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Right circulant matrices with generalized Fibonacci and Lucas polynomials and coding theory

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Abstract

In the present paper, we consider two new coding algorithms by means of right circulant matrices with elements generalized Fibonacci and Lucas polynomials. To that end, we study basic properties of right circulant matrices using generalized Fibonacci polynomials $F_{p,q,n}(x)$*, generalized Lucas polynomials* $L_{p,q,n}(x)$ *and geometric sequences.*

Keywords: *Coding/decoding method, right circulant matrix, generalized Fibonacci polynomials, generalized Lucas polynomials.*

Genelleştirilmiş Fibonacci ve Lucas polinomlarıyla birlikte sağ devirsel matrisler ve kodlama teorisi

Özet

Bu çalışmada elemanları genelleştirilmiş Fibonacci ve Lucas polinomları olan devirsel matrisler kullanılarak iki yeni kodlama algoritması vereceğiz. Bu amaçla, genelleşirilmiş Fibonacci polinomları $F_{p,q,n}(x)$*, genelleştirilmiş Lucas polimomları* $L_{p,q,n}(x)$ *ve geometrik diziler kullanılarak sağ devirsel matrislerin bazı temel özelliklerini çalışacağız.*

Anahtar Kelimeler: *Kodlama/kod çözme metodu, genelleştirilmiş Fibonacci polinomları, genelleştirilmiş Lucas polinomları, sağ devirsel matris.*

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1 Introduction

There are many studies including Fibonacci, Fibonacci quaternion, Lucas, Pell, Pell *(p,i)*-numbers and their applications such as coding theory in the literature [1-7]. Here we consider two classes of right circulant matrices whose entries are generalized Fibonacci and Lucas polynomials to obtain new coding/decoding algorithms.

Let $n, g > 0$ be integers. From [8], we know that a g-circulant matrix with order n is a square matrix is of the form:

$$B_{g,n} = \begin{pmatrix} b_1 & b_2 & \cdots & b_n \\ b_{n-g+1} & b_{n-g+2} & \cdots & b_{n-g} \\ b_{n-2g+1} & b_{n-2g+2} & & b_{n-2g} \\ \vdots & \vdots & \ddots & \vdots \\ b_{g+1} & b_{g+2} & \cdots & b_g \end{pmatrix},$$

where each subscript is thought to be reduced modulo n. The first row of $B_{g,n}$ is $(b_1, b_2, ..., b_n)$ and its $(j+1)$ th row is gained by giving its j th row a right circulant shift by g positions.

It is clear that $g = 1$ or $g = n+1$ give the standart right circulant matrix, or easily, circulant matrix. Then a right circulant matrix can be given by

$$RCirc(b_1, b_2, ..., b_n) = \begin{pmatrix} b_1 & b_2 & \cdots & b_n \\ b_n & b_1 & \cdots & b_{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ b_2 & b_3 & \cdots & b_1 \end{pmatrix}.$$

A geometric sequence is known to be a sequence $\{a_k\}_{k=1}^{\infty}$ such that each term is given by a multiple of the previous one.

In [8], it was given a g-circulant, right circulant and left circulant matrices whose entries are $h(x)$-Fibonacci polynomials and presented the determinants of these matrices. In [9], it was introduced the right circulant matrices with ratio of the elements of Fibonacci and a geometric sequence and given eigenvalues, determinants, Euclidean norms and inverses of these matrices.

In [10], it has been dealt with circulant matrices with the Jacobsthal and Jacobsthal-Lucas numbers, studied the invertibility of these circulant matrices and presented the determinant and the inverse matrix. Similarly, in [11], it has been studied inverses and determinants of the circulant matrices related to Fibonacci and Lucas numbers. Furthermore, there are many applications of circulant matrices in the literature. For example, these matrices has been studied related to models and several differential equations such as fractional order models for nonlocal epidemics, differential delay equations (for more details one can see [12-16] and the references therein).

Recently, $h(x)$-Fibonacci polynomials are given by $F_{h,0}(x)=0$, $F_{h,1}(x)=1$ and $F_{h,n+1}(x)=h(x)F_{h,n}(x)+F_{h,n-1}(x)$ for $n\geq 1$, and then it was given some properties of them in [17].

Let $p(x)$ and $q(x)$ be polynomials with real coefficients, $p(x)\neq 0, q(x)\neq 0$ and $p^2(x)+4q(x)>0.$ In [18], it was defined generalized Fibonacci polynomials $F_{p,q,n}(x)$ by

$$F_{p,q,n+1}(x)=p(x)F_{p,q,n}(x)+q(x)F_{p,q,n-1}(x),\ n\geq 1 \tag{1}$$

with the initial values $F_{p,q,0}(x)=0, F_{p,q,1}(x)=1$ and generalized Lucas polynomials $L_{p,q,n}(x)$ were given by

$$L_{p,q,n+1}(x)=p(x)L_{p,q,n}(x)+q(x)L_{p,q,n-1}(x),\ n\geq 1 \tag{2}$$

with the initial values $L_{p,q,0}(x)=2, L_{p,q,1}(x)=p(x)$.
It is known that the polynomial $F_{p,q,n}(x)$ generalizes classical Fibonacci numbers, k-Fibonacci numbers, generalized Fibonacci numbers, Catalan's Fibonacci polynomials (for more details see [19-24]). Similarly, the polynomial $L_{p,q,n}(x)$ comprises generalized Lucas numbers, k-Lucas numbers, classical Lucas numbers (for more details see [22-24] and [27-28]).
Considering the recurrence relation (1), let $\alpha(x)$ and $\beta(x)$ be the roots of the following characteristic equation

$$v^2-vp(x)-q(x)=0.$$

It is known that the Binet formulas for the generalized Fibonacci polynomials $F_{p,q,n}(x)$ and generalized Lucas polynomials $L_{p,q,n}(x)$ are of the following forms, respectively [18]:

$$F_{p,q,n}(x)=\frac{\alpha^n(x)-\beta^n(x)}{\alpha(x)-\beta(x)}, \text{for } n\geq 0$$

and

$$L_{p,q,n}(x)=\alpha^n(x)+\beta^n(x), \text{for } n\geq 0,$$

where

$$\alpha(x)=\frac{p(x)+\sqrt{p^2(x)+4q(x)}}{2} \text{ and } \beta(x)=\frac{p(x)-\sqrt{p^2(x)+4q(x)}}{2}.$$

It is clear that $\alpha(x)+\beta(x)=p(x)$, $\alpha(x)\beta(x)=-q(x)$ and $\alpha(x)-\beta(x)=\sqrt{p^2(x)+4q(x)}$.

In this study, we investigate right circulant matrices using generalized Fibonacci polynomials, generalized Lucas polynomials and geometric sequences. In Section 2, we give the eigenvalues and determinants of the right circulant matrices whose entries are the ratio of the elements of generalized Fibonacci sequence and some geometric sequences. In Section 3, we give right circulant matrices whose entries are the generalized Fibonacci and Lucas polynomials and calculate the determinants of these matrices. In Section 4, we give some applications of right circulant matrices to coding theory.

From now on, we shortly denote $\alpha(x)$ by α, $\beta(x)$ by β, $p(x)$ by p and $q(x)$ by q.

2 Right circulant matrices with generalized Fibonacci polynomials and geometric sequences

Let $\{f_n\}_{n=1}^{\infty}$ be the sequence of the form

$$f_n = \frac{F_{p,q,n}(x)}{ar^n},$$

where $F_{p,q,n}(x)$ is the n-th generalized Fibonacci polynomial and ar^n is the n-th element of any geometric sequence.

Using these types of sequences, we consider the following right circulant matrix $\mathcal{F}_n$:

$$\mathcal{F}_n = \begin{pmatrix} f_0 & f_1 & f_2 & \cdots & f_{n-2} & f_{n-1} \\ f_{n-1} & f_0 & f_1 & \cdots & f_{n-3} & f_{n-2} \\ f_{n-2} & f_{n-1} & f_0 & \cdots & f_{n-4} & f_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ f_2 & f_3 & f_4 & \cdots & f_0 & f_1 \\ f_1 & f_2 & f_3 & \cdots & f_{n-1} & f_0 \end{pmatrix}.$$

Theorem 2.1 The eigenvalues of the matrix $\mathcal{F}_n$ are as follows:

$$\lambda_m = \frac{-rF_{p,q,n}(x) - w^{-m}\left(qF_{p,q,n-1}(x) - r^n\right)}{ar^{n-1}\left(r-\alpha w^{-m}\right)\left(r-\beta w^{-m}\right)},$$

where $m = 0,1,\ldots,n-1$, $\alpha = \dfrac{p+\sqrt{p^2+4q}}{2}$, $\beta = \dfrac{p-\sqrt{p^2+4q}}{2}$ and $w = e^{\frac{2\pi i}{n}}$.

Proof. From [25], we know that the eigenvalues of a right circulant matrix $\mathcal{F}_n$ are

$$\lambda_m = \sum_{k=0}^{n-1} f_k w^{-mk}, \tag{3}$$

where $m = e^{\frac{2\pi i}{n}}$ and $m = 0,1,2,...,n-1$. Using the equation (3) and the Binet's formula for the generalized Fibonacci polynomials $F_{p,q,n}(x)$, we get

$$\lambda_m = \sum_{k=0}^{n-1} \frac{F_{p,q,k}(x)}{ar^k} w^{-mk} = \sum_{k=0}^{n-1} \frac{\alpha^k - \beta^k}{(\alpha-\beta) ar^k} w^{-mk},$$

where $\alpha = \frac{p+\sqrt{p^2+4q}}{2}, \beta = \frac{p-\sqrt{p^2+4q}}{2}$. Then using the properties of the geometric series, we obtain

$$\begin{aligned}
\lambda_m &= \frac{1}{a(\alpha-\beta)}\left(\frac{1-(\alpha/r)^n}{1-\alpha w^{-m}/r} - \frac{1-(\beta/r)^n}{1-\beta w^{-m}/r}\right) \\
&= \frac{1}{ar^{n-1}(\alpha-\beta)}\left(\frac{r^n-\alpha^n}{r-\alpha w^{-m}} - \frac{r^n-\beta^n}{r-\beta w^{-m}}\right) \\
&= \frac{1}{ar^{n-1}(\alpha-\beta)}\left(\frac{(r^n-\alpha^n)(r-\beta w^{-m})-(r-\alpha w^{-m})(r^n-\beta^n)}{(r-\alpha w^{-m})(r-\beta w^{-m})}\right).
\end{aligned}$$

By the fact $\alpha\beta = -q$, we find

$$\begin{aligned}
\lambda_m &= \frac{-r(\alpha^n-\beta^n)+r^n w^{-m}(\alpha-\beta)-w^{-m}(-\beta\alpha^n+\alpha\beta^n)}{ar^{n-1}(\alpha-\beta)(r-\alpha w^{-m})(r-\beta w^{-m})} \\
&= \frac{-r(\alpha^n-\beta^n)+r^n w^{-m}(\alpha-\beta)-w^{-m}q(\alpha^{n-1}-\beta^{n-1})}{ar^{n-1}(\alpha-\beta)(r-\alpha w^{-m})(r-\beta w^{-m})} \\
&= \frac{-rF_{p,q,n}(x)+r^n w^{-m}F_{p,q,1}(x)-w^{-m}qF_{p,q,n-1}(x)}{ar^{n-1}(r-\alpha w^{-m})(r-\beta w^{-m})}
\end{aligned}$$

and so

$$\lambda_m = \frac{-rF_{p,q,n}(x)-w^{-m}(qF_{p,q,n-1}(x)-r^n)}{ar^{n-1}(r-\alpha w^{-m})(r-\beta w^{-m})}.$$

Now we present the below theorem.

Theorem 2.2 The determinant of the matrix $\mathcal{F}_n$ is

$$\det(\mathcal{F}_n)=\frac{(-1)^n r^n F_{p,q,n}^n(x)-\left(qF_{p,q,n-1}(x)-r^n\right)^n}{a^n r^{n(n-1)}-\left(r^{2n}-r^n L_{p,q,n}(x)+(-q)^n\right)}.$$

Proof. Because the product of eigenvalues of a matrix gives its determinant, from Theorem 2.1 we obtain

$$\det(\mathcal{F}_n)=\prod_{m=0}^{n-1}\frac{-rF_{p,q,n}(x)-w^{-m}\left(qF_{p,q,n-1}(x)-r^n\right)}{ar^{n-1}\left(r-\alpha w^{-m}\right)\left(r-\beta w^{-m}\right)}. \tag{4}$$

From the complex analysis, we know

$$\prod_{m=0}^{n-1}\left(x-yw^{-m}\right)=x^n-y^n \tag{5}$$

Applying the equation (5) to the equation (4), we have

$$\begin{aligned}\det(\mathcal{F}_n)&=\frac{(-1)^n r^n F_{p,q,n}^n(x)-\left(qF_{p,q,n-1}(x)-r^n\right)^n}{a^n r^{n(n-1)}\left(r^n-\alpha^n\right)\left(r^n-\beta^n\right)}\\&=\frac{(-1)^n r^n F_{p,q,n}^n(x)-\left(qF_{p,q,n-1}(x)-r^n\right)^n}{a^n r^{n(n-1)}\left(r^{2n}-r^n\left(\alpha^n+\beta^n\right)+(-q)^n\right)}.\end{aligned}$$

Using the Binet's formulas for the generalized Lucas polynomials $L_{p,q,n}(x)$, we get

$$\det(\mathcal{F}_n)=\frac{(-1)^n r^n F_{p,q,n}^n(x)-\left(qF_{p,q,n-1}(x)-r^n\right)^n}{a^n r^{n(n-1)}\left(r^{2n}-r^n L_{p,q,n}(x)+(-q)^n\right)}.$$

Notice that for $p=x$ and $q=1$; for $p=2x$ and $q=1$; $p=k$ and $q=t$; $p=k$ and $q=1$; $p=q=1$ in Theorem 2.1 and Theorem 2.2, we have similar theorems for Catalan's Fibonacci polynomials $F_n(x)$, Byrd's polynomials $\varphi_n(x)$, generalized Fibonacci numbers U_n, k-Fibonacci numbers $F_{k,n}$, classical Fibonacci numbers F_n, respectively. Also, in [26], the right circulant matrix with Fibonacci sequence was defined and given eigenvalues, Euclidean norm of this matrix.

3 Right circulant matrices with generalized Fibonacci and Lucas polynomials

In this part, we give the determinant of a right circulant matrix whose elements are generalized Fibonacci polynomials $F_{p,q,n}(x)$ and generalized Lucas polynomials $L_{p,q,n}(x)$.

Theorem 3.1 Let G_n be a right circulant matrix of the following form:

$$G_n = \begin{pmatrix} F_{p,q,1}(x) & F_{p,q,2}(x) & F_{p,q,3}(x) & \cdots & F_{p,q,n-1}(x) & F_{p,q,n}(x) \\ F_{p,q,n}(x) & F_{p,q,1}(x) & F_{p,q,2}(x) & \cdots & F_{p,q,n-2}(x) & F_{p,q,n-1}(x) \\ F_{p,q,n-1}(x) & F_{p,q,n}(x) & F_{p,q,1}(x) & \cdots & F_{p,q,n-3}(x) & F_{p,q,n-2}(x) \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ F_{p,q,4}(x) & F_{p,q,5}(x) & F_{p,q,6}(x) & \cdots & F_{p,q,2}(x) & F_{p,q,3}(x) \\ F_{p,q,3}(x) & F_{p,q,4}(x) & F_{p,q,5}(x) & \cdots & F_{p,q,1}(x) & F_{p,q,2}(x) \\ F_{p,q,2}(x) & F_{p,q,3}(x) & F_{p,q,4}(x) & \cdots & F_{p,q,n}(x) & F_{p,q,1}(x) \end{pmatrix}. \tag{6}$$

Then we have

$$\det(G_n) = \left(1 - F_{p,q,n+1}(x)\right)^{n-1} + \left(qF_{p,q,n}(x)\right)^{n-2} \sum_{k=1}^{n-1} \left(\frac{1 - F_{p,q,n+1}(x)}{qF_{p,q,n}(x)} \right)^{k-1} qF_{p,q,k}(x). \tag{7}$$

Proof. For $n = 1$, $\det(G_1) = 1$ satisfies the equation (7). Let us consider the case $n \geq 2$ and we focus on the following matrices:

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ -p & 0 & 0 & 0 & \cdots & 0 & 0 & 1 \\ -q & 0 & 0 & 0 & \cdots & 0 & 1 & -p \\ \vdots & \vdots & \vdots & & \ddots & \vdots & \vdots & \\ 0 & 0 & 0 & 1 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 1 & -p & \cdots & 0 & 0 & 0 \\ 0 & 1 & -p & -q & \cdots & 0 & 0 & 0 \end{pmatrix}_{n \times n} \tag{8}$$

and

$$B = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & \left(\frac{qF_{p,q,n}(x)}{1 - F_{p,q,n+1}(x)} \right)^{n-2} & 0 & \cdots & 0 & 1 \\ 0 & \left(\frac{qF_{p,q,n}(x)}{1 - F_{p,q,n+1}(x)} \right)^{n-3} & 0 & \cdots & 1 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \\ 0 & \left(\frac{qF_{p,q,n}(x)}{1 - F_{p,q,n+1}(x)} \right) & 1 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \end{pmatrix}_{n \times n}.$$

Notice that

$$\det(A)=\det(B)=(-1)^{\frac{(n-1)(n-2)}{2}}.$$

If we multiply the matrices A, G_n and B we have the following product matrices:

$$AG_nB=\begin{pmatrix} F_{p,q,1}(x) & \alpha_n & F_{p,q,n-1}(x) & \cdots & F_{p,q,3}(x) & F_{p,q,2}(x) \\ 0 & \beta_n & F_{p,q,n-2}(x) & \cdots & qF_{p,q,2}(x) & qF_{p,q,1}(x) \\ 0 & 0 & F_{p,q,1}(x)-F_{p,q,n+1}(x) & \cdots & & \\ 0 & 0 & -qF_{p,q,n}(x) & & & \\ \vdots & \vdots & & \ddots & & \\ 0 & 0 & & & \ddots & \\ 0 & 0 & 0 & & -qF_{p,q,n}(x) & F_{p,q,1}(x)-F_{p,q,n+1}(x) \end{pmatrix},$$

where

$$\alpha_{n,p,q}=\sum_{k=1}^{n-1}\left(\frac{qF_{p,q,n}(x)}{F_{p,q,1}(x)-F_{p,q,n+1}(x)}\right)^{n-(k+1)}F_{p,q,k+1}(x) \tag{9}$$

and

$$\beta_{n,p,q}=\left(1-F_{p,q,n+1}(x)\right)+\sum_{k=1}^{n-1}\left(\frac{qF_{p,q,n}(x)}{F_{p,q,1}(x)-F_{p,q,n+1}(x)}\right)^{n-(k+1)}qF_{p,q,k}(x). \tag{10}$$

Then we have

$$\det(AG_nB)=F_{p,q,1}(x)\beta_{n,p,q}\left(F_{p,q,1}(x)-F_{p,q,n+1}(x)\right)^{n-2}.$$

Using the equation (10), we get

$$\det(AG_nB)=\left(1-F_{p,q,n+1}(x)\right)^{n-1}+\left(qF_{p,q,n}(x)\right)^{n-2}\sum_{k=1}^{n-1}\left(\frac{1-F_{p,q,n+1}(x)}{qF_{p,q,n}(x)}\right)^{k-1}qF_{p,q,k}(x).$$

Since $\det(AG_nB)=\det(G_n)$, we find

$$\det(G_n)=\left(1-F_{p,q,n+1}(x)\right)^{n-1}+\left(qF_{p,q,n}(x)\right)^{n-2}\sum_{k=1}^{n-1}\left(\frac{1-F_{p,q,n+1}(x)}{qF_{p,q,n}(x)}\right)^{k-1}qF_{p,q,k}(x).$$

Now we give the following theorem for generalized Lucas polynomials $L_{p,q,n}(x)$.

Theorem 3.2 Let H_n be a right circulant matrix of the following form:

$$H_n = \begin{pmatrix} L_{p,q,1}(x) & L_{p,q,2}(x) & L_{p,q,3}(x) & \cdots & L_{p,q,n-1}(x) & L_{p,q,n}(x) \\ L_{p,q,n}(x) & L_{p,q,1}(x) & L_{p,q,2}(x) & \cdots & L_{p,q,n-2}(x) & L_{p,q,n-1}(x) \\ L_{p,q,n-1}(x) & L_{p,q,n}(x) & L_{p,q,1}(x) & \cdots & L_{p,q,n-3}(x) & L_{p,q,n-2}(x) \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ L_{p,q,4}(x) & L_{p,q,5}(x) & L_{p,q,6}(x) & \cdots & L_{p,q,2}(x) & L_{p,q,3}(x) \\ L_{p,q,3}(x) & L_{p,q,4}(x) & L_{p,q,5}(x) & \cdots & L_{p,q,1}(x) & L_{p,q,2}(x) \\ L_{p,q,2}(x) & L_{p,q,3}(x) & L_{p,q,4}(x) & \cdots & L_{p,q,n}(x) & L_{p,q,1}(x) \end{pmatrix}. \tag{11}$$

Then we have

$$\begin{aligned} \det(H_n) &= L_{p,q,1}(x)\left(L_{p,q,1}(x) - L_{p,q,n+1}(x)\right)^{n-1} \\ &+ L_{p,q,1}(x) q^{n-1}\left(L_{p,q,n}(x) - 2\right)^{n-2} \sum_{k=1}^{n-1} \left(\frac{L_{p,q,1}(x) - L_{p,q,n+1}(x)}{qL_{p,q,n}(x) - 2q} \right)^{k-1} L_{p,q,k}(x) \\ &- 2q^{n-1}\left(L_{p,q,n}(x) - 2\right)^{n-2} \sum_{k=1}^{n-1} \left(\frac{L_{p,q,1}(x) - L_{p,q,n+1}(x)}{qL_{p,q,n}(x) - 2q} \right)^{k-1} L_{p,q,k+1}(x). \end{aligned} \tag{12}$$

Proof. For $n=1$, $\det(H_1) = p$ satisfies the equation (12). Let us consider the case $n \geq 2$. Let A be a matrix of the form given in (8) and D be a matrix of the following form:

$$D = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & \left(\frac{qL_{p,q,n}(x) - 2q}{L_{p,q,1}(x) - L_{p,q,n+1}(x)} \right)^{n-2} & 0 & \cdots & 0 & 1 \\ 0 & \left(\frac{qL_{p,q,n}(x) - 2q}{L_{p,q,1}(x) - L_{p,q,n+1}(x)} \right)^{n-3} & 0 & \cdots & 1 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \\ 0 & \left(\frac{qL_{p,q,n}(x) - 2q}{L_{p,q,1}(x) - L_{p,q,n+1}(x)} \right) & 1 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \end{pmatrix}_{n \times n}.$$

Using the properties of determinants and multiplying these matrices A, H_n and D, the proof can be fulfilled by an analogous way applied in the proof of the above theorem.

4 An Application to coding theory

In this section, we give two new coding/decoding methods using the right circulant matrices G_n and H_n for $p = q = 1$. At first, we give an algorithm by means of the generalized Fibonacci polynomials. Following the notations in [5], we give generalized Fibonacci and Lucas blocking algorithms with transformations

$$M \times G_n = E, M \times H_n = E$$

and

$$E \times (G_n)^{-1} = M, E \times (H_n)^{-1} = M,$$

where M is nonsingular square message matrix, E is a code matrix, G_n is coding matrix and the inverse matrix $(G_n)^{-1}$ is decoding matrix.

We put our message in a matrix adding zero between two words and end of the message until we obtain the size of the message matrix is $3m$. Dividing the message square matrix M into the block matrices, named B_i ($1 \le i \le m^2$), of size 3×3, from left to right, we can construct a new coding method.

Now we explain the symbols of our coding method. Suppose that matrices B_i and E_i are of the following forms:

$$B_i = \begin{bmatrix} b_1^i & b_2^i & b_3^i \\ b_4^i & b_5^i & b_6^i \\ b_7^i & b_8^i & b_9^i \end{bmatrix} \text{ and } E_i = \begin{bmatrix} e_1^i & e_2^i & e_3^i \\ e_4^i & e_5^i & e_6^i \\ e_7^i & e_8^i & e_9^i \end{bmatrix}.$$

We use the matrix G_n given in (6) for $p = q = 1$ and rewrite the elements of this matrix as $G_n = \begin{bmatrix} g_1 & g_2 & g_3 \\ g_3 & g_1 & g_2 \\ g_2 & g_3 & g_1 \end{bmatrix}$. The number of the block matrices B_i is denoted by b. In accordance with b, we choose the number n as follows:

$$n = \begin{cases} 3 & , \quad b = 1 \\ 3b & , \quad b \neq 1 \end{cases}.$$

Using the chosen n, we write the following character table according to $mod\,27$ (this table can be enlarged according to the used characters in the message matrix). We begin the "n" for the first character.

A	B	C	D	E	F	G	H	I
n	n+1	n+2	n+3	n+4	n+5	n+6	n+7	n+8
J	K	L	M	N	O	P	Q	R
n+9	n+10	n+11	n+12	n+13	n+14	n+15	n+16	n+17
S	T	U	V	W	X	Y	Z	0
n+18	n+19	n+20	n+21	n+22	n+23	n+24	n+25	n+26

(13)

Generalized Fibonacci Blocking Algorithm
Coding Algorithm

Step 1. Divide the matrix M into blocks B_i $\left(1 \le i \le m^2\right)$.

Step 2. Choose n.

Step 3. Determine b_j^i $\left(1 \le j \le 9\right)$.

Step 4. Compute $\det(B_i) \to d_i$.

Step 5. Construct $K = \left[d_i, b_k^i\right]_{k \in \{1,2,3,4,6,7,8,9\}}$.

Step 6. End of algorithm.

Decoding Algorithm

Step 1. Compute G_n.

Step 2. Determine g_j $(1 \le j \le 3)$.

Step 3. Compute $g_1 b_1^i + g_3 b_2^i + g_2 b_3^i \to e_1^i,$ $\left(1 \le i \le m^2\right)$.

$$g_2 b_1^i + g_1 b_2^i + g_3 b_3^i \to e_2^i,$$
$$g_3 b_1^i + g_2 b_2^i + g_1 b_3^i \to e_3^i,$$
$$g_1 b_7^i + g_3 b_8^i + g_2 b_9^i \to e_7^i,$$
$$g_2 b_7^i + g_1 b_8^i + g_3 b_9^i \to e_8^i,$$
$$g_3 b_7^i + g_2 b_8^i + g_1 b_9^i \to e_9^i.$$

Step 4. Solve

$$\begin{aligned}\det\left(G_3\right) \times d_i = {} & e_1^i e_9^i (g_2 b_4^i + g_1 x_i + g_3 b_6^i) + e_8^i e_3^i (g_1 b_4^i + g_3 x_i + g_2 b_6^i) \\ & + e_7^i e_2^i (g_3 b_4^i + g_2 x_i + g_1 b_6^i) - (e_3^i e_7^i (g_2 b_4^i + g_1 x_i + g_3 b_6^i) \\ & + e_8^i e_1^i (g_3 b_4^i + g_2 x_i + g_1 b_6^i) + e_9^i e_2^i (g_1 b_4^i + g_3 x_i + g_2 b_6^i).\end{aligned}$$

Step 5. Substitute for $x_i = b_5^i$.

Step 6. Construct B_i.

Step 7. Construct M.

Step 8. End of algorithm.

We give an application of the above generalized Fibonacci blocking algorithm in the following example for $b = 1$.

Example 4.1 *Let us consider the message matrix for the following message text*:

"SUMEYRA"

Using the message text, we get the following message matrix M :

$$M = \begin{bmatrix} S & U & M \\ E & Y & R \\ A & 0 & 0 \end{bmatrix}_{3\times 3}.$$

Coding Algorithm:
Step 1. We construct the message text M of size 3×3, named B_1:

$$B_1 = \begin{bmatrix} S & U & M \\ E & Y & R \\ A & 0 & 0 \end{bmatrix}.$$

Step 2. Since $b=1$, we calculate $n=3$. For $n=3$, we use the following "letter table" for the message matrix M:

S	U	M	E	Y	R	A	0
21	23	15	7	27	20	3	2

Step 3. We have the elements of the block B_1 as follows:

$b_1^1 = 21$	$b_2^1 = 23$	$b_3^1 = 15$
$b_4^1 = 7$	$b_5^1 = 27$	$b_6^1 = 20$
$b_7^1 = 3$	$b_8^1 = 2$	$b_9^1 = 2.$

Step 4. Now we calculate the determinant d_1 of the block B_1:

$$d_1 = \det(B_1) = 347.$$

Step 5. Using Step 3 and Step 4, we obtain the following matrix K:

$$K = \begin{bmatrix} 347 & 21 & 23 & 15 & 7 & 20 & 3 & 2 & 2 \end{bmatrix}.$$

Step 6. End of algorithm.

Decoding algorithm:
Step 1. By (6), we know that

$$G_3 = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 1 \\ 1 & 2 & 1 \end{bmatrix}.$$

Step 2. The elements of G_3 are denoted by

$$g_1 = 1, g_2 = 1 \text{ and } g_3 = 2.$$

Step 3. We compute the elements $e_1^1, e_2^1, e_3^1, e_4^1, e_7^1, e_8^1, e_9^1$ to construct the matrix E_1:

$e_1^1 = 82$, $e_2^1 = 74$, $e_3^1 = 80$, $e_7^1 = 9$, $e_8^1 = 9$ and $e_9^1 = 10$.

Step 4. We calculate the elements x_1:

$$4 \times 347 = 80624 + 2926x_1 - 78912 - 2938x_1$$
$$\Rightarrow x_1 = 27.$$

Step 5. We rename x_1 as follows:

$$x_1 = b_5^1 = 27.$$

Step 6. We construct the block matrix B_1:

$$B_1 = \begin{bmatrix} 21 & 23 & 15 \\ 7 & 27 & 20 \\ 3 & 2 & 2 \end{bmatrix}.$$

Step 7. We obtain the message matrix M:

$$M = \begin{bmatrix} 21 & 23 & 15 \\ 7 & 27 & 20 \\ 3 & 2 & 2 \end{bmatrix} = \begin{bmatrix} S & U & M \\ E & Y & R \\ A & 0 & 0 \end{bmatrix}.$$

Step 8. End of algorithm.

Now, we give another blocking algorithm benefiting from the generalized Lucas polynomials $L_{p,q,n}(x)$. Let's suppose

$$B_i = \begin{bmatrix} b_1^i & b_2^i \\ b_3^i & b_4^i \end{bmatrix} \text{ and } E_i = \begin{bmatrix} e_1^i & e_2^i \\ e_3^i & e_4^i \end{bmatrix}.$$

We use the matrix H_n given in (11) for $p = q = 1$ and we rewrite the elements of this matrix as $H_n = \begin{bmatrix} h_1 & h_2 \\ h_2 & h_1 \end{bmatrix}$. Similarly, the number of the block matrices B_i is denoted by b. According to b, we choose the number n as follows:

$$n = \begin{cases} 2 & , \quad b = 1 \\ 2b & , \quad b \neq 1 \end{cases}.$$

Using the chosen n, we write the character table given in (13) according to $mod\,27$ or we can differently array this table. For example, we begin the "n" for the first, second,central, last character etc.

Generalized Lucas Blocking Algorithm
Coding Algorithm
Step 1. Divide the matrix M into blocks B_i $\left(1 \le i \le m^2\right)$.
Step 2. Choose n.
Step 3. Determine b_j^i $\left(1 \le j \le 4\right)$.
Step 4. Compute $\det(B_i) \to d_i$.
Step 5. Construct $K = \left[d_i, b_k^i\right]_{k\in\{1,3,4\}}$.
Step 6. End of algorithm.
Decoding Algorithm
Step 1. Compute H_n.
Step 2. Determine h_j $(1 \le j \le 2)$.
Step 3. Compute $h_1 b_3^i + h_2 b_4^i \to e_3^i$, $\left(1 \le i \le m^2\right)$.

$$h_2 b_3^i + h_1 b_4^i \to e_4^i.$$

Step 4. Solve $\det\left(H_2\right) \times d_i = e_4^i (h_1 b_1^i + h_2 x_i) - e_3^i (h_2 b_1^i + h_1 x_i)$.
Step 5. Substitute for $x_i = b_2^i$.
Step 6. Construct B_i.
Step 7. Construct M.
Step 8. End of algorithm.

We give following example as an application of the generalized Lucas blocking algorithm for $b = 1$.

Example 4.2 *Let us consider the message matrix for the following message text:*

"GOOD"

Using the message text, we get the following message matrix M:

$$M = \begin{bmatrix} G & O \\ O & D \end{bmatrix}_{2\times 2}.$$

Coding Algorithm:
Step 1. We construct the message text M of size 2×2, named B_1:

$$B_1 = \begin{bmatrix} G & O \\ O & D \end{bmatrix}.$$

Step 2. Since $b = 1$, we calculate $n = 2$. For $n = 2$, we use the following "letter table" for the message matrix M:

G	O	O	D
8	16	16	5

Step 3. We have the elements of the block B_1 as follows:

$b_1^1 = 8$	$b_2^1 = 16$	$b_3^1 = 16$	$b_4^1 = 5.$

Step 4. Now we calculate the determinants d_1 of the block B_1:

$$d_1 = \det(B_1) = -216.$$

Step 5. Using Step 3 and Step 4 we obtain the following matrix K:

$$K = \begin{bmatrix} -216 & 8 & 16 & 5 \end{bmatrix}.$$

Step 6. End of algorithm.
Decoding algorithm:
Step 1. By (11), we know that

$$H_2 = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}.$$

Step 2. The elements of H_2 are denoted by

$h_1 = 1$ and $h_2 = 3.$

Step 3. We compute the elements e_3^1, e_4^1 to construct the matrix E_1:

$$e_3^1 = 31, e_4^1 = 53.$$

Step 4. We calculate the elements x_1:
$$(-8)\times(-216) = 424 + 159x_1 - 744 - 31x_1$$
$$\Rightarrow x_1 = 16.$$

Step 5. We rename x_1 as follows:

$$x_1 = b_2^1 = 16.$$

Step 6. We construct the block matrix B_1:

$$B_1 = \begin{bmatrix} 8 & 16 \\ 16 & 5 \end{bmatrix}.$$

Step 7. We obtain the message matrix M:

$$M = \begin{bmatrix} 8 & 16 \\ 16 & 5 \end{bmatrix} = \begin{bmatrix} G & O \\ O & D \end{bmatrix}.$$

Step 8. End of algorithm.

5 Conclusions

We have presented two new coding/decoding algorithms by means of the blocks of sizes 3×3 and 2×2. Since the determinant of the matrix G_2 is 0, we study the matrix G_n for $n \geq 3$ in the generalized Fibonacci blocking algorithm, although we can study with the matrix H_n for $n \geq 2$ in the generalized Lucas blocking algorithm.

By differently taking p and q, we can obtain different algorithms. Furthermore it can be mixed the above new blocking methods with the previous methods given in [5-7]. It is possible to produce new blocking methods similar to minesweeper algorithm given in [7].

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