New Trends in Mathematical Sciences
http://dx.doi.org/10.20852/ntmsci.2017.204

On the algebraic properties of the univalent functions in class S

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Received: 15 August 2017, Accepted: 12 September 2017
Published online: 28 September 2017.

Abstract: This work is shown below, the algebraic sum of the two functions selected from class S of univalent functions which is a subclass of this class A of functions $f(z)$ satisfy the conditions analiytic in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$ normalized with $f(0) = 0$ and $f\prime(0) = 1$is not univalent.

Keywords: Algebraic sum, analitik functions, univalent functions.

1 Introduction

A single-valuable function f is saide to be univalent(or schlicht) in a domain $D \subset \mathbb{C}$ if it never takes the same value twice; that is ,if $f(z_1) \neq f(z_2)$ for all points z_1 and z_2 in D with $z_{1\neq}z_2$. The function f is said to be locally univalent at a point $z_0 \in D$ if it is univalent in some neighbornhood of z_0. For analytic functions f, the condition $f'(z_0) \neq 0$ is equivlent to local univalence at z_0. An analytic univalent function is called a conformal mapping because of its angle-preserving property.

We shall be concerned primarily with the class S of functions f analytic and univalent in the unit disk $D = \{z : |z| < 1\}$, normalized by the conditions $f(0) = 0$ and $f'(0) = 1$. Thus each $f \in S$ has a Taylor series expansion of the form

$$f(z) = z + a_2z^2 + a_3z^3 + a_4z^4 + ..., |z| < 1.$$

In view of the Riemann mapping theorem, most of the geometric theorems concerning functions of class S are readily translates to statements about univalent functions in arbitrary simply connectes domains with more than one boundary point.

Definition 1. *The leading example of a function of class S in the Koebe function*

$$k(z) = z(1-z)^{-2} = z + 2z^2 + 3z^3 + ...,$$

the Koebe function maps the disk D onto the entire plane minus the part of the negative real axis from $-\frac{1}{4}$ *to infinity.This is best seen by writing*

$$k(z) = \frac{1}{4}\left(\frac{1+z}{1-z}\right)^2 - \frac{1}{4}$$

and observing that the function

$$w = \frac{1+z}{1-z}$$

maps D conformally onto the right half-plane Re $w > 0$.

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Examples of functions in S;

(1) $f(z) = z$ the identity mapping,
(2) $f(z) = z(1-z)^{-1}$ which maps D conformally onto the half-plane $Re\ w > -\frac{1}{2}$;
(3) $f(z) = z(1-z)^{-1}$, which maps D onto the entire plane minus the two half-lines $\frac{1}{2} \leq x < \infty$ and $-\infty < x \leq -\frac{1}{2}$;
(4) $f(z) = \frac{1}{2}\log\left[\frac{(1+z)}{(1-z)}\right]$, which maps D onto the horizontal strip $-\frac{\pi}{4} < Im\ w < \frac{\pi}{4}$;
(5) $f(z) = z - \frac{1}{2}z^2 = \frac{1}{2}\left[1-(1-z)^2\right]$, which maps D onto the interior of cardioid.

Theorem 1. *(Rouche's Theorem) Let f and g be analytic inside and on a rectifiable Jordan Curve C, with $|g(z)| < |f(z)|$ on C. Then $(f+g)$ have same number of zeros, counted according to multiplicity, inside C.*

Proof. $\Delta_c \arg(f+g) = \Delta_c \arg f + \Delta_c g(1+g/f) = \Delta_c \arg f$. If a squence $\{f_n\}$ of functions analytic in domain D converges uniformly on each compact subset of D to a function f, then f is also analytic in D. This is easily proved with aid of Cauchy integral formula. Hurwitz's theorem establishes a close connection between the zeros of f and the zeros of the function f_n.

Theorem 2. *(Hurwitz's Theorem) Let f_n be analytic in a domain D, and suppose $f_n(z) \longrightarrow f(z)$ as $n \longrightarrow \infty$, uniformaly on each compact subset of D.Then either $f(z) \equiv 0$ in D, or every zero of f is a limit-point of a squence of zeroes of the function f_n.*

Proof. Suppose $f(z_0) = 0$ but $f(z) \neq 0$. It is enough to show that every neighborhood of z_0 contains a zero of some function f_n. Choose $\delta > 0$ so small that disk $|z - z_0| = \delta$. Let m be the minimum of $|f(z)|$ on C. Then for all $n \geq N$,

$$|f_n(z) - f(z)| < m \leq |f(z)|$$

on C. Thus by Rouche's theorem, f_n has the same number of zeroes as f does inside C. In other words, $f_n(z)$ must vanish at least once inside C whenever $n \geq N$.

A function f analytic in a domain D is said to be univalent there if it does not take the same value twice: $f(z_1) \neq f(z_2)$ for all pairs of distinct points z_1 and z_2 in D.

Theorem 3. *Let f_n be analytic and univalent in a domain D, and suppose $f_n(z) \longrightarrow f(z)$ as $n \longrightarrow \infty$, uniformaly on each compact subset of D. Then f is either univalent or constant in D.*

Proof. Suppose, on the contrary, that $f(z_1) = f(z_2) = \alpha$ for some pair of distinct point z_1 and z_2 in D.Then if $f(z) \neq \alpha$, that for $n \geq N$ the funct,on $f_n/(z) - \alpha$ vanishes in prescibed neighborhoods of both z_1 and z_2. This violetes the univlence of f_n so $f(z) \equiv \alpha$.

Alternatively; the theorem can be proved by direct appeal to Rouche's theorem. It should be remarked that the limit function can actually be constant. For example, let $f_n(z) = \frac{z}{n}$.

Theorem 4. *(Riemann Maping Theorem) Let D be a simply connected domain which is a proper subset of complex plane. Let ζ be a given point in D. Then there is a unique function f which maps D conformally onto the unit disk and has the properties $f(\zeta) = 0$ and $f'(\zeta) > 0$.*

Proof. The hypothesis that D not be whole plane is essential because of Liouville's theorem that every bounded entire function is constant. The uniqueness assertion is easily established. Indeed if g is another mapping with the given properties, the function $h = g \circ f^{-1}$ is a conformal mapping of the unit disk onto itself and is therefore linear fractional mapping of the form displayed. But $h(0) = 0$ and $h\prime(0) > 0$,so h is the identity. Thus $f = g$ and the mapping is unique. We now turn to the proof of existence. Consider the family $\mathscr{F}$ of all functions f analytic and univalent in D, ith $f(\zeta)0, f\prime(\zeta) > 0$ and $|f(z)| < 1$ for all $z \in D$. This is the family of all normalized conformal mappings of D into the unit disk . According to Montel's theorem, $\mathscr{F}$ is a normal family. To see that $\mathscr{F}$ is nonempty, choose a finite point $\alpha \notin D$ and consider the function $g(z) = (z-\alpha)^{1/2}$. Since D is simply connected, g has a single-valued branch.

This functiong is analytic and univalent in D, and $g(z_1) \neq -g(z_2)$ for all points z_1 and z_2 in D. Thus because g assumes all values in some disk $|w - g(\zeta)| \leq \varepsilon$ it must omit the entire disk $|w + g(\zeta)| \leq \varepsilon$. Let ψ be the linear fractional mapping of the region $|w + g(\zeta)| > \varepsilon$ onto the unit disk with $\psi(g(\zeta)) = 0$ and $\psi'(g(\zeta)) > 0$. Then $\psi \circ g \in \mathscr{F}$.

Now let $sup_{f\in\mathscr{F}} f'(\zeta) = M \leq \infty$, and choose a squence of functions $f_n \in \mathscr{F}$ for which $f\prime_n(\zeta) \to M$. Since $\mathscr{F}$ is a normal family, some subsequence converges uniformly on compact sets to an analytic function f which is either univalent or constant. The limit function has properties $f(\zeta) = 0$ and $f\prime(\zeta) = M > 0$. In particular, $M < \infty$ and f is not constant, so $f \in \mathscr{F}$.

The extremal function f is actually the required conformal mapping of D onto the unit disk. If not, then f omits some point w$\in$ D, some branch of

$$F(z) = \left\{\frac{f(z)-w}{1-\overline{w}f(z)}\right\}^{1/2}$$

is analytic and single-valued in D. Furthermore, F is univalent in D and $|F(z)| < 1$ there. The function

$$G(z) = e^{-i\theta} = \frac{F(z)-F(\zeta)}{1-\overline{F(\zeta)}F(z)},$$

where $e^{i\theta} = F'(\zeta)/|F'(\zeta)|$, therefore belongs to $\mathscr{F}$. However, a straightforward calculation gives and $soG\prime(\zeta) > f'(\zeta)$. This contradiction to the extremal property of f shows that f cannot omit any point in the unit disk. The proof is complete.

Theorem 5. *(Bieberbach's Theorem) If $f \in S$, then $|a_2| \leq 2$, with equality if and only if f is a rotation of the Koebe function.*

Proof. A square-root transformation and an inversion applied to f$\in$ S will produce a function

$$g(z) = \left\{f(1/z^2)\right\}^{-1/2} = z - (a_2/2)z^{-1} + \ldots$$

of class Σ.

Thus $|a_2| \leq 2$, by the corollary by the corollary to the area theorem. Equality occurs if and only if g has the form

$$g(z) = z - e^{i\theta}/z.$$

A simple calculation shows that this is equivalent to

$$f(\zeta) = \zeta(1-e^{i\theta}\zeta)^{-2} = e^{-i\theta}k(e^{i\theta}\zeta),$$

a rotation of Koebe function.

As a first application of Bieberbach's theorem, we shall now prove a famous covering theorem due to Koebe. Each function $f \in S$ is an open mapping with $f(0) = 0$, so its range contains some disk centered at the origin.

Theorem 6. *For each $f \in S$,*

$$\left|\frac{zf''(z)}{f'(z)} - \frac{2r^2}{1-r^2}\right| \leq \frac{4r}{1-r^2}, \quad |z| = r < 1. \tag{1}$$

Proof. Given $f \in S$, fix $\zeta \in D$ and perform a disk automorphism to construct

$$F(z) = \frac{f\left(\frac{z+\zeta}{1+\overline{\zeta}z}\right) - f(\zeta)}{(1-|\zeta|^2)f'(\zeta)} = z + A_2(\zeta)z^2 + \ldots \tag{2}$$

Then $F \in S$ and a calculation gives

$$A_2(\zeta) = \frac{1}{2}\left\{(1-|\zeta|^2)\frac{f''(\zeta)}{f'(\zeta)} - 2\overline{\zeta}\right\}.$$

But Bieberbach'stheorem, $|A_2(\zeta)| \leq 2$. Simplifying this inequality and replacing ζ by z, we obtain the inequality (1). A suitable rotation of the Koebe function shows that the estimate is sharp for each $z \in D$.

Theorem 7. *(Main Theorem)* $f(z)=\frac{1}{2}\left[z(1-z)^{-2}+z(1+z)^{-2}\right]$ *is the average of two functions in S but is not univalent*

Proof. If

$$g(z)=z(1-z)^{-2} \text{ and } h(z)=z(1+z)^{-2}$$

lets form the following sum

$$f(z)=\frac{1}{2}\left[g(z)+h(z)\right].$$

Firstly $g(z)=\frac{z}{(1-z)^{-2}}\in$ using Koebe function

$$z+\sum_{n=2}^{\infty}nz^{n}=z+2z^{2}+3z^{3}+4z^{4}+\ldots \tag{3}$$

$g(0)=0$, $g\prime(0)=1$ and $g(z)\in A$.

Let's see if g(z) function is univalent. If $z_1\neq z_2$ then, $g(z_1)-g(z_2)\neq 0$ in the event of g(z) function is univalent. If $z_1-z_2\neq 0$ then,

$$\begin{aligned}
g(z_1)-g(z_2)&=z_1+\sum_{n=2}^{\infty}nz_1^{n}-z_2-\sum_{n=2}^{\infty}nz_2^{n}\\
&=z_1-z_2+\sum_{n=2}^{\infty}nz_1^{n}-\sum_{n=2}^{\infty}nz_1^{n}\\
&=z_1-z_2+2z_1^2+3z_1^3+4z_1^4+\ldots-2z_2^2+3z_1^3-4z_2^4-\ldots\\
&=z_1-z_2+2z_1^2-2z_2^2+3z_1^3-3z_2^3+4z_1^4-4z_2^4+\ldots\\
&=z_1-z_2+2(z_1^2-z_2^2)+3(z_1^3-z_2^3)+\ldots\\
&=z_1-z_2+2(z_1-z_2)(z_1+z_2)+3(z_1-z_2)(z_1^2+z_1z_2+z_2^2)+\ldots\\
&=(z_1-z_2)\left[1+2(z_1+z_2)+3(z_1^2+z_1z_2+z_2^2)+\ldots\right]\neq 0.
\end{aligned}$$

So $g(z)$ function is univalent. Now let's find out where the image of the function g turns.

$$\begin{aligned}
&w=g(z)=\frac{z}{(1-z)^{-2}}=\frac{z}{z^2-2z+1}\Longrightarrow wz^2-2wz+w=z,\\
&wz^2-2wz+w-z=0\\
&wz^2-(2w+1)z+w=0.
\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}
z_{1,2}&=\frac{-b\pm\sqrt{\Delta}}{2a}=\frac{2w+1\pm\sqrt{4w+1}}{2w}\\
&\sqrt{4w+1}\rightarrow\sqrt{4(u+iv)+1}=\sqrt{4u+4iv+1}\\
&4u+4iv+1\geq 0\\
&u+iv+\frac{1}{4}\geq 0.
\end{aligned}$$

Then

$$u+\frac{1}{4}\geq 0\Longrightarrow u\geq-\frac{1}{4}$$

Thus, $g(z)$ function in unit disk is maps to Re $u \geq -\frac{1}{4}$ and $h(z) = \frac{z}{(1+z)^{-2}}$ using Binom expansion

$$\frac{1}{(1+z)^{-2}} = 1^{-2} + \frac{(-2)1^{-3}z}{1!} + \frac{(-2)(-3)1^{-4}z^2}{2!} + \frac{(-2)(-3)(-4)1^{-5}z^3}{3!} + \frac{(-2)(-3)(-4)(-5)1^{-6}z^4}{4!}$$
$$= 1 - 2z + 3z^2 - 4z^3 + 5z^4 + \ldots$$
$$z\frac{1}{(1+z)^{-2}} = z(1 - 2z + 3z^2 - 4z^3 + 5z^4 + \ldots) = z - 2z^2 + 3z^3 - 4z^4 + 5z^5 + \ldots = z + \sum_{n=2}^{\infty} (-1)^{n-1} nz^n, \quad (4)$$

$h(0) = 0$, $h'(0) = 1$, and $h(z) \in A$. Let's see if $h(z)$ function is univalent.

If $z_1 \neq z_2$ then $h(z_1) - h(z_2) \neq 0$ in the event of $h(z)$ function is univalent.

if $z_1 - z_2 \neq 0$ then

$$\begin{aligned} h(z_1) - h(z_2) &= z + \sum_{n=2}^{\infty} (-1)^{n-1} nz^n - z - \sum_{n=2}^{\infty} (-1)^{n-1} nz^n \\ &= z_1 - 2z_1^2 + 3z_1^3 - 4z_1^4 + 5z_1^5 + \ldots - z_2 + 2z_2^2 - 3z_2^3 + 4z_2^4 - 5z_2^5 + \ldots \\ &= z_1 - z_2 - 2z_1^2 + 2z_2^2 + 3z_1^3 + 3z_2^3 - 4z_1^4 + 4z_2^4 \ldots \\ &= z_1 - z_2 - 2(z_1^2 - z_2^2) - 3(z_1^3 + z_2^3) - 4(\ldots) \\ &= (z_1 - z_2)\left[1 - 2(z_1^2 - z_2^2) - 3(z_1^2 \ldots\right] \neq 0. \end{aligned}$$

So $h(z)$ function is univalent.

Now let's find out where the image of the function h turns.

$$w = h(z) = \frac{z}{(1+z)^{-2}} = \frac{z}{z^2 + 2z + 1} \Longrightarrow wz^2 + 2wz + w = z$$
$$wz^2 + (2w - 1)z + w = 0.$$

Thus

$$z_{1,2} = \frac{-b \pm \sqrt{\Delta}}{2a} = \frac{-2w + 1 \pm \sqrt{-4w+1}}{2w} \Longrightarrow \sqrt{-4w+1} = \sqrt{-4u - 4iv + 1} \Longrightarrow -4u - 4iv + 1 \geq 0$$
$$= -u - v + \frac{1}{4} \geq 0,$$

we look reel part $\Longrightarrow u \leq \frac{1}{4}$, Thus, h(z) function in unit disk is maps to Re $u \leq \frac{1}{4}$.

We proved that $g(z)$ and $h(z) \in A$ and function g, h is univalent then $g(z), h(z) \in S$.

Now we replace (3) and (4) in $f(z)$ function

$$f(z) = \frac{1}{2}[g(z) + h(z)] = \frac{1}{2}\left(2z + 6z^3 + 10z^5 + \ldots\right) = z + 3z^3 + 5z^5 + \ldots$$

Corollary. $f(z)$ is odd fonksiyon.

Now, let us take this statement

$$f(-z) = -z - 3z^3 - 5z^5 + \ldots \Longrightarrow -f(-z) = z + 3z^3 + 5z^5 + \ldots$$
$$f(z) = z + 3z^3 + 5z^5 + \ldots$$

So $-f(-z) = f(z)$.

If we take the derivatives of $f(z)$, then $f\prime(z) = 1 + 9z^2 + 25z^4 + \ldots$, $f(0) = 0$ and $f\prime(0) = 1$. This means that $f(z)$ is an analytic function and $f(z) \in A$. We try to prove function $f(z)$ is an univalent function,

$$f(z) = z + 3z^3 + 5z^5 + \ldots = z + \sum_{n=2}^{\infty}(2n+1)z^{(2n+1)}.$$

If $z_1 - z_2 \neq 0$ then $f(z_1) - f(z_2) \neq 0$ in the event of $f(z)$ function is univalent.

If $z_1 - z_2 \neq 0$ then

$$\begin{aligned} f(z_1) - f(z_2) &= z_1 + \sum_{n=2}^{\infty}(2n+1)z_1^{(2n+1)} - z_2 - \sum_{n=2}^{\infty}(2n+1)z_2^{(2n+1)} \\ &= z_1 - z_2 + \sum_{n=2}^{\infty}(2n+1)z_1^{(2n+1)} - \sum_{n=2}^{\infty}(2n+1)z_2^{(2n+1)} \\ &= z_1 - z_2 + 3z_1^3 + 5z_1^5 + \ldots - 3z_2^3 - 5z_2^5 - \ldots \\ &= z_1 - z_2 + 3z_1^3 - 3z_2^3 + 5z_1^5 - 5z_2^5 + \ldots \\ &= z_1 - z_2 + 3(z_1^3 - z_2^3) + 5(z_1^5 - z_2^5) + \ldots \\ &= (z_1 - z_2)\left[1 + 3(z_1^3 + z_1z_2 + z_2^2) + 5\left[(z_1 - z_2)^4 - 5z_1z_2(z_1 + z_2) - 10z_1z_2\ldots\right]\right] + \ldots \end{aligned}$$

$(z_1 - z_2) \neq 0$ and the equation $\left[1 + 3(z_1^3 + z_1z_2 + z_2^2) + 5\left[(z_1 - z_2)^4 - 5z_1z_2(z_1 + z_2) - 10z_1z_2\ldots\right]\right] + \ldots$ may not always be zero. This is not always the case $f(z_1) - f(z_2) \neq 0$. So $f(z)$ is not univalent function. This completes the proof.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

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