{N}$, *it holds*

$$\left|\int_a^b f(x)\,dx - \sum_{k=0}^{n-1}\frac{1}{(k+1)!}\left(\frac{b-a}{N}\right)^{k+1}\left[j^{k+1}f^{(k)}(a) + (-1)^k (N-j)^{k+1} f^{(k)}(b)\right]\right| \tag{1.4}$$
$$\leq \frac{\|f^{(n)}\|_{L_\infty([a,b])}}{(n+1)!}\left(\frac{b-a}{N}\right)^{n+1}\left[j^{n+1} + (N-j)^{n+1}\right],$$

(v) *if* $f^{(k)}(a) = f^{(k)}(b) = 0$, $k = 1, ..., n-1$, *from (1.4) we get:*

$$\left|\int_a^b f(x)\,dx - \left(\frac{b-a}{N}\right)[jf(a) + (N-j)f(b)]\right| \tag{1.5}$$
$$\leq \frac{\|f^{(n)}\|_{L_\infty([a,b])}}{(n+1)!}\left(\frac{b-a}{N}\right)^{n+1}\left[j^{n+1} + (N-j)^{n+1}\right]$$

for $j = 0, 1, 2, ..., N \in \mathbb{N}$,

(vi) *when* $N = 2$ *and* $j = 1$, *(1.5) turns to*

$$\left|\int_a^b f(x)\,dx - \left(\frac{b-a}{2}\right)(f(a) + f(b))\right| \leq \frac{\|f^{(n)}\|_{L_\infty([a,b])}}{(n+1)!}\frac{(b-a)^{n+1}}{2^n}, \tag{1.6}$$

(vii) *when* $n = 1$ *(without any boundary conditions), we get from (1.6) that*

$$\left|\int_a^b f(x)\,dx - \left(\frac{b-a}{2}\right)(f(a) + f(b))\right| \leq \|f'\|_{[a,b],\infty}\frac{(b-a)^2}{4},$$

a similar to Iyengar inequality (1.1).

We mention here $L_1([a,b])$ is the normed space of integrable functions over $[a,b]$).

Theorem 1.7. ([3]) *Let* $f \in AC^n([a,b])$, $n \in \mathbb{N}$. *Then*

(i)

$$\left|\int_a^b f(x)\,dx - \sum_{k=0}^{n-1}\frac{1}{(k+1)!}\left[f^{(k)}(a)(t-a)^{k+1} + (-1)^k f^{(k)}(b)(b-t)^{k+1}\right]\right| \tag{1.7}$$
$$\leq \frac{\|f^{(n)}\|_{L_1([a,b])}}{n!}\left[(t-a)^n + (b-t)^n\right],$$

for all $t \in [a,b]$,

(ii) *at* $t = \frac{a+b}{2}$, *the right hand side of (1.7) is minimized, and we get:*

$$\left|\int_a^b f(x)\,dx - \sum_{k=0}^{n-1}\frac{1}{(k+1)!}\frac{(b-a)^{k+1}}{2^{k+1}}\left[f^{(k)}(a) + (-1)^k f^{(k)}(b)\right]\right|$$
$$\leq \frac{\|f^{(n)}\|_{L_1([a,b])}}{n!}\frac{(b-a)^n}{2^{n-1}},$$

(iii) if $f^{(k)}(a)=f^{(k)}(b)=0$, for all $k=0,1,...,n-1$, we obtain

$$\left|\int_a^b f(x)\,dx\right| \leq \frac{\left\|f^{(n)}\right\|_{L_1([a,b])}}{n!}\frac{(b-a)^n}{2^{n-1}},$$

which is a sharp inequality,

(iv) more generally, for $j=0,1,2,...,N\in\mathbb{N}$, it holds

$$\left|\int_a^b f(x)\,dx-\sum_{k=0}^{n-1}\frac{1}{(k+1)!}\left(\frac{b-a}{N}\right)^{k+1}\left[j^{k+1}f^{(k)}(a)+(-1)^k(N-j)^{k+1}f^{(k)}(b)\right]\right| \tag{1.8}$$
$$\leq \frac{\left\|f^{(n)}\right\|_{L_1([a,b])}}{n!}\left(\frac{b-a}{N}\right)^n\left[j^n+(N-j)^n\right],$$

(v) if $f^{(k)}(a)=f^{(k)}(b)=0$, $k=1,...,n-1$, from (1.8) we get:

$$\left|\int_a^b f(x)\,dx-\left(\frac{b-a}{N}\right)\left[jf(a)+(N-j)f(b)\right]\right| \tag{1.9}$$
$$\leq \frac{\left\|f^{(n)}\right\|_{L_1([a,b])}}{n!}\left(\frac{b-a}{N}\right)^n\left[j^n+(N-j)^n\right],$$

for $j=0,1,2,...,N\in\mathbb{N}$,

(vi) when $N=2$ and $j=1$, (1.9) turns to

$$\left|\int_a^b f(x)\,dx-\frac{(b-a)}{2}(f(a)+f(b))\right| \leq \frac{\left\|f^{(n)}\right\|_{L_1([a,b])}}{n!}\frac{(b-a)^n}{2^{n-1}}, \tag{1.10}$$

(vii) when $n=1$ (without any boundary conditions), we get from (1.10) that

$$\left|\int_a^b f(x)\,dx-\left(\frac{b-a}{2}\right)(f(a)+f(b))\right| \leq \left\|f'\right\|_{L_1([a,b])}(b-a).$$

We mention here $L_q([a,b])$ is the normed space of functions f such that $|f|^q$ is integrable over $[a,b]$)

Theorem 1.8. *([3]) Let $f\in AC^n([a,b])$, $n\in\mathbb{N}$; $p,q>1$ such that $\frac{1}{p}+\frac{1}{q}=1$, and $f^{(n)}\in L_q([a,b])$. Then*

(i)

$$\left|\int_a^b f(x)\,dx-\sum_{k=0}^{n-1}\frac{1}{(k+1)!}\left[f^{(k)}(a)(t-a)^{k+1}+(-1)^k f^{(k)}(b)(b-t)^{k+1}\right]\right| \tag{1.11}$$
$$\leq \frac{\left\|f^{(n)}\right\|_{L_q([a,b])}}{(n-1)!\left(n+\frac{1}{p}\right)(p(n-1)+1)^{\frac{1}{p}}}\left[(t-a)^{n+\frac{1}{p}}+(b-t)^{n+\frac{1}{p}}\right],$$

for all $t\in[a,b]$,

(ii) *at* $t = \frac{a+b}{2}$, *the right hand side of* (1.11) *is minimized, and we get:*

$$\left| \int_a^b f(x)\,dx - \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \frac{(b-a)^{k+1}}{2^{k+1}} \left[f^{(k)}(a) + (-1)^k f^{(k)}(b) \right] \right|$$

$$\leq \frac{\left\| f^{(n)} \right\|_{L_q([a,b])}}{(n-1)!\left(n+\frac{1}{p}\right)(p(n-1)+1)^{\frac{1}{p}}} \frac{(b-a)^{n+\frac{1}{p}}}{2^{n-\frac{1}{q}}},$$

(iii) *if* $f^{(k)}(a) = f^{(k)}(b) = 0$, *for all* $k = 0, 1, ..., n-1$, *we obtain*

$$\left| \int_a^b f(x)\,dx \right| \leq \frac{\left\| f^{(n)} \right\|_{L_q([a,b])}}{(n-1)!\left(n+\frac{1}{p}\right)(p(n-1)+1)^{\frac{1}{p}}} \frac{(b-a)^{n+\frac{1}{p}}}{2^{n-\frac{1}{q}}},$$

which is a sharp inequality,

(iv) *more generally, for* $j = 0, 1, 2, ..., N \in \mathbb{N}$, *it holds*

$$\left| \int_a^b f(x)\,dx - \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \left(\frac{b-a}{N} \right)^{k+1} \left[j^{k+1} f^{(k)}(a) + (-1)^k (N-j)^{k+1} f^{(k)}(b) \right] \right| \tag{1.12}$$

$$\leq \frac{\left\| f^{(n)} \right\|_{L_q([a,b])}}{(n-1)!\left(n+\frac{1}{p}\right)(p(n-1)+1)^{\frac{1}{p}}} \left(\frac{b-a}{N} \right)^{n+\frac{1}{p}} \left[j^{n+\frac{1}{p}} + (N-j)^{n+\frac{1}{p}} \right],$$

(v) *if* $f^{(k)}(a) = f^{(k)}(b) = 0$, $k = 1, ..., n-1$, *from* (1.12) *we get:*

$$\left| \int_a^b f(x)\,dx - \left(\frac{b-a}{N} \right) [j f(a) + (N-j) f(b)] \right| \tag{1.13}$$

$$\leq \frac{\left\| f^{(n)} \right\|_{L_q([a,b])}}{(n-1)!\left(n+\frac{1}{p}\right)(p(n-1)+1)^{\frac{1}{p}}} \left(\frac{b-a}{N} \right)^{n+\frac{1}{p}} \left[j^{n+\frac{1}{p}} + (N-j)^{n+\frac{1}{p}} \right],$$

for $j = 0, 1, 2, ..., N \in \mathbb{N}$,

(vi) *when* $N = 2$ *and* $j = 1$, (1.13) *turns to*

$$\left| \int_a^b f(x)\,dx - \frac{(b-a)}{2} (f(a) + f(b)) \right| \leq \frac{\left\| f^{(n)} \right\|_{L_q([a,b])}}{(n-1)!\left(n+\frac{1}{p}\right)(p(n-1)+1)^{\frac{1}{p}}} \frac{(b-a)^{n+\frac{1}{p}}}{2^{n-\frac{1}{q}}}, \tag{1.14}$$

(vii) *when* $n = 1$ *(without any boundary conditions), we get from* (1.14) *that*

$$\left| \int_a^b f(x)\,dx - \left(\frac{b-a}{2} \right) (f(a) + f(b)) \right| \leq \frac{\|f'\|_{L_q([a,b])}}{\left(1+\frac{1}{p}\right)} \frac{(b-a)^{1+\frac{1}{p}}}{2^{\frac{1}{p}}}.$$

We need

Remark 1.1. *We define the ball* $B(0,R) = \left\{ x \in \mathbb{R}^N : |x| < R \right\} \subseteq \mathbb{R}^N$, $N \geq 2$, $R > 0$, *and the sphere*

$$S^{N-1} := \left\{ x \in \mathbb{R}^N : |x| = 1 \right\},$$

where $|\cdot|$ *is the Euclidean norm. Let* $d\omega$ *be the element of surface measure on* S^{N-1} *and*

$$\omega_N = \int_{S^{N-1}} d\omega = \frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}$$

is the area of S^{N-1}.
For $x \in \mathbb{R}^N - \{0\}$ *we can write uniquely* $x = r\omega$, *where* $r = |x| > 0$ *and* $\omega = \frac{x}{r} \in S^{N-1}$, $|\omega| = 1$.
Note that $\int_{B(0,R)} dy = \frac{\omega_N R^N}{N}$ *is the Lebesgue measure on the ball, that is the volume of* $B(0,R)$, *which exactly is* $Vol(B(0,R)) = \frac{\pi^{\frac{N}{2}} R^N}{\Gamma\left(\frac{N}{2}+1\right)}$.
Following [8, pp. 149-150, exercise 6], *and* [9, pp. 87-88, Theorem 5.2.2] *we can write for* $F : \overline{B(0,R)} \to \mathbb{R}$ *a Lebesgue integrable function that*

$$\int_{B(0,R)} F(x)\,dx = \int_{S^{N-1}} \left(\int_0^R F(r\omega)\, r^{N-1} dr \right) d\omega, \tag{1.15}$$

and we use this formula a lot.
Typically here the function $f : \overline{B(0,R)} \to \mathbb{R}$ *is not radial. A radial function* f *is such that there exists a function* g *with* $f(x) = g(r)$, *where* $r = |x|$, $r \in [0,R]$, *for all* $x \in \overline{B(0,R)}$.

We need

Remark 1.2. *Let the spherical shell* $A := B(0,R_2) - \overline{B(0,R_1)}$, $0 < R_1 < R_2$, $A \subseteq \mathbb{R}^N$, $N \geq 2$, $x \in \overline{A}$. *Consider that* $f : \overline{A} \to \mathbb{R}$ *is not radial. A radial function* f *is such that there exists a function* g *with* $f(x) = g(r)$, $r = |x|$, $r \in [R_1, R_2]$, *for all* $x \in \overline{A}$. *Here* x *can be written uniquely as* $x = r\omega$, *where* $r = |x| > 0$ *and* $\omega = \frac{x}{r} \in S^{N-1}$, $|\omega| = 1$, *see* ([8], *p. 149-150 and* [2], *p. 421), furthermore for* $F : \overline{A} \to \mathbb{R}$ *a Lebesgue integrable function we have that*

$$\int_A F(x)\,dx = \int_{S^{N-1}} \left(\int_{R_1}^{R_2} F(r\omega)\, r^{N-1} dr \right) d\omega. \tag{1.16}$$

Here

$$Vol(A) = \frac{\omega_N \left(R_2^N - R_1^N\right)}{N} = \frac{\pi^{\frac{N}{2}} \left(R_2^N - R_1^N\right)}{\Gamma\left(\frac{N}{2}+1\right)}.$$

In this article we derive general multivariate Iyengar type inequalities on the shell and ball of $\mathbb{R}^N$, $N \geq 2$, for not necessarily radial functions. Our results are based on Theorems 1.1-1.8.

2. Main Results

We present the following non-radial multivariate Iyengar type inequalities:
We start with

Theorem 2.9. *Let the spherical shell* $A := B(0,R_2) - \overline{B(0,R_1)}$, $0 < R_1 < R_2$, $A \subseteq \mathbb{R}^N$, $N \geq 2$. *Consider* $f : \overline{A} \to \mathbb{R}$ *that is not necessarily radial, and that* $f \in C^1\left(\overline{A}\right)$. *Assume that* $\left|\frac{\partial f(s\omega)}{\partial s}\right| \leq M_1$, *for all* $s \in [R_1, R_2]$, *and for all* $\omega \in S^{N-1}$, *where* $M_1 > 0$.
Then

$$\left| \int_A f(y)\,dy - \frac{(R_2 - R_1)}{2} \left(R_1^{N-1} \int_{S^{N-1}} f(R_1\omega)\,d\omega + R_2^{N-1} \int_{S^{N-1}} f(R_2\omega)\,d\omega \right) \right|$$

$$\leq \frac{M_1 \pi^{\frac{N}{2}} (R_2 - R_1)^2}{2\Gamma\left(\frac{N}{2}\right)} - \frac{\int_{S^{N-1}} \left(f(R_2\omega) R_2^{N-1} - f(R_1\omega) R_1^{N-1} \right)^2 d\omega}{4M_1}.$$

Proof. Here $f(s\omega)s^{N-1} \in C^1([R_1, R_2])$, $N \geq 2$, for all $\omega \in S^{N-1}$. By (1.1) we get

$$\left| \int_{R_1}^{R_2} f(s\omega) s^{N-1} ds - \frac{1}{2}(R_2 - R_1)\left(f(R_1\omega) R_1^{N-1} + f(R_2\omega) R_2^{N-1}\right) \right|$$
$$\leq \frac{M_1 (R_2 - R_1)^2}{4} - \frac{\left(f(R_2\omega) R_2^{N-1} - f(R_1\omega) R_1^{N-1}\right)^2}{4M_1} =: \lambda_1(\omega),$$

for all $\omega \in S^{N-1}$.
Equivalently, we have

$$-\lambda_1(\omega) \leq \int_{R_1}^{R_2} f(s\omega) s^{N-1} ds - \frac{1}{2}(R_2 - R_1)\left(f(R_1\omega) R_1^{N-1} + f(R_2\omega) R_2^{N-1}\right) \leq \lambda_1(\omega),$$

for all $\omega \in S^{N-1}$.
Hence it holds

$$- \int_{S^{N-1}} \lambda_1(\omega) d\omega \leq \int_{S^{N-1}} \left(\int_{R_1}^{R_2} f(s\omega) s^{N-1} ds \right) d\omega$$
$$- \frac{1}{2}(R_2 - R_1)\left(R_1^{N-1} \int_{S^{N-1}} f(R_1\omega) d\omega + R_2^{N-1} \int_{S^{N-1}} f(R_2\omega) d\omega \right)$$
$$\leq \int_{S^{N-1}} \lambda_1(\omega) d\omega.$$

That is (by (1.16))

$$- \left[\frac{\pi^{\frac{N}{2}} M_1 (R_2 - R_1)^2}{2\Gamma\left(\frac{N}{2}\right)} - \frac{\int_{S^{N-1}} \left(f(R_2\omega) R_2^{N-1} - f(R_1\omega) R_1^{N-1}\right)^2 d\omega}{4M_1} \right]$$
$$\leq \int_A f(y) dy - \frac{(R_2 - R_1)}{2} \left(R_1^{N-1} \int_{S^{N-1}} f(R_1\omega) d\omega + R_2^{N-1} \int_{S^{N-1}} f(R_2\omega) d\omega \right)$$
$$\leq \frac{\pi^{\frac{N}{2}} M_1 (R_2 - R_1)^2}{2\Gamma\left(\frac{N}{2}\right)} - \frac{\int_{S^{N-1}} \left(f(R_2\omega) R_2^{N-1} - f(R_1\omega) R_1^{N-1}\right)^2 d\omega}{4M_1},$$

proving the claim. □

We continue with

Theorem 2.10. *Let the spherical shell* $A := B(0, R_2) - \overline{B(0, R_1)}$, $0 < R_1 < R_2$, $A \subseteq \mathbb{R}^N$, $N \geq 2$. *Consider* $f : \overline{A} \to \mathbb{R}$ *that is not necessarily radial, and that* $f \in C^2(\overline{A})$. *Assume that* $\left| \frac{\partial^2 f(s\omega)}{\partial s^2} \right| \leq M_2$, *for all* $s \in [R_1, R_2]$, *and for all* $\omega \in S^{N-1}$, *where* $M_2 > 0$.
Set

$$\Delta_1(\omega) := \left(f(s\omega) s^{N-1}\right)'(R_1) - \frac{2\left(f(R_2\omega) R_2^{N-1} - f(R_1\omega) R_1^{N-1}\right)}{R_2 - R_1}$$
$$+ \left(f(s\omega) s^{N-1}\right)'(R_2), \quad \forall\, \omega \in S^{N-1}.$$

Then

$$\begin{aligned}
&\left|\int_A f(y)\,dy - \frac{(R_2-R_1)}{2}\left(R_1^{N-1}\int_{S^{N-1}} f(R_1\omega)\,d\omega + R_2^{N-1}\int_{S^{N-1}} f(R_2\omega)\,d\omega\right)\right. \\
&+ \left.\frac{(R_2-R_1)^2}{8}\left[\int_{S^{N-1}} \left(f(s\omega)s^{N-1}\right)'(R_2)\,d\omega - \int_{S^{N-1}} \left(f(s\omega)s^{N-1}\right)'(R_1)\,d\omega\right]\right| \\
&\leq \frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\frac{M_2}{12}(R_2-R_1)^3 - \frac{(R_2-R_1)}{16M_2}\int_{S^{N-1}} \Delta_1^2(\omega)\,d\omega.
\end{aligned}$$

Proof. Here $f(s\omega)s^{N-1} \in C^2([R_1,R_2])$, $N \geq 2$, for all $\omega \in S^{N-1}$. By (1.2) we get

$$\begin{aligned}
&\left|\int_{R_1}^{R_2} f(s\omega)s^{N-1}ds - \frac{1}{2}(R_2-R_1)\left(f(R_1\omega)R_1^{N-1} + f(R_2\omega)R_2^{N-1}\right)\right. \\
&+ \left.\frac{1}{8}(R_2-R_1)^2\left(\left(f(s\omega)s^{N-1}\right)'(R_2) - \left(f(s\omega)s^{N-1}\right)'(R_1)\right)\right| \\
&\leq \frac{M_2}{24}(R_2-R_1)^3 - \frac{(R_2-R_1)}{16M_2}\Delta_1^2(\omega) =: \lambda_2(\omega),
\end{aligned}$$

for all $\omega \in S^{N-1}$.
Equivalently, we have

$$\begin{aligned}
&-\lambda_2(\omega) \leq \int_{R_1}^{R_2} f(s\omega)s^{N-1}ds - \frac{(R_2-R_1)}{2}\left(f(R_1\omega)R_1^{N-1} + f(R_2\omega)R_2^{N-1}\right) \\
&+ \frac{1}{8}(R_2-R_1)^2\left(\left(f(s\omega)s^{N-1}\right)'(R_2) - \left(f(s\omega)s^{N-1}\right)'(R_1)\right) \leq \lambda_2(\omega),
\end{aligned}$$

for all $\omega \in S^{N-1}$.
Hence it holds

$$\begin{aligned}
&-\int_{S^{N-1}} \lambda_2(\omega)\,d\omega \leq \int_{S^{N-1}}\left(\int_{R_1}^{R_2} f(s\omega)s^{N-1}ds\right)d\omega \\
&- \frac{(R_2-R_1)}{2}\left(R_1^{N-1}\int_{S^{N-1}} f(R_1\omega)\,d\omega + R_2^{N-1}\int_{S^{N-1}} f(R_2\omega)\,d\omega\right) \\
&+ \frac{(R_2-R_1)^2}{8}\left[\int_{S^{N-1}} \left(f(s\omega)s^{N-1}\right)'(R_2)\,d\omega - \int_{S^{N-1}} \left(f(s\omega)s^{N-1}\right)'(R_1)\,d\omega\right] \\
&\leq \int_{S^{N-1}} \lambda_2(\omega)\,d\omega.
\end{aligned}$$

That is (by (1.16))

$$\begin{aligned}
&-\left[\frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\frac{M_2}{12}(R_2-R_1)^3 - \frac{(R_2-R_1)}{16M_2}\int_{S^{N-1}} \Delta_1^2(\omega)\,d\omega\right] \\
&\leq \int_A f(y)\,dy - \frac{(R_2-R_1)}{2}\left(R_1^{N-1}\int_{S^{N-1}} f(R_1\omega)\,d\omega + R_2^{N-1}\int_{S^{N-1}} f(R_2\omega)\,d\omega\right) \\
&+ \frac{(R_2-R_1)^2}{8}\left[\int_{S^{N-1}} \left(f(s\omega)s^{N-1}\right)'(R_2)\,d\omega - \int_{S^{N-1}} \left(f(s\omega)s^{N-1}\right)'(R_1)\,d\omega\right] \\
&\leq \frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\frac{M_2}{12}(R_2-R_1)^3 - \frac{(R_2-R_1)}{16M_2}\int_{S^{N-1}} \Delta_1^2(\omega)\,d\omega,
\end{aligned}$$

proving the claim. □

We give

Theorem 2.11. *Let the spherical shell* $A := B(0, R_2) - \overline{B(0, R_1)}$, $0 < R_1 < R_2$, $A \subseteq \mathbb{R}^N$, $N \geq 2$. *Consider* $f : \overline{A} \to \mathbb{R}$ *that is not necessarily radial, and that* $f \in C^1(\overline{A})$. *Let* $M > m$ *and assume that* $m \leq \frac{\partial f(s\omega)}{\partial s} \leq M$, *for all* $s \in [R_1, R_2]$, *and for all* $\omega \in S^{N-1}$.
Then

$$\begin{aligned}
&\left| \int_A f(y)\,dy - \frac{(R_2 - R_1)}{2} \left(R_1^{N-1} \int_{S^{N-1}} f(R_1\omega)\,d\omega + R_2^{N-1} \int_{S^{N-1}} f(R_2\omega)\,d\omega \right) \right| \\
\leq\; & \frac{1}{2(M-m)} \int_{S^{N-1}} \left[\left(f(R_2\omega) R_2^{N-1} - f(R_1\omega) R_1^{N-1} - m(R_2 - R_1) \right) \right. \\
\times\; & \left. \left(M(R_2 - R_1) - f(R_2\omega) R_2^{N-1} + f(R_1\omega) R_1^{N-1} \right) \right] d\omega.
\end{aligned}$$

Proof. Similar to the proof of Theorem 2.9 by using Theorem 1.3 and (1.16). □

We give

Theorem 2.12. *Let the spherical shell* $A := B(0, R_2) - \overline{B(0, R_1)}$, $0 < R_1 < R_2$, $A \subseteq \mathbb{R}^N$, $N \geq 2$. *Consider* $f : \overline{A} \to \mathbb{R}$ *that is not necessarily radial, and that* $f \in C^2(\overline{A})$. *Assume that* $\left| \frac{\partial^2 f(s\omega)}{\partial s^2} \right| \leq M_3$, *for all* $s \in [R_1, R_2]$, *and for all* $\omega \in S^{N-1}$, *where* $M_3 > 0$.
Set

$$\begin{aligned}
&Q_1^2(\omega) \\
:=\; & \frac{\left[\left(f(s\omega) s^{N-1} \right)'(R_1) + \left(f(s\omega) s^{N-1} \right)'(R_2) - 2\left(\frac{f(R_2\omega) R_2^{N-1} - f(R_1\omega) R_1^{N-1}}{R_2 - R_1} \right) \right]^2}{\left[M_3^2 (R_2 - R_1)^2 - \left(\left(f(s\omega) s^{N-1} \right)'(R_2) - \left(f(s\omega) s^{N-1} \right)'(R_1) \right)^2 \right]},
\end{aligned}$$

for all $\omega \in S^{N-1}$.
Then

$$\begin{aligned}
&\left| \int_A f(y)\,dy - \frac{(R_2 - R_1)}{2} \left(R_1^{N-1} \int_{S^{N-1}} f(R_1\omega)\,d\omega + R_2^{N-1} \int_{S^{N-1}} f(R_2\omega)\,d\omega \right) \right. \\
+\; & \left. \frac{(R_2 - R_1)^2}{8} \int_{S^{N-1}} \left(1 + Q_1^2(\omega)\right) \left(\left(f(s\omega) s^{N-1} \right)'(R_2) - \left(f(s\omega) s^{N-1} \right)'(R_1) \right) d\omega \right| \\
\leq\; & \frac{M_3 (R_2 - R_1)^3}{24} \int_{S^{N-1}} \left(1 - 3Q_1^2(\omega)\right) d\omega.
\end{aligned}$$

Proof. Similar to the proof of Theorem 2.10 by using Theorem 1.4 and (1.16). □

We continue with

Theorem 2.13. *Here all as in Theorem 2.9, and let* $M_1 > m_1$. *Assume that*

$$m_1 \leq \frac{\left(f(s\omega) s^{N-1} \right)'(x) - \left(f(s\omega) s^{N-1} \right)'(R_1)}{x - R_1} \leq M_1,$$

and

$$m_1 \leq \frac{\left(f(s\omega) s^{N-1} \right)'(R_2) - \left(f(s\omega) s^{N-1} \right)'(x)}{R_2 - x} \leq M_1,$$

for all $x \in [R_1, R_2]$, *for all* $\omega \in S^{N-1}$.

Set

$$P_1^2(\omega) = \frac{\left[\left(f(s\omega)s^{N-1}\right)'(R_1) + \left(f(s\omega)s^{N-1}\right)'(R_2) - 2\left(\frac{f(R_2\omega)R_2^{N-1} - f(R_1\omega)R_1^{N-1}}{R_2 - R_1}\right)\right]^2}{\left(\frac{M_1 - m_1}{2}\right)^2 (R_2 - R_1)^2 - \left[\left(f(s\omega)s^{N-1}\right)'(R_2) - \left(f(s\omega)s^{N-1}\right)'(R_1) - \left(\frac{m_1 + M_1}{2}\right)(R_2 - R_1)\right]^2},$$

for all $\omega \in S^{N-1}$.
Then

$$\left| \int_A f(y)\,dy - \left(\frac{R_2 - R_1}{2}\right)\left(R_1^{N-1}\int_{S^{N-1}} f(R_1\omega)\,d\omega + R_2^{N-1}\int_{S^{N-1}} f(R_2\omega)\,d\omega\right) \right.$$
$$+ \frac{(R_2 - R_1)^2}{8}\int_{S^{N-1}} \left(1 + P_1^2(\omega)\right)\left(\left(f(s\omega)s^{N-1}\right)'(R_2) - \left(f(s\omega)s^{N-1}\right)'(R_1)\right)d\omega$$
$$\left. - \frac{(R_2 - R_1)^3}{48}(m_1 + M_1)\int_{S^{N-1}} \left(1 + 3P_1^2(\omega)\right)d\omega \right|$$
$$\leq \frac{(M_1 - m_1)(R_2 - R_1)^3}{48}\int_{S^{N-1}} \left(1 - 3P_1^2(\omega)\right)d\omega.$$

Proof. Similar to the proof of Theorem 2.10 by using Theorem 1.5 and (1.16). □

We present

Theorem 2.14. *Consider* $f : \overline{A} \to \mathbb{R}$ *be Lebesgue integrable, which is not necessarily radial. Assume that* $f(s\omega)s^{N-1} \in AC^n([R_1, R_2])$ *(i.e.* $\left(f(s\omega)s^{N-1}\right)^{(n-1)} \in AC([R_1, R_2])$ *absolutely continuous functions), for all* $\omega \in S^{N-1}$, $N \geq 2$. *We assume that* $\left(f(s\omega)s^{N-1}\right)^{(n)} \in L_\infty([R_1, R_2])$, *for all* $\omega \in S^{N-1}$. *There exists* $K_1 > 0$ *such that* $\left\|\left(f(s\omega)s^{N-1}\right)^{(n)}\right\|_{L_\infty([R_1,R_2])} \leq K_1$, *where* $s \in [R_1, R_2]$, *for all* $\omega \in S^{N-1}$.
Then

(i)

$$\left| \int_A f(y)\,dy - \sum_{k=0}^{n-1} \frac{1}{(k+1)!}\left[\left(\int_{S^{N-1}} \left(f(s\omega)s^{N-1}\right)^{(k)}(R_1)\,d\omega\right)(t - R_1)^{k+1} \right.\right.$$
$$\left.\left. + (-1)^k \left(\int_{S^{N-1}} \left(f(s\omega)s^{N-1}\right)^{(k)}(R_2)\,d\omega\right)(R_2 - t)^{k+1}\right]\right|$$
$$\leq \frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\frac{K_1}{(n+1)!}\left[(t - R_1)^{n+1} + (R_2 - t)^{n+1}\right], \tag{2.17}$$

for all $t \in [R_1, R_2]$,

(ii) at $t = \frac{R_1 + R_2}{2}$, *the right hand side of (2.17) is minimized, and we get:*

$$\left| \int_A f(y)\,dy - \sum_{k=0}^{n-1} \frac{1}{(k+1)!}\frac{(R_2 - R_1)^{k+1}}{2^{k+1}}\left[\int_{S^{N-1}} \left(f(s\omega)s^{N-1}\right)^{(k)}(R_1)\,d\omega \right.\right.$$
$$\left.\left. + (-1)^k \int_{S^{N-1}} \left(f(s\omega)s^{N-1}\right)^{(k)}(R_2)\,d\omega\right]\right|$$
$$\leq \frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\frac{K_1}{(n+1)!}\frac{(R_2 - R_1)^{n+1}}{2^{n-1}},$$

(iii) if $\left(f(s\omega)s^{N-1}\right)^{(k)}(R_1) = \left(f(s\omega)s^{N-1}\right)^{(k)}(R_2) = 0$*, for all* $\omega \in S^{N-1}$*, (i.e.* $\frac{\partial^k\left(f(s\omega)s^{N-1}\right)}{\partial s^k}$ *vanish on* $\partial B(0,R_1)$ *and* $\partial B(0,R_2)$*) for all* $k = 0,1,...,n-1$*, we obtain*

$$\left|\int_A f(y)\,dy\right| \le \frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\frac{K_1}{(n+1)!}\frac{(R_2-R_1)^{n+1}}{2^{n-1}},$$

which is a sharp inequality,

(iv) more generally, for $j = 0,1,2,...,\overline{N} \in \mathbb{N}$*, it holds*

$$\left|\int_A f(y)\,dy - \sum_{k=0}^{n-1}\frac{1}{(k+1)!}\left(\frac{R_2-R_1}{\overline{N}}\right)^{k+1}\left[j^{k+1}\left(\int_{S^{N-1}}\left(f(s\omega)s^{N-1}\right)^{(k)}(R_1)\,d\omega\right)\right.\right.$$
$$\left.\left.+\quad(-1)^k\left(\overline{N}-j\right)^{k+1}\left(\int_{S^{N-1}}\left(f(s\omega)s^{N-1}\right)^{(k)}(R_2)\,d\omega\right)\right]\right|$$
$$\le\quad\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\frac{K_1}{(n+1)!}\left(\frac{R_2-R_1}{\overline{N}}\right)^{n+1}\left[j^{n+1}+\left(\overline{N}-j\right)^{n+1}\right], \tag{2.18}$$

(v) if $\left(f(s\omega)s^{N-1}\right)^{(k)}(R_1) = \left(f(s\omega)s^{N-1}\right)^{(k)}(R_2) = 0$*, for all* $\omega \in S^{N-1}$*, (i.e.* $\frac{\partial^k\left(f(s\omega)s^{N-1}\right)}{\partial s^k}$ *vanish on* $\partial B(0,R_1)$ *and* $\partial B(0,R_2)$*) for* $k = 1,...,n-1$*, from (2.18) we get:*

$$\left|\int_A f(y)\,dy - \left(\frac{R_2-R_1}{\overline{N}}\right)\left[jR_1^{N-1}\left(\int_{S^{N-1}}f(R_1\omega)\,d\omega\right)\right.\right.$$
$$\left.\left.+\quad\left(\overline{N}-j\right)R_2^{N-1}\left(\int_{S^{N-1}}f(R_2\omega)\,d\omega\right)\right]\right| \le \frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\cdot$$
$$\times\quad\frac{K_1}{(n+1)!}\left(\frac{R_2-R_1}{\overline{N}}\right)^{n+1}\left[j^{n+1}+\left(\overline{N}-j\right)^{n+1}\right], \tag{2.19}$$

for $j = 0,1,2,...,\overline{N} \in \mathbb{N}$*,*

(vi) when $\overline{N} = 2$ *and* $j = 1$*, (2.19) turns to*

$$\left|\int_A f(y)\,dy - \left(\frac{R_2-R_1}{2}\right)\left(R_1^{N-1}\int_{S^{N-1}}f(R_1\omega)\,d\omega + R_2^{N-1}\int_{S^{N-1}}f(R_2\omega)\,d\omega\right)\right|$$
$$\le\quad\frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\frac{K_1}{(n+1)!}\frac{(R_2-R_1)^{n+1}}{2^{n-1}}, \tag{2.20}$$

(vii) when $n = 1$ *(without any boundary conditions), we get from (2.20) that*

$$\left|\int_A f(y)\,dy - \left(\frac{R_2-R_1}{2}\right)\left(R_1^{N-1}\int_{S^{N-1}}f(R_1\omega)\,d\omega + R_2^{N-1}\int_{S^{N-1}}f(R_2\omega)\,d\omega\right)\right|$$
$$\le\quad\frac{\pi^{\frac{N}{2}}K_1}{2\Gamma\left(\frac{N}{2}\right)}(R_2-R_1)^2.$$

Proof. Similar to the proof of Theorem 2.9. We apply Theorem 1.6 along with (1.16). □

We continue with

Theorem 2.15. *Consider* $f : \overline{A} \to \mathbb{R}$ *be Lebesgue integrable, which is not necessarily radial. Assume that* $f(s\omega)s^{N-1} \in AC^n([R_1,R_2])$ *(i.e.* $\left(f(s\omega)s^{N-1}\right)^{(n-1)} \in AC([R_1,R_2])$ *absolutely continuous functions), for all* $\omega \in S^{N-1}$*,* $N \ge 2$*. Here there exists* $K_2 > 0$ *such that* $\left\|\left(f(s\omega)s^{N-1}\right)^{(n)}\right\|_{L_1([R_1,R_2])} \le K_2$*, where* $s \in [R_1,R_2]$*, for all* $\omega \in S^{N-1}$*.*

Then

(i)

$$\left| \int_A f(y)\,dy - \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \left[\left(\int_{S^{N-1}} \left(f(s\omega) s^{N-1} \right)^{(k)} (R_1)\,d\omega \right) (t-R_1)^{k+1} \right. \right. \tag{2.21}$$

$$\left. \left. + \quad (-1)^k \left(\int_{S^{N-1}} \left(f(s\omega) s^{N-1} \right)^{(k)} (R_2)\,d\omega \right) (R_2-t)^{k+1} \right] \right|$$

$$\leq \quad \frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \frac{K_2}{n!} \left[(t-R_1)^n + (R_2-t)^n \right],$$

for all $t \in [R_1, R_2]$,

(ii) at $t = \frac{R_1+R_2}{2}$, *the right hand side of (2.21) is minimized, and we get:*

$$\left| \int_A f(y)\,dy - \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \frac{(R_2-R_1)^{k+1}}{2^{k+1}} \left[\int_{S^{N-1}} \left(f(s\omega) s^{N-1} \right)^{(k)} (R_1)\,d\omega \right. \right.$$

$$\left. \left. + \quad (-1)^k \int_{S^{N-1}} \left(f(s\omega) s^{N-1} \right)^{(k)} (R_2)\,d\omega \right] \right|$$

$$\leq \quad \frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \frac{K_2}{n!} \frac{(R_2-R_1)^n}{2^{n-2}},$$

(iii) if $\left(f(s\omega) s^{N-1} \right)^{(k)} (R_1) = \left(f(s\omega) s^{N-1} \right)^{(k)} (R_2) = 0$, *for all* $\omega \in S^{N-1}$, *(i.e.* $\frac{\partial^k \left(f(s\omega) s^{N-1} \right)}{\partial s^k}$ *vanish on* $\partial B(0, R_1)$ *and* $\partial B(0, R_2)$*) for all* $k = 0, 1, ..., n-1$, *we obtain*

$$\left| \int_A f(y)\,dy \right| \leq \frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \frac{K_2}{n!} \frac{(R_2-R_1)^n}{2^{n-2}},$$

which is a sharp inequality,

(iv) more generally, for $j = 0, 1, 2, ..., \overline{N} \in \mathbb{N}$, *it holds*

$$\left| \int_A f(y)\,dy - \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \left(\frac{R_2-R_1}{\overline{N}} \right)^{k+1} \left[j^{k+1} \left(\int_{S^{N-1}} \left(f(s\omega) s^{N-1} \right)^{(k)} (R_1)\,d\omega \right) \right. \right. \tag{2.22}$$

$$\left. \left. + \quad (-1)^k \left(\overline{N} - j \right)^{k+1} \left(\int_{S^{N-1}} \left(f(s\omega) s^{N-1} \right)^{(k)} (R_2)\,d\omega \right) \right] \right|$$

$$\leq \quad \frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \frac{K_2}{n!} \left(\frac{R_2-R_1}{\overline{N}} \right)^n \left[j^n + \left(\overline{N} - j \right)^n \right],$$

(v) if $\left(f(s\omega) s^{N-1} \right)^{(k)} (R_1) = \left(f(s\omega) s^{N-1} \right)^{(k)} (R_2) = 0$, *for all* $\omega \in S^{N-1}$, *(i.e.* $\frac{\partial^k \left(f(s\omega) s^{N-1} \right)}{\partial s^k}$ *vanish on* $\partial B(0, R_1)$ *and* $\partial B(0, R_2)$*) for* $k = 1, ..., n-1$, *from (2.22) we get:*

$$\left| \int_A f(y)\,dy - \left(\frac{R_2-R_1}{\overline{N}} \right) \left[j R_1^{N-1} \left(\int_{S^{N-1}} f(R_1\omega)\,d\omega \right) \right. \right. \tag{2.23}$$

$$\left. \left. + \quad \left(\overline{N} - j \right) R_2^{N-1} \left(\int_{S^{N-1}} f(R_2\omega)\,d\omega \right) \right] \right| \leq \frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}$$

$$\times \quad \frac{K_2}{n!} \left(\frac{R_2-R_1}{\overline{N}} \right)^n \left[j^n + \left(\overline{N} - j \right)^n \right],$$

for $j = 0, 1, 2, ..., \overline{N} \in \mathbb{N}$, *(vi) when* $\overline{N}=2$ *and* $j=1$, (2.23) *turns to*

$$\left|\int_A f(y)\,dy-\left(\frac{R_2-R_1}{2}\right)\left(R_1^{N-1}\int_{S^{N-1}} f(R_1\omega)\,d\omega+R_2^{N-1}\int_{S^{N-1}} f(R_2\omega)\,d\omega\right)\right| \leq \frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\frac{K_2}{n!}\frac{(R_2-R_1)^n}{2^{n-2}}, \tag{2.24}$$

(vii) when $n=1$ *(without any boundary conditions), we get from* (2.24) *that*

$$\left|\int_A f(y)\,dy-\left(\frac{R_2-R_1}{2}\right)\left(R_1^{N-1}\int_{S^{N-1}} f(R_1\omega)\,d\omega+R_2^{N-1}\int_{S^{N-1}} f(R_2\omega)\,d\omega\right)\right| \leq \frac{2\pi^{\frac{N}{2}}K_2}{\Gamma\left(\frac{N}{2}\right)}(R_2-R_1).$$

Proof. Similar to the proof of Theorem 2.9. We apply Theorem 1.7 along with (1.16). □

We continue with

Theorem 2.16. *Let* $p,q>1:\frac{1}{p}+\frac{1}{q}>1$. *Consider* $f:\overline{A}\to\mathbb{R}$ *be Lebesgue integrable, which is not necessarily radial. Assume that* $f(s\omega)s^{N-1}\in AC^n([R_1,R_2])$ *(i.e.* $\left(f(s\omega)s^{N-1}\right)^{(n-1)}\in AC([R_1,R_2])$ *absolutely continuous functions), for all* $\omega\in S^{N-1}$, $N\geq 2$. *We assume that* $\left(f(s\omega)s^{N-1}\right)^{(n)}\in L_q([R_1,R_2])$, *for all* $\omega\in S^{N-1}$. *There exists* $K_3>0$ *such that* $\left\|\left(f(s\omega)s^{N-1}\right)^{(n)}\right\|_{L_q([R_1,R_2])}\leq K_3$, *where* $s\in[R_1,R_2]$, *for all* $\omega\in S^{N-1}$.
Then

(i)

$$\Bigg|\int_A f(y)\,dy-\sum_{k=0}^{n-1}\frac{1}{(k+1)!}\Bigg[\left(\int_{S^{N-1}}\left(f(s\omega)s^{N-1}\right)^{(k)}(R_1)\,d\omega\right)(t-R_1)^{k+1} + (-1)^k\left(\int_{S^{N-1}}\left(f(s\omega)s^{N-1}\right)^{(k)}(R_2)\,d\omega\right)(R_2-t)^{k+1}\Bigg]\Bigg| \leq \frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\frac{K_3}{(n-1)!\left(n+\frac{1}{p}\right)(p(n-1)+1)^{\frac{1}{p}}}\left[(t-R_1)^{n+\frac{1}{p}}+(R_2-t)^{n+\frac{1}{p}}\right], \tag{2.25}$$

for all $t\in[R_1,R_2]$,

(ii) at $t=\frac{R_1+R_2}{2}$, *the right hand side of* (2.25) *is minimized, and we get:*

$$\Bigg|\int_A f(y)\,dy-\sum_{k=0}^{n-1}\frac{1}{(k+1)!}\frac{(R_2-R_1)^{k+1}}{2^{k+1}} \times\left[\int_{S^{N-1}}\left(f(s\omega)s^{N-1}\right)^{(k)}(R_1)\,d\omega+(-1)^k\int_{S^{N-1}}\left(f(s\omega)s^{N-1}\right)^{(k)}(R_2)\,d\omega\right]\Bigg| \leq \frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\frac{K_3}{(n-1)!\left(n+\frac{1}{p}\right)(p(n-1)+1)^{\frac{1}{p}}}\frac{(R_2-R_1)^{n+\frac{1}{p}}}{2^{n-1-\frac{1}{q}}},$$

(iii) if $\left(f(s\omega)s^{N-1}\right)^{(k)}(R_1) = \left(f(s\omega)s^{N-1}\right)^{(k)}(R_2) = 0$, *for all* $\omega \in S^{N-1}$, *(i.e.* $\frac{\partial^k\left(f(s\omega)s^{N-1}\right)}{\partial s^k}$ *vanish on* $\partial B(0,R_1)$ *and* $\partial B(0,R_2)$*) for all* $k = 0,1,...,n-1$*, we obtain*

$$\left|\int_A f(y)\,dy\right| \leq \frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \frac{K_3}{(n-1)!\left(n+\frac{1}{p}\right)(p(n-1)+1)^{\frac{1}{p}}} \frac{(R_2-R_1)^{n+\frac{1}{p}}}{2^{n-1-\frac{1}{q}}},$$

which is a sharp inequality,

(iv) more generally, for $j = 0,1,2,...,\overline{N} \in \mathbb{N}$*, it holds*

$$\left|\int_A f(y)\,dy - \sum_{k=0}^{n-1} \frac{1}{(k+1)!}\left(\frac{R_2-R_1}{\overline{N}}\right)^{k+1} \left[j^{k+1}\left(\int_{S^{N-1}} \left(f(s\omega)s^{N-1}\right)^{(k)}(R_1)\,d\omega\right)\right.\right.$$
$$\left.\left.+ (-1)^k\left(\overline{N}-j\right)^{k+1}\left(\int_{S^{N-1}} \left(f(s\omega)s^{N-1}\right)^{(k)}(R_2)\,d\omega\right)\right]\right|$$
$$\leq \frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \frac{K_3}{(n-1)!\left(n+\frac{1}{p}\right)(p(n-1)+1)^{\frac{1}{p}}} \left(\frac{R_2-R_1}{\overline{N}}\right)^{n+\frac{1}{p}} \left[j^{n+\frac{1}{p}} + \left(\overline{N}-j\right)^{n+\frac{1}{p}}\right], \tag{2.26}$$

(v) if $\left(f(s\omega)s^{N-1}\right)^{(k)}(R_1) = \left(f(s\omega)s^{N-1}\right)^{(k)}(R_2) = 0$, *for all* $\omega \in S^{N-1}$, *(i.e.* $\frac{\partial^k\left(f(s\omega)s^{N-1}\right)}{\partial s^k}$ *vanish on* $\partial B(0,R_1)$ *and* $\partial B(0,R_2)$*) for* $k = 1,...,n-1$*, from (2.26) we get:*

$$\left|\int_A f(y)\,dy - \left(\frac{R_2-R_1}{\overline{N}}\right)\left[jR_1^{N-1}\left(\int_{S^{N-1}} f(R_1\omega)\,d\omega\right)\right.\right.$$
$$\left.\left.+ \left(\overline{N}-j\right)R_2^{N-1}\left(\int_{S^{N-1}} f(R_2\omega)\,d\omega\right)\right]\right| \leq \frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}$$
$$\times \frac{K_3}{(n-1)!\left(n+\frac{1}{p}\right)(p(n-1)+1)^{\frac{1}{p}}} \left(\frac{R_2-R_1}{\overline{N}}\right)^{n+\frac{1}{p}} \left[j^{n+\frac{1}{p}} + \left(\overline{N}-j\right)^{n+\frac{1}{p}}\right], \tag{2.27}$$

for $j = 0,1,2,...,\overline{N} \in \mathbb{N}$*,*

(vi) when $\overline{N} = 2$ *and* $j = 1$*, (2.27) turns to*

$$\left|\int_A f(y)\,dy - \left(\frac{R_2-R_1}{2}\right)\left(R_1^{N-1}\int_{S^{N-1}} f(R_1\omega)\,d\omega + R_2^{N-1}\int_{S^{N-1}} f(R_2\omega)\,d\omega\right)\right|$$
$$\leq \frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \frac{K_3}{(n-1)!\left(n+\frac{1}{p}\right)(p(n-1)+1)^{\frac{1}{p}}} \frac{(R_2-R_1)^{n+\frac{1}{p}}}{2^{n-1-\frac{1}{q}}}, \tag{2.28}$$

(vii) when $n = 1$ *(without any boundary conditions), we get from (2.28) that*

$$\left|\int_A f(y)\,dy - \left(\frac{R_2-R_1}{2}\right)\left(R_1^{N-1}\int_{S^{N-1}} f(R_1\omega)\,d\omega + R_2^{N-1}\int_{S^{N-1}} f(R_2\omega)\,d\omega\right)\right|$$
$$\leq \frac{2^{\frac{1}{q}}\pi^{\frac{N}{2}}K_3}{\Gamma\left(\frac{N}{2}\right)\left(1+\frac{1}{p}\right)}(R_2-R_1)^{1+\frac{1}{p}}.$$

Proof. Similar to the proof of Theorem 2.9. We apply Theorem 1.8 along with (1.16). □

We continue with results on the ball. We present

Theorem 2.17. *Consider* $f : \overline{B(0,R)} \to \mathbb{R}$ *be Lebesgue integrable, which is not necessarily radial. Assume that* $f(s\omega)s^{N-1} \in AC([0,R])$*, for all* $\omega \in S^{N-1}$*,* $N \geq 2$*. We further assume* *that* $\frac{\partial f(s\omega)s^{N-1}}{\partial s} \in L_\infty([0,R])$, *for all* $\omega \in S^{N-1}$. *Suppose there exists* $M_1 > 0$ *such that* $\left\|\frac{\partial f(s\omega)s^{N-1}}{\partial s}\right\|_{\infty,(s\in[0,R])} \leq M_1$, *for all* $\omega \in S^{N-1}$.

Then

(i)

$$\left|\int_{B(0,R)} f(y)\,dy - \left(\int_{S^{N-1}} f(R\omega)\,d\omega\right) R^{N-1}(R-t)\right| \leq \frac{\pi^{\frac{N}{2}} M_1}{\Gamma\left(\frac{N}{2}\right)}\left[t^2+(R-t)^2\right], \tag{2.29}$$

for all $t \in [0,R]$,

(ii) at $t=\frac{R}{2}$, *the right hand side of (2.29) is minimized, and we get:*

$$\left|\int_{B(0,R)} f(y)\,dy - \left(\int_{S^{N-1}} f(R\omega)\,d\omega\right)\frac{R^N}{2}\right| \leq \frac{\pi^{\frac{N}{2}} M_1 R^2}{2\Gamma\left(\frac{N}{2}\right)},$$

(iii) if $f(R\omega)=0$, *for all* $\omega \in S^{N-1}$, *(i.e.* $f(\cdot\omega)$ *vanishes on* $\partial B(0,R)$*), we obtain*

$$\left|\int_{B(0,R)} f(y)\,dy\right| \leq \frac{\pi^{\frac{N}{2}} M_1 R^2}{2\Gamma\left(\frac{N}{2}\right)},$$

which is a sharp inequality,

(iv) more generally, for $j=0,1,2,\ldots,\overline{N}\in\mathbb{N}$, *it holds*

$$\left|\int_{B\ (0,R)} f(y)\,dy - \frac{R^N}{\overline{N}}\left(\overline{N}-j\right)\int_{S^{N-1}} f(R\omega)\,d\omega\right| \leq \frac{\pi^{\frac{N}{2}} M_1}{\Gamma\left(\frac{N}{2}\right)}\left(\frac{R}{\overline{N}}\right)^2\left[j^2+\left(\overline{N}-j\right)^2\right], \tag{2.30}$$

(v) when $\overline{N}=2$ *and* $j=1$, *(2.30) turns to*

$$\left|\int_{B(0,R)} f(y)\,dy - \frac{R^N}{2}\int_{S^{N-1}} f(R\omega)\,d\omega\right| \leq \frac{\pi^{\frac{N}{2}} M_1 R^2}{2\Gamma\left(\frac{N}{2}\right)}.$$

Proof. Same as the proof of Theorem 2.14, just set there $R_1=0$ and $R_2=R$ and use (1.15). □

We continue with

Theorem 2.18. *Consider* $f:\overline{B(0,R)}\to\mathbb{R}$ *be Lebesgue integrable, which is not necessarily radial. Assume that* $f(s\omega)s^{N-1}\in AC([0,R])$, *for all* $\omega\in S^{N-1}$, $N\geq 2$. *Suppose there exists* $M_2>0$ *such that* $\left\|\frac{\partial f(s\omega)s^{N-1}}{\partial s}\right\|_{L_1([0,R])} \leq M_2$, *for all* $\omega \in S^{N-1}$.

Then

(i)

$$\left|\int_{B(0,R)} f(y)\,dy - \left(\int_{S^{N-1}} f(R\omega)\,d\omega\right) R^{N-1}(R-t)\right| \leq \frac{2\pi^{\frac{N}{2}} M_2 R}{\Gamma\left(\frac{N}{2}\right)}, \tag{2.31}$$

for all $t \in [0,R]$,

(ii) if $f(R\omega)=0$, *for all* $\omega \in S^{N-1}$, *(i.e.* $f(\cdot\omega)$ *vanishes on* $\partial B(0,R)$*) from (2.31), we obtain*

$$\left|\int_{B(0,R)} f(y)\,dy\right| \leq \frac{2\pi^{\frac{N}{2}} M_2 R}{\Gamma\left(\frac{N}{2}\right)},$$

which is a sharp inequality,

(iii) more generally, for $j = 0, 1, 2, ..., \overline{N} \in \mathbb{N}$, it holds

$$\left| \int_{B\ (0,R)} f(y)\, dy - \frac{R^N}{\overline{N}} \left(\overline{N} - j\right) \int_{S^{N-1}} f(R\omega)\, d\omega \right| \leq \frac{2\pi^{\frac{N}{2}} M_2 R}{\Gamma\left(\frac{N}{2}\right)}, \tag{2.32}$$

(iv) when $\overline{N} = 2$ and $j = 1$, (2.32) turns to

$$\left| \int_{B(0,R)} f(y)\, dy - \frac{R^N}{2} \int_{S^{N-1}} f(R\omega)\, d\omega \right| \leq \frac{2\pi^{\frac{N}{2}} M_2 R}{\Gamma\left(\frac{N}{2}\right)}.$$

Proof. Same as the proof of Theorem 2.15, just set there $R_1 = 0$ and $R_2 = R$ and use (1.15). ☐

We continue with

Theorem 2.19. *Let $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. Consider $f : \overline{B(0,R)} \to \mathbb{R}$ be Lebesgue integrable, which is not necessarily radial. Assume that $f(s\omega) s^{N-1} \in AC([0,R])$, for all $\omega \in S^{N-1}$, $N \geq 2$. We further assume that $\frac{\partial f(s\omega) s^{N-1}}{\partial s} \in L_q([0,R])$, for all $\omega \in S^{N-1}$. Suppose there exists $M_3 > 0$ such that $\left\| \frac{\partial f(s\omega) s^{N-1}}{\partial s} \right\|_{L_q([0,R])} \leq M_3$, for all $\omega \in S^{N-1}$.*
Then

(i)

$$\left| \int_{B(0,R)} f(y)\, dy - \left(\int_{S^{N-1}} f(R\omega)\, d\omega \right) R^{N-1} (R - t) \right| \tag{2.33}$$
$$\leq \frac{2\pi^{\frac{N}{2}} M_3}{\Gamma\left(\frac{N}{2}\right) \left(1 + \frac{1}{p}\right)} \left[t^{1+\frac{1}{p}} + (R - t)^{1+\frac{1}{p}} \right],$$

for all $t \in [0, R]$,

(ii) at $t = \frac{R}{2}$, the right hand side of (2.33) is minimized, and we get:

$$\left| \int_{B(0,R)} f(y)\, dy - \left(\int_{S^{N-1}} f(R\omega)\, d\omega \right) \frac{R^N}{2} \right| \leq \frac{2^{\frac{1}{q}} \pi^{\frac{N}{2}} M_3 R^{1+\frac{1}{p}}}{\Gamma\left(\frac{N}{2}\right)},$$

(iii) if $f(R\omega) = 0$, for all $\omega \in S^{N-1}$, (i.e. $f(\cdot\omega)$ vanishes on $\partial B(0,R)$), we obtain

$$\left| \int_{B(0,R)} f(y)\, dy \right| \leq \frac{2^{\frac{1}{q}} \pi^{\frac{N}{2}} M_3 R^{1+\frac{1}{p}}}{\Gamma\left(\frac{N}{2}\right)},$$

which is a sharp inequality,

(iv) more generally, for $j = 0, 1, 2, ..., \overline{N} \in \mathbb{N}$, it holds

$$\left| \int_{B\ (0,R)} f(y)\, dy - \frac{R^N}{\overline{N}} \left(\overline{N} - j\right) \int_{S^{N-1}} f(R\omega)\, d\omega \right| \tag{2.34}$$
$$\leq \frac{2\pi^{\frac{N}{2}} M_3}{\left(1 + \frac{1}{p}\right) \Gamma\left(\frac{N}{2}\right)} \left(\frac{R}{\overline{N}}\right)^{1+\frac{1}{p}} \left[j^{1+\frac{1}{p}} + \left(\overline{N} - j\right)^{1+\frac{1}{p}} \right],$$

(v) when $\overline{N} = 2$ and $j = 1$, (2.34) turns to

$$\left| \int_{B(0,R)} f(y)\, dy - \frac{R^N}{2} \int_{S^{N-1}} f(R\omega)\, d\omega \right| \leq \frac{2^{\frac{1}{q}} \pi^{\frac{N}{2}} M_3 R^{1+\frac{1}{p}}}{\left(1 + \frac{1}{p}\right) \Gamma\left(\frac{N}{2}\right)}.$$

Proof. Same as the proof of Theorem 2.16, just set there $R_1 = 0$ and $R_2 = R$ and use (1.15). $\square$

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