TURKISH JOURNAL OF SCIENCE
VOLUME IV, ISSUE I, 30-38
ISSN: 2587-0971
http://dergipark.gov.tr/tjos

İki-Boyutlu Harmonik Konveks Fonksiyonlar ve İlgili Genelleştirilmiş Eşitsizlikler

Two-Dimensional Operator Harmonically Convex Functions and Related Generalized Inequalities

Nurgül OKUR[1] and Fatma Buğlem YALÇIN [2*]

[1]*Giresun University, Faculty of Science and Arts,*
Department of Statistics, Giresun-TURKEY
[2] *Ordu University, Faculty of Science and Arts,*
Department of Mathematics, Ordu-TURKEY
**Sorumlu yazar / Corresponding Author*: yalcinfatmabuglem@gmail.com

Geliş Tarihi / Received Date: 12 March 2019
Kabul Tarihi / Accepted Date: 25 April 2019

Öz: Bu çalışmada, fonksiyonların harmonik konveksliği üzerine çalıştık. İlk olarak, reel sayı doğrusu üzerinde bu fonksiyonlar için bazı yeni genelleştirilmiş Hadamard tipi ve Ostrowski tipi eşitsizlikleri elde ettik. Ayrıca, harmonik konveks fonksiyonlar için iki-boyutlu operatörü kullanarak yukarıda söz edilen eşitsizlikleri genelleştirdik.

Anahtar Kelimeler — harmonik konveks fonksiyon, Hermite-Hadamard tipli eşitsizlikler, Ostrowski tipli eşitsizlikler.

Abstract: In this study, we studied on the harmonically convexity of functions. Firstly, we obtained some new generalized Hadamard's type and Ostrowski's type inequalities for these functions on the real number line. Besides, we generalized the above mentioned inequalities using two-dimensional operator for harmonically convex functions.

Keywords — harmonically convex function, Hermite-Hadamard type inequalities, Ostrowski's type inequalities.

1. Introduction and Preliminaries

It becomes famous the Hermite-Hadamard inequality for convex functions [1] in the recent years. Alomari [2] generalized this classical Hermite-Hadamard type inequality for every convex function and obtained the Ostrowski's type inequalitiy for every positive convex function on $[a, b]$. Recently, Dragomir [3] proved the classical Hermite-Hadamard type inequality for convex functions on the coordinates. Like as Alomari [2], Mwaeze [4] proved some generalizations inequalities of the above mentioned inequalities on the coordinates.

The Hermite-Hadamard type integral inequality for convex functions has received renewed attention in recent years and the remarkable varieties of refinements and generalizations have been found in [1]-[13]. In this context, the Hermite-Hadamard type inequalities for harmonically convex functions were obtained by many researchers in literature (see, [8]-[13]). Also, İşcan [8] showed definition of harmonically convex as follows:

Definition 1.1 ([8]). *Let $I \subset \mathbb{R} \setminus \{0\}$ be a real interval. A function $f: I \subset \mathbb{R} \setminus \{0\} \to \mathbb{R}$ is said to be harmonically convex, if*

$$f\left(\frac{xy}{\lambda x+(1-\lambda)y}\right)\leq \lambda f(y)+(1-\lambda)f(x)$$

for all $x,y\in I,\quad \lambda\in[0,1]$. *If the above inequality is reversed, then* f *is said to be harmonically concave.*

Proposition 1.1 ([8]). *Let* $f: I\subset \mathbb{R}\setminus\{0\}\to\mathbb{R}$ *be a function then*

i) if $I\subset(0,\infty)$ *and* f *is convex and nondecreasing function then* f *is harmonically convex,*

ii) if $I\subset(0,\infty)$ *and* f *is harmonically convex and nondecreasing function then* f *is convex,*

iii) if $I\subset(0,\infty)$ *and* f *is harmonically convex and nondecreasing function then* f *is convex,*

iv) if $I\subset(0,\infty)$ *and* f *is convex and nondecreasing function then* f *is harmonically convex.*

Theorem 1.1 ([8]). *Let* $f: I\subset \mathbb{R}\setminus\{0\}\to\mathbb{R}$ *be harmonically convex function and* $a,b\in I$ *with* $a<b$. *If* $f\in L[a,b]$ *then the following inequality holds*

$$f\left(\frac{2ab}{a+b}\right)\leq \frac{ab}{b-a}\int_a^b f(x)dx\leq \frac{f(a)+f(b)}{2}.$$

Now, Let us see definition of two-dimensional harmonically convex function by Noor et al. [9]. Consider $\Delta=[a,b]\times[c,d]\subset\mathbb{R}^2\setminus\{0\}$ with $a<b$ and $c<d$.

Definition 1.2 ([9]). *A function* $f:\Delta\to\mathbb{R}$ *is said to be a harmonically convex on* Δ, *if the following inequality holds*

$$f\left(\frac{ac}{\lambda a+(1-\lambda)c},\frac{bd}{\lambda b+(1-\lambda)d}\right)\leq(1-\lambda)f(a,b)+\lambda f(c,d)$$

for all $(a,b),(c,d)\in\Delta$ *and* $\lambda\in[0,1]$. *If the above inequality is reversed, then* f *is said to be a harmonically concave on* Δ.

Definition 1.3 ([9]). *A function* $f:\Delta\subset\mathbb{R}^2\setminus\{0\}\to\mathbb{R}$ *is said to be a harmonically convex on the co-ordinates if the partial mappings* $f_s:[a,b]\subset\mathbb{R}\setminus\{0\}\to\mathbb{R}, f_s(u):=f(u,s)$ *and* $f_t:[c,d]\subset\mathbb{R}\setminus\{0\}\to\mathbb{R}, f_t(v):=f(t,v)$ *defined for all* $t\in[a,b]$ *and* $s\in[c,d]$ *are harmonically convex.*

Theorem 1.2 ([9]). *Suppose that* $f:\Delta\subset\mathbb{R}^2\setminus\{0\}\to\mathbb{R}$ *is harmonically convex on the co-ordinates. Then the following inequality holds*

$$f\left(\frac{2ab}{a+b},\frac{2cd}{c+d}\right)\leq\frac{1}{2}\left[\frac{ab}{b-a}\int_a^b\frac{1}{t^2}f\left(t,\frac{2cd}{c+d}\right)dt+\frac{cd}{d-c}\int_c^d\frac{1}{s^2}f\left(\frac{2ab}{a+b},s\right)ds\right]$$

$$\leq\frac{abcd}{(b-a)(d-c)}\int_a^b\int_c^d\frac{1}{(ts)^2}f(t,s)dtds$$

$$\leq\frac{1}{4}\left[\frac{ab}{b-a}\int_a^b\frac{1}{t^2}\big(f(t,c)+f(t,d)\big)dt+\frac{cd}{d-c}\int_c^d\frac{1}{s^2}\big(f(a,s)+f(b,s)\big)ds\right]$$

$$\leq\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}.$$

Concordantly, in present study, we obtained some important generalized inequalities for harmonically convex functions on real number line and on the coordinates, respectively.

2. Main Results

This section contains two sub-sections. In the first sub-section we obtained the generalized Hadamard's type and Ostrowski's type inequalities for harmonically convex functions on the real number line. In the second sub-section we verified the above mentioned inequalities for harmonically convex functions on the coordinates.

2.1. Generalized Inequalities for Harmonically Convex Functions on the Real Line

Let us obtain the generalized Hadamard's type inequality for related functions.

Theorem 2.1. *Let $f: I \subset \mathbb{R} \setminus \{0\} \to \mathbb{R}$ be a harmonically convex function and $f \in L[a,b]$. Then the double inequality holds*

$$\frac{b-a}{nab}\sum_{k=1}^{n} f\left(\frac{2t_{k-1}t_k}{t_{k-1}+t_k}\right) \le \int_a^b \frac{f(t)}{t^2}dt \le \frac{b-a}{2nab}\left[f(a)+2\sum_{k=1}^{n-1} f(t_k)+f(b)\right] \tag{2.1}$$

where $t_k = a + k\frac{b-a}{n}$, $k = 1,2,\dots,n;\ n \in N$.

Proof. Using harmonically convexity of f on each sub-interval $[t_{k-1}, t_k] \subseteq [a,b]$, $k = 1,2,\dots,n$, then for all $\lambda \in [0,1]$

$$f\left(\frac{t_{k-1}t_k}{\lambda t_{k-1}+(1-\lambda)t_k}\right) \le \lambda f(t_k) + (1-\lambda)f(t_{k-1}). \tag{2.2}$$

Integrating (2.2) with respect to λ on $[0,1]$

$$\int_0^1 f\left(\frac{t_{k-1}t_k}{\lambda t_{k-1}+(1-\lambda)t_k}\right)d\lambda \le \frac{f(t_{k-1})+X(t_k)}{2}. \tag{2.3}$$

Changing of variable $t = \frac{t_{k-1}t_k}{\lambda t_{k-1}+(1-\lambda)t_k}$ in (2.3)

$$\int_{t_{k-1}}^{t_k} \frac{f(t)}{t^2}dt \le \frac{t_k - t_{k-1}}{2t_{k-1}t_k}\big(f(t_{k-1})+f(t_k)\big).$$

Taking the sum over k from 1 to n, we get

$$\sum_{k=1}^{n}\int_{t_{k-1}}^{t_k} \frac{f(t)}{t^2}dt = \int_a^b \frac{f(t)}{t^2}dt \le \sum_{k=1}^{n}\frac{t_k - t_{k-1}}{2t_{k-1}t_k}\big(f(t_{k-1})+f(t_k)\big).$$

$$\le \frac{1}{2}\max_k\left\{\frac{t_k - t_{k-1}}{t_{k-1}t_k}\right\}\sum_{k=1}^{n}\big(f(t_{k-1})+f(t_k)\big)$$

$$= \frac{b-a}{2nab}\left[f(t_0)+f(t_1)+\sum_{k=2}^{n-1}\big(f(t_{k-1})+f(t_k)\big)+f(t_{n-1})+f(t_n)\right]$$

$$= \frac{b-a}{2nab}\left[f(a) + 2\sum_{k=1}^{n-1} f(t_k) + f(b)\right]. \quad (2.4)$$

Because of harmonically convexity of f on $[t_{k-1}, t_k]$, then for $\lambda \in [0,1]$

$$f\left(\frac{2t_{k-1}t_k}{t_{k-1}+t_k}\right) \leq \frac{1}{2}\left[f\left(\frac{t_{k-1}t_k}{\lambda t_{k-1} + (1-\lambda)t_k}\right) + f\left(\frac{t_{k-1}t_k}{\lambda t_{k-1} + (1-\lambda)t_k}\right)\right]$$

$$\leq \frac{1}{2}[f(\lambda t_k + (1-\lambda)t_{k-1}) + f((1-\lambda)t_k + \lambda t_{k-1})]. \quad (2.5)$$

Applying on (2.5) by using similar way in (2.2)-(2.4)

$$\frac{b-a}{nab}\sum_{k=1}^{n} f\left(\frac{2t_{k-1}t_k}{t_{k-1}+t_k}\right) \leq \int_a^b \frac{f(t)}{t^2}dt. \quad (2.6)$$

From (2.4) and (2.6), we have (2.1).

Let us give firstly the generalized Ostrowki's type inequality for related functions

Theorem 2.2. *Let* $f: I \subset \mathbb{R} \setminus \{0\} \to \mathbb{R}_+$ *be a positive harmonically convex function and* $f \in L[a,b]$. *Then the inequality holds*

$$\int_a^b \frac{f(t)}{t^2}dt - \left(\frac{b-a}{2ab}\right)f(s) \leq \frac{b-a}{2nab}\left[f(a) + 2\sum_{k=1}^{n-1} f(t_k) + f(b)\right] \quad (2.7)$$

for all $s \in [a,b]$, *where* $t_k = a + k\frac{b-a}{n}$, $k = 1,2,\ldots,n$; $n \in N$.

Proof. Fix $s \in [t_{k-1}, t_k]$, $k = 1,2,\ldots,n$. Since f is harmonically convex on $[a,b]$, then f so is harmonically convex on each subinterval $[t_{k-1}, t_k]$, in particular on $[t_{k-1}, s]$, then

$$f\left(\frac{t_{k-1}s}{\lambda t_{k-1} + (1-\lambda)s}\right) \leq \lambda f(s) + (1-\lambda)f(t_{k-1}), k = 1,2,\ldots,n \quad (2.8)$$

Integrating (2.8) with respect to λ on $[0,1]$ we get

$$\int_0^1 f\left(\frac{t_{k-1}s}{\lambda t_{k-1} + (1-\lambda)s}\right)d\lambda \leq \frac{f(t_{k-1}) + f(s)}{2}. \quad (2.9)$$

Changing of variable $t = \frac{t_{k-1}s}{\lambda t_{k-1} + (1-\lambda)s}$ in (2.9), then

$$\int_{t_{k-1}}^{s} \frac{f(t)}{t^2}dt \leq \frac{s - t_{k-1}}{2t_{k-1}s}\big(f(t_{k-1}) + f(s)\big). \quad (2.10)$$

Similarly for $[s, t_k]$

$$\int_s^{t_k} \frac{f(t)}{t^2}dt \leq \frac{t_k - s}{2t_{k-1}s}\big(f(s) + f(t_k)\big). \quad (2.11)$$

Adding the inequalities (2.10) and (2.11)

$$\int_{t_{k-1}}^{s}\frac{f(t)}{t^2}dt+\int_{s}^{t_k}\frac{f(t)}{t^2}dt=\int_{t_{k-1}}^{t_k}\frac{f(t)}{t^2}dt$$

$$\leq\frac{s-t_{k-1}}{2t_{k-1}s}\big(f(t_{k-1})+f(s)\big)+\frac{t_k-s}{2t_k s}\big(f(s)+f(t_k)\big)$$

$$\leq\frac{t_k-t_{k-1}}{2t_{k-1}t_k}\{f(t_{k-1})+f(t_k)\}+\frac{b-a}{2ab}f(s). \tag{2.12}$$

Taking the sum over k from 1 to n, we get

$$\sum_{k=1}^{n}\int_{t_{k-1}}^{t_k}\frac{f(t)}{t^2}dt=\int_{a}^{b}\frac{f(t)}{t^2}dt\leq\sum_{k=1}^{n}\frac{t_k-t_{k-1}}{2t_{k-1}t_k}\big(f(t_{k-1})+f(t_k)\big)+\sum_{k=1}^{n}\frac{b-a}{2ab}f(s).$$

$$\leq\frac{1}{2}\max_k\left\{\frac{t_k-t_{k-1}}{t_{k-1}t_k}\right\}\sum_{k=1}^{n}\big(f(t_{k-1})+f(t_k)\big)+\frac{b-a}{2ab}f(s)$$

$$=\frac{b-a}{2nab}\left[f(t_0)+f(t_1)+\sum_{k=2}^{n-1}\big(f(t_{k-1})+f(t_k)\big)+f(t_{n-1})+f(t_n)\right]+\frac{b-a}{2ab}f(s)$$

$$=\frac{b-a}{2nab}\left[f(a)+2\sum_{k=1}^{n-1}f(t_k)+f(b)\right]+\frac{b-a}{2ab}f(s).$$

Remark 2.2. In Theorem 2.1 for $n=1$, then the clasicall Hermite-Hadamard inequality for harmonically convex functions is obtained.

2.2. Generalized Inequalities for Two-dimensional Harmonically Convex Functions

Throughout this section, consider $\Delta:=[a,b]\times[c,d]\subset\mathbb{R}^2\setminus\{0\}$ with $a<b$ and $c<d$. Let us obtain the generalized Hadamard's inequality for related functions:

Theorem 2.3. *Let $f:\Delta\subset\mathbb{R}^2\setminus\{0\}\to\mathbb{R}$ be harmonically convex on the coordinates on Δ. The following inequality satisfies*

$$\frac{d-c}{2ncd}\sum_{k=1}^{n}\int_{a}^{b}\frac{\left(t,\frac{2s_k s_{k-1}}{s_{k-1}+s_k}\right)}{t^2}dt+\frac{b-a}{2nab}\sum_{k=1}^{n}\int_{c}^{d}\frac{\left(\frac{2t_{k-1}t_k}{t_{k-1}+t_k},s\right)}{s^2}ds$$

$$\leq\frac{abcd}{(b-a)(d-c)}\int_{a}^{b}\int_{c}^{d}\frac{f(t,s)}{(ts)^2}dtds$$

$$\leq\frac{d-c}{4ncd}\int_{a}^{b}\frac{[f(t,c)+(t,d)]}{t^2}dt+\frac{b-a}{4nab}\int_{c}^{d}\frac{[(a,s)+f(b,s)]}{s^2}ds$$

$$+\frac{d-c}{2ncd}\sum_{k=1}^{n-1}\int_a^b \frac{f(t,s_k)}{t^2}dt+\frac{b-a}{2nab}\sum_{k=1}^{n-1}\int_c^d \frac{f(t_k,s)}{s^2}ds, \tag{2.13}$$

where $t_k = a + k\frac{b-a}{n}$, $s_k = c + k\frac{d-c}{n}$, $k = 1,2,\ldots,n;\ n \in N$.

Proof. Using Theorem 2.1 for $f_t : [c,d] \subset \mathbb{R}\setminus\{0\} \to \mathbb{R}$, $f_t(s) = f\big((t,s)\big)$

$$\frac{d-c}{ncd}\sum_{k=1}^{n} f_t\left(\frac{2s_ks_{k-1}}{s_{k-1}+s_k}\right) \le \int_c^d \frac{f_t(s)}{s^2}ds \le \frac{d-c}{2ncd}\left[f_t(c)+2\sum_{k=1}^{n-1} f_t(s_k)+f_t(d)\right].$$

Thus

$$\frac{d-c}{ncd}\sum_{k=1}^{n} f\left(t,\frac{2s_ks_{k-1}}{s_{k-1}+s_k}\right) \le \int_c^d \frac{f(t,s)}{s^2}ds$$

$$\le \frac{d-c}{2ncd}\left[f(t,c)+f(t,d)+2\sum_{k=1}^{n-1} f(t,s_k)\right]. \tag{2.14}$$

Integrating all sides of (2.14) on $[a,b]$ and multiplying $\frac{1}{t^2}$, we have

$$\frac{d-c}{ncd}\sum_{k=1}^{n}\int_a^b \frac{f\left(t,\frac{2s_ks_{k-1}}{s_{k-1}+s_k}\right)}{t^2}dt \le \int_a^b\int_c^d \frac{f(t,s)}{(ts)^2}dtds$$

$$\le \frac{d-c}{2ncd}\left[\int_a^b \frac{f(t,c)}{t^2}dt+\int_a^b \frac{f(t,d)}{t^2}dt+2\sum_{k=1}^{n-1}\int_a^b \frac{f(t,s_k)}{t^2}dt\right]. \tag{2.15}$$

Similarly

$$\frac{b-a}{nab}\sum_{k=1}^{n}\int_c^d \frac{f\left(\frac{2t_{k-1}t_k}{t_{k-1}+t_k},s\right)}{s^2}ds \le \int_a^b\int_c^d \frac{f(t,s)}{(ts)^2}dtds$$

$$\le \frac{b-a}{2nab}\left[\int_c^d \frac{f(u_1,s)}{s^2}ds+\int_c^d \frac{f(u_2,s)}{s^2}ds+2\sum_{k=1}^{n-1}\int_c^d \frac{f(t_k,s)}{s^2}ds\right]. \tag{2.16}$$

Adding (2.15) and (2.16), one gets (2.13). That completes the proof.

Remark 2.3. *Under the assumptions of Theorem 2.3, the classical Hermite-Hadamard inequality for harmonically convex functions is obtained on the coordinates and we have the following inequality*

$$\sum_{k=1}^{n} f\left(\frac{2ab}{a+b},\frac{2s_ks_{k-1}}{s_{k-1}+s_k}\right)+\sum_{k=1}^{n} f\left(\frac{2t_{k-1}t_k}{t_{k-1}+t_k},\frac{2cd}{c+d}\right)$$

$$\leq \frac{ncd}{d-c}\int_c^d \frac{f\left(\frac{2ab}{a+b},s\right)}{s^2}ds + \frac{nab}{b-a}\int_a^b \frac{f\left(t,\frac{2cd}{c+d}\right)}{t^2}dt;$$

$$\frac{ncd}{d-c}\int_c^d \frac{[f(a,s)+f(b,s)]}{s^2}ds + \frac{nab}{b-a}\int_a^b \frac{[f(t,c)+f(t,d)]}{t^2}dt$$

$$\leq f(a,c)+f(a,d)+f(b,c)+f(b,d)+\sum_{k=1}^{n-1}[f(a,s_k)+f(b,s_k)+f(t_k,c)+f(t_k,d)].$$

Let us prove the generalized Ostrowki's type inequality for related functions.

Theorem 2.4. *Let* $f\colon \Delta \subset \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}_+$ *be harmonically* convex *on the coordinates. Then the following inequality holds*

$$\int_a^b\int_c^d \frac{f(t,s)}{(ts)^2}dtds$$

$$\leq \frac{b-a}{4nab}\left[(n+1)\int_c^d \frac{f(a,s)}{s^2}ds + +2\sum_{k=1}^{n-1}\int_c^d \frac{f(t_k,s)}{s^2}ds\,(n+1)\int_c^d \frac{f(b,s)}{s^2}ds\right]$$

$$+\frac{d-c}{4ncd}\left[(n+1)\int_a^b \frac{f(t,c)}{t^2}dt + +2\sum_{k=1}^{n-1}\int_a^b \frac{f(t,s_k)}{t^2}dt\,(n+1)\int_a^b \frac{f(t,d)}{t^2}dt\right],$$

where t_k *and* s_k *are defined as in Theorem 2.3.*

Proof. Using Theorem 2.2 for $f_s\colon [a,b] \subset \mathbb{R}\setminus\{0\} \to \mathbb{R}$, $f_s(t) = f((t,s))$ at $t=b$,

$$\int_a^b \frac{f_s(t)}{t^2}dt - \left(\frac{b-a}{2ab}\right)f_s(b) \leq \frac{b-a}{2nab}\left[f_s(a) + 2\sum_{k=1}^{n-1} f_s(t_k) + f_s(b)\right].$$

Thus

$$\int_a^b \frac{f(t,s)}{t^2}dt - \left(\frac{b-a}{2ab}\right)f(b,s) \leq \frac{b-a}{2nab}\left[f(a,s) + 2\sum_{k=1}^{n-1} f(t_k,s) + f(b,s)\right]. \quad (2.17)$$

Integrating all sides of (2.17) on $[c,d]$ and multiplying $\frac{1}{s^2}$, we get

$$\int_a^b\int_c^d \frac{f(t,s)}{(ts)^2}dtds$$

$$\leq \frac{b-a}{2nab}\left[\int_c^d \frac{f(a,s)}{s^2}ds + (1+2n)\int_c^d \frac{f(b,s)}{s^2}ds + 2\sum_{k=1}^{n-1}\int_c^d \frac{f(t_k,s)}{s^2}ds\right]. \quad (2.18)$$

Using Theorem 2.2 for the mapping f_s at $t=a$ and integrating over $[c,d]$

$$\int_a^b\int_c^d \frac{f(t,s)}{(ts)^2}dtds$$
$$\leq \frac{b-a}{2nab}\left[(1+2n)\int_c^d \frac{f(a,s)}{s^2}ds+\int_c^d \frac{f(b,s)}{s^2}ds+2\sum_{k=1}^{n-1}\int_c^d \frac{f(t_k,s)}{s^2}ds\right] \tag{2.19}$$

Using (2.18) and (2.19)

$$\int_a^b\int_c^d \frac{f(t,s)}{(ts)^2}dtds \leq \frac{b-a}{2nab}\left[\begin{array}{c}(n+1)\int_c^d \frac{f(a,s)}{s^2}ds+(n+1)\int_c^d \frac{f(b,s)}{s^2}ds \\ +2\sum_{k=1}^{n-1}\int_c^d \frac{f(t_k,s)}{s^2}ds\end{array}\right]. \tag{2.20}$$

Similarly

$$\int_a^b\int_c^d \frac{f(t,s)}{(ts)^2}dtds \leq \frac{d-c}{2ncd}\left[\begin{array}{c}(n+1)\int_a^b \frac{f(t,c)}{t^2}dt+(n+1)\int_a^b \frac{f(t,d)}{t^2}dt \\ +2\sum_{k=1}^{n-1}\int_{u_1}^{u_2} \frac{f(t,s_k)}{t^2}dt\end{array}\right]. \tag{2.21}$$

The desired inequality is obtained by adding (2.20) and (2.21).

Remark 2.4. *Under the assumptions of Theorem 2.4, then*

$$\int_a^b\int_c^d \frac{f(t,s)}{(ts)^2}dtds \leq \frac{d-c}{2cd}\int_a^b \frac{[f(t,c)+f(t,d)]}{t^2}dt+\frac{b-a}{2ab}\int_c^d \frac{[f(a,s)+f(b,s)]}{s^2}ds;$$

$$\int_a^b\int_c^d \frac{f(t,s)}{(ts)^2}dtds \leq \frac{d-c}{8cd}\int_a^b \frac{\left[3f(t,c)+2f\left(t,\frac{2cd}{c+d}\right)+3f(t,d)\right]}{t^2}dt$$
$$+\frac{b-a}{8ab}\int_c^d \frac{\left[3f(a,s)+2f\left(\frac{2ab}{a+b},s\right)+3f(b,s)\right]}{s^2}ds.$$

3. Acknowledgements

The authors would like to thank to Assoc. Prof. Erhan SET from Ordu University and Assoc. Prof. İmdat İŞCAN from Giresun University owing to their ideas and comments.

4. Conclusion

In this paper, we obtained important generalized inequalities for harmonically functions on the real number line and on the coordinates. One can verify some generalized inequalities for various classes of convex functions.

References

[1] J. Hadamard, *Étude sur les propriétés des fonctions entières et en particulier d.une function considerée par Riemann*, J. Math Pures Appl., 58 (1893), 171-215.

[2] M.W. Alomari, *A generalization of Hermite–Hadamard's inequality*, Kragujevac J. Math., 41 (2) (2017), 313-328.

[3] S.S. Dragomir, *On Hadamards inequality for convex functions on the coordinates in a rectangle from the plane*, Taiwanese J. Math., 4 (2001), 775–788.

[4] E.R. Mwaeze, *Generalized Hermite–Hadamard's inequality for functions convex on the coordinates*. Applied Mathematics E-Notes, 18 (2018), 275-283.

[5] S.S. Dragomir, *Two Mappings in Connection to Hadamard's Inequalities*, Journal of Mathematıcal Analysis and Applications 167 (1992), 49-56.

[6] G.S. Yang, M.C Hong, *A note on Hadamard's inequality*, Tamkang J. Math. 28(1) (1997), 33–37.

[7] X. Gao, *A Note on the Hermite–Hadamard Inequality*, Journal Of Mathematical Inequalities, 4(4) (2010), 587–591.

[8] İ. İşcan, *Hermite-Hadamard Inequalities for Harmonically Convex Functions.* Hacet. J. Math. Stat. 43 (6) (2014), 935-942.

[9] M.A. Noor, KI. Noor and M.U. Awan, *Integral inequalities for coordinated harmonically convex functions*, Complex Var. Elliptic Equat., 60(6) (2015), 776–786.

[10] M.A. Noor, K.I. Noor and M. U. Awan. *Some characterizations of harmonically log-convex functions*. Proc. Jangjeon. Math. Soc., 17(1) (2014), 51-61.

[11] M.A. Noor, K I. Noor, M.U. Awan and S. Costache, *Some integral inequalities for harmonically h-convex functions*. U.P.B. Sci. Bull. Serai A, 77(1) (2015), 5-16.

[12] M. A. Noor, K. I. Noor and S. Iftikhar, *Hermite-Hadamard inequalities for harmonic nonconvex functions*, MAGNT Research Report, 4(1) (2016), 24-40.

[13] J.M. Viloria and M.V. Cortez, *Hermite-Hadamard type inequalities for harmonically convex functions on n-coordinates*, Appl. Math. Inf. Sci. Lett., 6(2) (2018), 1-6.