



INCLUSION THEOREMS FOR (C, k) SUMMABILITY METHOD OF SEQUENCES OF FUZZY NUMBERS

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ABSTRACT

We introduce (C, k) summability method of sequences of fuzzy numbers and compare (C, k) summability method with Abel summability method, Hölder summability method and with higher order Cesàro method of sequences of fuzzy numbers. Some Tauberian conditions under which (C, k) summability of sequences of fuzzy numbers imply convergence in fuzzy number space are also obtained as corollaries.

Keywords: Sequences of fuzzy numbers, Inclusion theorems, Tauberian theorems

1. INTRODUCTION

Events and problems occurring in real world generally include noncategorical objects and incomplete information. Human cognitive processes such as approximate reasoning, decision making play essential role in coping with these event and problems involving such vagueness. Motivated by this issue, Zadeh[1] put forward the concept of fuzzy sets as a formal mathematical system to model human reasoning and decision making processes in uncertain environments. Zadeh's invention attracted many researchers in different fields of science and found numerous applications ranging from control theory to artificial intelligence. In mathematics, sequences and series of fuzzy numbers are defined and related convergence properties are investigated[2-6]. Besides, various summability methods are defined to handle sequences of fuzzy numbers which fail to converge in fuzzy number space and Tauberian conditions are given to achieve the convergence[7-9]. In this study we introduce Cesàro summability method of order k , or (C, k) method, of sequences of fuzzy numbers and prove inclusion theorems between (C, k) method and Abel, Hölder summability methods of sequences of fuzzy numbers. We also prove that (C, k) summability method becomes more effective in summing up divergent sequences of fuzzy numbers as the order k increases. As corollaries, some Tauberian conditions under which convergence of sequences of fuzzy numbers follows from (C, k) summability are obtained. Before to continue with main results we give some preliminaries concerning the concept of fuzzy sets.

A *fuzzy number* is a fuzzy set on the real axis, i.e. u is normal, fuzzy convex, upper semi-continuous and $\text{supp } u = \{t \in \mathbb{R}: u(t) > 0\}$ is compact [1]. E^1 denotes the space of fuzzy numbers. α -level set $[u]_\alpha$ is defined by

$$[u]_\alpha := \begin{cases} \overline{\{t \in \mathbb{R}: u(t) \geq \alpha\}} & , \text{ if } 0 < \alpha \leq 1, \\ \overline{\{t \in \mathbb{R}: u(t) > \alpha\}} & , \text{ if } \alpha = 0. \end{cases}$$

$r \in \mathbb{R}$ may be seen as a fuzzy number \bar{r} defined by

$$\bar{r}(t) := \begin{cases} 1 & , \text{ if } t = r, \\ 0 & , \text{ if } t \neq r. \end{cases}$$

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Let $u, v \in E^1$ and $k \in \mathbb{R}$. The addition and scalar multiplication are defined by

$$[u + v]_\alpha = [u]_\alpha + [v]_\alpha = [u_\alpha^- + v_\alpha^-, u_\alpha^+ + v_\alpha^+] \quad , \quad [ku]_\alpha = k[u]_\alpha$$

where $[u]_\alpha = [u_\alpha^-, u_\alpha^+]$, for all $\alpha \in [0,1]$.

Fuzzy number $\bar{0}$ is identity element in $(E^1, +)$ and none of $u \neq \bar{r}$ has inverse in $(E^1, +)$. For any $k_1, k_2 \in \mathbb{R}$ with $k_1 k_2 \geq 0$, distribution property $(k_1 + k_2)u = k_1 u + k_2 u$ holds but for general $k_1, k_2 \in \mathbb{R}$ it fails to hold. On the other hand properties $k(u + v) = ku + kv$ and $k_1(k_2 u) = (k_1 k_2)u$ holds for any $k, k_1, k_2 \in \mathbb{R}$ [10]. It should be noted that E^1 with addition and scalar multiplication defined above is not a linear space over \mathbb{R} .

The metric D on E^1 is defined as

$$D(u, v) := \sup_{\alpha \in [0,1]} \max\{|u_\alpha^- - v_\alpha^-|, |u_\alpha^+ - v_\alpha^+|\}$$

and it has the following properties [10]

$$D(ku, kv) = |k|D(u, v), \quad D(u + v, w + z) \leq D(u, w) + D(v, z)$$

where $u, v, w, z \in E^1$ and $k \in \mathbb{R}$. By $w(F)$, we denote the set of all sequences of fuzzy numbers.

Definition 1. [11] Let (u_k) be a sequence of fuzzy numbers. Denote $s_n = \sum_{k=0}^n u_k$ for all $n \in \mathbb{N}$, if the sequence (s_n) converges to a fuzzy number u then we say that the series $\sum u_k$ of fuzzy numbers converges to u and write $\sum u_k = u$ which implies that

$$\sum_{k=0}^n u_k^-(\alpha) \rightarrow u^-(\alpha) \quad \text{and} \quad \sum_{k=0}^n u_k^+(\alpha) \rightarrow u^+(\alpha) \quad (n \rightarrow \infty)$$

uniformly in $\alpha \in [0,1]$. Conversely, for the sequence (u_k) of fuzzy numbers if $\sum_{k=0}^\infty u_k^-(\alpha) = \beta(\alpha)$ and $\sum_{k=0}^\infty u_k^+(\alpha) = \gamma(\alpha)$ converge uniformly in α , then $\{(\beta(\alpha), \gamma(\alpha)) : \alpha \in [0,1]\}$ defines a fuzzy number u represented by $[u]_\alpha = [\beta(\alpha), \gamma(\alpha)]$ and $\sum u_k = u$.

Remark 2. [12] Let (u_n) be a sequence of fuzzy numbers. If (x_n) is a sequence of non-negative real numbers, then

$$\sum_{k=0}^n x_k \sum_{m=0}^k u_m = \sum_{m=0}^n u_m \sum_{k=m}^n x_k.$$

Theorem 3. [13] Let $\sum_{n=0}^\infty u_n$ be a convergent series of fuzzy numbers. If $\sum_{n=0}^\infty x_n$ is a convergent series with non-negative real terms, then

$$\left(\sum_{n=0}^\infty x_n\right) \left(\sum_{n=0}^\infty u_n\right) = \sum_{n=0}^\infty \sum_{k=0}^n u_k x_{n-k}.$$

Theorem 4. [14] If $\sum D(u_k, \bar{0}) < \infty$, then series $\sum u_k$ is convergent.

Definition 5. [15] Let (u_n) be a sequence of fuzzy numbers. Hölder means of (u_n) of order k is defined by

$$H_n^k = \frac{1}{n+1} \sum_{v=0}^n H_v^{k-1}$$

where $H_n^0 = u_n$. We say that sequence (u_n) is (H, k) summable to fuzzy number μ if $\lim_{n \rightarrow \infty} H_n^k = \mu$.

Theorem 6. [16] If sequence (u_n) of fuzzy numbers is (H, k) summable to fuzzy number μ and (u_n) is slowly decreasing then (u_n) converges to μ .

Theorem 7. [16] If sequence (u_n) of fuzzy numbers is (H, k) summable to fuzzy number μ and if there exists an $H > 0$ such that $nu_n \geq nu_{n-1} - \bar{H}$ then (u_n) converges to μ .

2. MAIN RESULTS

Definition 8. A sequence (u_n) of fuzzy numbers is said to be (C, k) summable to fuzzy number μ if

$$\lim_{n \rightarrow \infty} \frac{u_n^k}{\binom{n+k}{n}} = \mu,$$

where $u_n^0 = u_n$ and $u_n^k = \sum_{v=0}^n u_v^{k-1}$.

We note that if sequence (u_n) of fuzzy numbers is (C, k) summable then $D(u_n^k, \bar{0}) = O(n^k)$ and hence $\sum_{n=0}^{\infty} u_n^k x^n$ exists for $x \in (0, 1)$ by Theorem 4. So we get

$$\sum_{n=0}^{\infty} u_n x^n = (1-x) \sum_{n=0}^{\infty} u_n^1 x^n = (1-x)^2 \sum_{n=0}^{\infty} u_n^2 x^n = \dots = (1-x)^k \sum_{n=0}^{\infty} u_n^k x^n \quad (1)$$

by Theorem 3 and followingly we obtain

$$\left\{ \frac{1}{(1-x)^k} \right\} \left\{ \sum_{n=0}^{\infty} u_n x^n \right\} = \sum_{n=0}^{\infty} u_n^k x^n. \quad (2)$$

Then again from Theorem 3 we have

$$\begin{aligned} \left\{ \frac{1}{(1-x)^k} \right\} \left\{ \sum_{n=0}^{\infty} u_n x^n \right\} &= \left\{ \sum_{n=0}^{\infty} \binom{n+k-1}{n} x^n \right\} \left\{ \sum_{n=0}^{\infty} u_n x^n \right\} \\ &= \sum_{n=0}^{\infty} \left\{ \sum_{v=0}^n \binom{n-v+k-1}{n-v} u_v \right\} x^n. \end{aligned} \quad (3)$$

So by (2) and (3) we get

$$u_n^k = \sum_{v=0}^n \binom{n-v+k-1}{n-v} u_v,$$

which yields the following alternative definition for (C, k) summability of sequences of fuzzy numbers.

Definition 9. Let (u_n) be a sequence of fuzzy numbers. Cesàro means of (u_n) of order k is defined by

$$C_n^k = \frac{1}{\binom{n+k}{n}} \sum_{v=0}^n \binom{n-v+k-1}{n-v} u_v.$$

We say that sequence (u_n) is (C, k) summable to fuzzy number μ if $\lim_{n \rightarrow \infty} C_n^k = \mu$.

Theorem 10. If sequence (u_n) of fuzzy numbers converges to $\mu \in E^1$, then (u_n) is (C, k) summable to μ .

Proof. Let $u_n \rightarrow \mu$. For any $\varepsilon > 0$ there exists $n_0 = n_0(\varepsilon)$ such that $D(u_n, \mu) < \varepsilon/2$ whenever $n > n_0$, and for $n \leq n_0$ there exists $M > 0$ such that $D(u_n, \mu) \leq M$. So we get

$$\begin{aligned} D(C_n^k, \mu) &= D\left(\frac{1}{\binom{n+k}{n}} \sum_{v=0}^n \binom{n-v+k-1}{n-v} u_v, \frac{1}{\binom{n+k}{n}} \sum_{v=0}^n \binom{n-v+k-1}{n-v} \mu\right) \\ &\leq \frac{1}{\binom{n+k}{n}} \sum_{v=0}^n \binom{n-v+k-1}{n-v} D(u_v, \mu) \\ &= \frac{1}{\binom{n+k}{n}} \sum_{v=0}^{n_0} \binom{n-v+k-1}{n-v} D(u_v, \mu) + \frac{1}{\binom{n+k}{n}} \sum_{v=n_0+1}^n \binom{n-v+k-1}{n-v} D(u_v, \mu) \\ &< \frac{M}{\binom{n+k}{n}} \sum_{v=0}^{n_0} \binom{n-v+k-1}{n-v} + \frac{\varepsilon}{2}. \end{aligned}$$

There also exists $n_1 = n_1(\varepsilon)$ such that

$$\frac{1}{\binom{n+k}{n}} \sum_{v=0}^{n_0} \binom{n-v+k-1}{n-v} < \frac{\varepsilon}{2M}$$

whenever $n > n_1$. Then we conclude that $D(C_n^k, \mu) < \varepsilon$ whenever $n > \max\{n_0, n_1\}$, which completes the proof.

Theorem 11. Let $A^{k,m} = (a_{nv}^{k,m})$ be a matrix mapping from $w(F)$ to $w(F)$ such that

$$a_{nv}^{k,m} = \begin{cases} \binom{n-v+m-1}{n-v} \binom{v+k}{v} / \binom{n+k+m}{n} & v \leq n \\ 0 & v > n \end{cases}$$

where $k, m > 0$ and (u_n) be a (C, k) summable sequence of fuzzy numbers. Then, $A^{k,m}$ - transform of the sequence C_n^k is the sequence C_n^{k+m} .

Proof. Let (u_n) be a (C, k) summable sequence of fuzzy numbers. In view of (1) we have

$$\begin{aligned} \sum_{n=0}^{\infty} u_n^{k+m} x^n &= (1-x)^{-m} \sum_{n=0}^{\infty} u_n^k x^n = \left\{ \sum_{n=0}^{\infty} \binom{n+m-1}{n} x^n \right\} \left\{ \sum_{n=0}^{\infty} u_n^k x^n \right\} \\ &= \sum_{n=0}^{\infty} \left\{ \sum_{v=0}^n \binom{n-v+m-1}{n-v} u_v^k \right\} x^n \end{aligned}$$

which implies $u_n^{k+m} = \sum_{v=0}^n \binom{n-v+m-1}{n-v} u_v^k$. Then we get

$$\begin{aligned}
 C_n^{k+m} &= \frac{u_n^{k+m}}{\binom{n+k+m}{n}} = \frac{1}{\binom{n+k+m}{n}} \sum_{v=0}^n \binom{n-v+m-1}{n-v} u_v^k \\
 &= \frac{1}{\binom{n+k+m}{n}} \sum_{v=0}^n \binom{n-v+m-1}{n-v} \binom{v+k}{v} C_v^k,
 \end{aligned}$$

which completes the proof.

Theorem 12. Let (u_n) be a sequence of fuzzy numbers and $k' > k$. If sequence (u_n) is (C, k) summable to a fuzzy number μ then it is (C, k') summable to μ .

Proof. Let sequence (u_n) of fuzzy numbers be (C, k) summable to fuzzy number μ . Then sequence C_n^k converges to μ . Since $C_n^{k'} = A^{k, k'-k}(C_n^k)$, sequence $C_n^{k'}$ also converges to μ in view of the regularity of $A^{k, k'-k}$ -transform which satisfies Theorem 4.6 in [14]. This completes the proof.

For $k' > k$, (C, k') summability of a sequence of fuzzy numbers does not imply (C, k) summability which can be seen by the following example.

Example 13. Let (u_n) be a sequence of fuzzy numbers such that

$$u_n(t) = \begin{cases} t - a_n, & a_n \leq t \leq a_n + 1 \\ 2 - t + a_n, & a_n + 1 \leq t \leq a_n + 2 \\ 0, & (\text{otherwise}) \end{cases}$$

where $a_n = \sum_{v=0}^n (-1)^v \binom{v+k}{v}$ for every $k \in \mathbb{N}$. Sequence (u_n) is $(C, 2k+1)$ summable to

$$\mu(t) = \begin{cases} t - \frac{1}{2^{k+1}}, & \frac{1}{2^{k+1}} \leq t \leq \frac{1}{2^{k+1}} + 1 \\ 2 - t + \frac{1}{2^{k+1}}, & \frac{1}{2^{k+1}} + 1 \leq t \leq \frac{1}{2^{k+1}} + 2 \\ 0, & (\text{otherwise}), \end{cases}$$

but not summable (C, k) to any fuzzy number since $\lim_{n \rightarrow \infty} \frac{D(u_n, \bar{0})}{n^k} = 1/2k! \neq 0$ (see Theorem 14).

Theorem 14. If sequence (u_n) of fuzzy numbers is (C, k) summable, then $D(u_n, \bar{0}) = o(n^k)$ and this estimate is best possible.

Proof. Let sequence (u_n) of fuzzy numbers be (C, k) summable to a fuzzy number μ . Then Cesàro means C_n^k of (u_n) converges to μ . Since

$$\begin{aligned}
 D(u_n^{k-1}, \bar{0}) &= D(u_n^k, u_{n-1}^k) \\
 &= D\left(\binom{n+k}{n} C_n^k, \binom{n+k-1}{n-1} C_{n-1}^k\right) \\
 &= D\left(\left\{\binom{n+k-1}{n-1} + \binom{n+k-1}{n}\right\} C_n^k, \binom{n+k-1}{n-1} C_{n-1}^k\right) \\
 &\leq \binom{n+k-1}{n-1} D(C_n^k, C_{n-1}^k) + \binom{n+k-1}{n} D(C_n^k, \bar{0})
 \end{aligned}$$

we get

$$\frac{D(u_n^{k-1}, \bar{0})}{n^k} \leq \frac{1}{n^k} \left\{ \binom{n+k-1}{k} D(C_n^k, C_{n-1}^k) + \binom{n+k-1}{k-1} D(C_n^k, \bar{0}) \right\}.$$

By the fact that $\binom{n+k}{k} \sim \frac{n^k}{k!}$ and by the convergence of sequence C_n^k we obtain $D(u_n^{k-1}, \bar{0}) = o(n^k)$. Furthermore, since

$$D(u_n^{k-2}, \bar{0}) = D(u_n^{k-1}, u_{n-1}^{k-1}) \leq D(u_n^{k-1}, \bar{0}) + D(u_{n-1}^{k-1}, \bar{0})$$

we get $D(u_n^{k-2}, \bar{0}) = o(n^k)$. Applying same procedure successively we conclude $D(u_n, \bar{0}) = o(n^k)$. This bound is best possible which can be seen by Theorem 3.1 in [13] even in the case $(C, 1)$.

Theorem 15. *If sequence (u_n) of fuzzy numbers is (C, k) summable to a fuzzy number μ , then it is Abel summable to μ .*

Proof. Let sequence (u_n) of fuzzy numbers be (C, k) summable to μ . Then $\lim_{n \rightarrow \infty} C_n^k = \mu$. We want to show that sequence (u_n) is Abel summable to μ . It is sufficient to show that $\sum_{n=0}^{\infty} u_n x^n$ exists for $x \in (0, 1)$ and $\lim_{x \rightarrow 1^-} (1-x) \sum_{n=0}^{\infty} u_n x^n = \mu$.

From Theorem 14 there exists $M > 0$ such that $D(u_n, \bar{0}) \leq Mn^k$ and hence we get

$$\sum_{n=0}^{\infty} D(u_n x^n, \bar{0}) = \sum_{n=0}^{\infty} D(u_n, \bar{0}) x^n \leq M \sum_{n=0}^{\infty} n^k x^n < \infty$$

which implies $\sum_{n=0}^{\infty} u_n x^n$ exists for $x \in (0, 1)$ by Theorem 4. On the other hand from the expression (1) we have

$$(1-x) \sum_{n=0}^{\infty} u_n x^n = (1-x)^{k+1} \sum_{n=0}^{\infty} u_n^k x^n = (1-x)^{k+1} \sum_{n=0}^{\infty} \binom{n+k}{n} C_n^k x^n.$$

Now for every $k \in \mathbb{N}$ using the regularity of the power series method (J, p) [17] where $p_n = \binom{n+k}{n}$, we conclude

$$\lim_{x \rightarrow 1^-} (1-x) \sum_{n=0}^{\infty} u_n x^n = \mu.$$

Abel summability of a sequence of fuzzy numbers does not imply its (C, k) summability which can be seen by the following example:

$$u_n(t) = \begin{cases} t - b_n^1, & b_n^1 \leq t \leq b_n^1 + 1 \\ 2 - t + b_n^1, & b_n^1 + 1 \leq t \leq b_n^1 + 2 \\ 0, & (\text{otherwise}). \end{cases}$$

where (b_n) is a sequence of real numbers such that $\sum_{n=0}^{\infty} b_n x^n = e^{1/1+x}$. Then sequence (u_n) is Abel summable to

$$\mu(t) = \begin{cases} t - e^{1/2}, & e^{1/2} \leq t \leq e^{1/2} + 1 \\ 2 - t + e^{1/2}, & e^{1/2} + 1 \leq t \leq e^{1/2} + 2 \\ 0, & (\text{otherwise}), \end{cases}$$

but not summable (C, k) to any fuzzy number.

Lemma 16. Let (u_n) be a sequence of fuzzy numbers and $p \geq 1$. Then

$$u_n^2 + \sum_{v=0}^n (v+p) u_v = (n+p+1)u_n^1.$$

Proof. Let (u_n) be a sequence of fuzzy numbers and $p \geq 1$. Then by Remark 2 we conclude

$$\begin{aligned} u_n^2 + \sum_{v=0}^n (v+p) u_v &= \sum_{r=0}^n \sum_{v=0}^r u_v + \sum_{v=0}^n (v+p) u_v = \sum_{v=0}^n (n-v+1) u_v + \sum_{v=0}^n (v+p) u_v \\ &= \sum_{v=0}^n (n+p+1) u_v = (n+p+1)u_n^1. \end{aligned}$$

Theorem 17. Let (u_n) be a sequence of fuzzy numbers and

$$h_n = \frac{u_0 + u_1 + \dots + u_n}{n+1}$$

Then limits $C_n^k(u) \rightarrow \mu$ and $C_n^{k-1}(h) \rightarrow \mu$ are equivalent.

Proof. Let (u_n) be a sequence of fuzzy numbers and (h_n) be the sequence arithmetical means of (u_n) . Since $u_n^2 = \sum_{v=0}^n u_v^1 = \sum_{v=0}^n (v+1) h_v$, we have $h_n^2 + u_n^2 = (n+2)h_n^1$ by Lemma 16. Applying the operation $\sum_{v=0}^n$ to both sides and using Lemma 16 again we get $2h_n^3 + u_n^3 = (n+3)h_n^2$. By successive application of this procedure we obtain

$$(k-1)h_n^k + u_n^k = (n+k)h_n^{k-1}. \tag{4}$$

So we conclude

$$(k-1)C_n^k(h) + C_n^k(u) = kC_n^{k-1}(h). \tag{5}$$

This shows that $C_n^{k-1}(h) \rightarrow \mu$ implies $C_n^k(u) \rightarrow \mu$ in view of Theorem 12.

Now making substitution $h_n^k = h_n^{k-1} + h_{n-1}^k$ in the expression (4) we get

$$(k-1)h_{n-1}^k + u_n^k = (n+1)h_n^{k-1}$$

and hence we have $(n+k)h_{n-1}^k + u_n^k = (n+1)h_n^k$. This implies that

$$nC_{n-1}^k(h) + C_n^k(u) = (n+1)C_n^k(h).$$

Then we get

$$\begin{aligned} C_0^k(u) &= C_0^k(h) \\ C_0^k(h) + C_1^k(u) &= 2C_1^k(h) \\ 2C_1^k(h) + C_2^k(u) &= 3C_2^k(h) \\ &\vdots \\ nC_{n-1}^k(h) + C_n^k(u) &= (n+1)C_n^k(h) \end{aligned}$$

and summing all rows we get $C_0^k(u) + C_1^k(u) + \dots + C_n^k(u) = (n+1)C_n^k(h)$. So we conclude

$$C_n^k(h) = \frac{1}{n+1} \sum_{v=0}^n C_v^k(u).$$

Hence $C_n^k(u) \rightarrow \mu$ implies $C_n^{k-1}(h) \rightarrow \mu$ by use of equality above and equality (5), respectively.

Theorem 18. *If sequence (u_n) of fuzzy numbers is (C, k) summable to a fuzzy number μ , then it is (H, k) summable to μ , and conversely. That is (C, k) and (H, k) methods are equivalent.*

Proof. Let (u_n) be a sequence of fuzzy numbers and consider the Cesàro means C_n^k and Hölder means H_n^k of (u_n) . By Theorem 17, the limits

$$C_n^k \rightarrow \mu, C_n^{k-1}(H^1) \rightarrow \mu, C_n^{k-2}(H^2) \rightarrow \mu, \dots, C_n^1(H^{k-1}) \rightarrow \mu, H_n^k \rightarrow \mu$$

are equivalent and this completes the proof.

By Theorem 10 and by Example 13, one can conclude that convergence of sequences of fuzzy numbers implies (C, k) summability but (C, k) summability of sequences of fuzzy numbers does not imply convergence in general. In view of Theorem 6, Theorem 7 and Theorem 18, we give the following Tauberian conditions sufficient for (C, k) summability of sequences of fuzzy numbers to imply convergence in fuzzy number space.

Theorem 19. *If sequence (u_n) of fuzzy numbers is (C, k) summable to fuzzy number μ and (u_n) is slowly decreasing then (u_n) converges to μ .*

Theorem 20. *If sequence (u_n) of fuzzy numbers is (C, k) summable to fuzzy number μ and if there exists an $H > 0$ such that $nu_n \geq nu_{n-1} - \bar{H}$ then (u_n) converges to μ .*

We give also the following comparison theorem between (H, k) summability method and Abel summability method of sequences of fuzzy numbers in view of Theorem 15 and Theorem 18.

Theorem 21. *If sequence (u_n) of fuzzy numbers is (H, k) summable to fuzzy number μ , then (u_n) is Abel summable to μ .*

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