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Attenuation of the Rayleigh Waves in a Covered Half-space Made of Viscoelastic Materials

Genelleştirilmiş Rayleigh Dalgaların Viskoelastik Tabakalı Yarı-uzay Ortamlarda Sönümlenmesi

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Abstract

This paper investigates the attenuation of the generalized Rayleigh waves propagating in a covered half-space made of viscoelastic materials. Exact equations of motion of the theory of linear viscoelasticity are utilized. The complex dispersion equation is obtained for an arbitrary type of hereditary operator of the viscoelastic materials and a solution algorithm is developed for obtaining numerical results on the attenuation of the waves under consideration. Viscoelasticity of the materials are described through fractional-exponential operators by Rabotnov. Attenuation curves are obtained and discussed for the dispersion curves which are limited by the dispersion curve constructed for the purely elastic cases with instantaneous and long-term values of the elastic constants. According to this discussion, the rules of the studied attenuation and the influence of the rheological parameters of the materials on this attenuation are established. In particular, it is established that a decrease in the values of the creep time of the viscoelastic materials causes an increase in the magnitude of the attenuation coefficient.

Keywords: *Generalized Rayleigh wave, Attenuation, Dispersion, Viscoelastic material, Rheological parameters, Fractional-exponential operator*

Öz

Bu makalede lineer viskoelastik malzemelerden oluşan tabakalı yarı uzay ortamlarda yayılan genelleştirilmiş Rayleigh dalgalarının sönümlenmesi incelmiştir. Araştırmalar doğrusal viskoelastisite teorisinin hareket denklemleri uygulayarak parçalı homojen modeli aracılığıyla ile yapılmıştır. Dispersiyon denklemi, elde edilmiş ve sayısal sonuçlar elde etmek için bir çözüm algoritması geliştirilmiştir. Farklı atenüasyon durumları için katmanların viskozitesinin etkisi, elastik sabitlerle karakterize eden reolojik parametreler aracılığıyla incelenmiştir. Malzemelerin viskoelastisite modelide Rabotnov 'un kısmi eksponansiyel operatörü ’nü kullanılmıştır. Özellikle, reolojik parametrelerin, incelenen dalgalarının atenüasyonunu nasıl etkilediği tespit edilmiştir.

Anahtar Kelimeler: *Genelleştirilmiş Rayleigh dalgası, Sönümlenme, Dispersiyon, Viskoelastik malzeme, Reolojik parametreler, Kısmi eksponansiyel operatörü*

1. Introduction

Viscoelastic Rayleigh wave research actually is very old and is going back to early 1940s. Later on, the study of wave propagation in attenuative materials has been a subject of extensive investigation in the literature. The review of the most related investigations before 1990s can be found in the paper by Carcione [1].

Until now, researchers have mainly used two different mathematical models to study the dispersion and the attenuation behavior of the guided waves in viscoelastic media. In fact, in most cases they either have described the viscoelasticity of the materials by simple spring-dashpot models such as the classical Kelvin-Voigt, Maxwell or some combinations of those models such as the standard linear solid model (see for instance [2-6]) or they have just replaced the real elasticity constants by the complex ones in the stress–strain relations of the viscoelastic materials (see for instance [7-10]). Yet, such simple mathematical models and numerical results obtained therein cannot illustrate the real and complex character of the viscoelastic materials and more importantly the influence of the rheological parameters on the corresponding wave dispersion and attenuation. Meral et al. [11, 12] recent efforts by utilizing fractional order Voigt model are more realistic model for the wave propagation and attenuation problems in viscoelastic media. In this way, by introducing a new rheological parameter, which is in fact, the order of the fractional derivatives, they got results which are agreed more accurately with experiments as compared with conventional models.

Now we consider a brief review of related investigations which are close to the studies carried out in this article. We begin this review with the paper by Carcione [1] which investigated the anelastic characteristics of the Rayleigh waves from the standpoint of balance energy and calculated the quality factors as a function of the frequency and depth. He showed that the viscoelastic properties calculated from energy consideration are consistent with those obtained from the Rayleigh secular equation.

Romeo [13] showed that the secular equation for Rayleigh waves propagating on a viscoelastic half-space always admits only one complex root corresponding to a surface wave. He obtained the roots in terms of complex integrals and showed that the wave solution represents an admissible surface wave for any viscoelastic relaxation kernel compatible with thermodynamics. Lai and Rix [14], based on the Cauchy residue theorem of complex analysis, presented a technique which permits simultaneous determination of the Rayleigh dispersion and attenuation curves for linear viscoelastic media with arbitrary values of material damping ratio. Jousset et al. [15] studied the magma properties and rheology and their impact on low-frequency volcanic earthquakes. They used linear viscoelastic theory and showed that volcanic media can be approximated by a standard linear solid (SLS) for seismic frequencies above 2 Hz. The results demonstrated that attenuation modifies both amplitudes and dispersive characteristics of low-frequency seismic waves. Fan [16] considered the nonlinear damping mechanism of seismic waves by applying the perturbation method and obtained the analytical solution of the Rayleigh wave propagation. Zhang et al. [17] investigated the dispersion of Rayleigh waves in viscoelastic media by applying pseudospectral modelling method to obtain high accuracy. In pseudospectral method the spatial derivatives in the vertical and horizontal directions are calculated using Chebyshev and Fourier difference operators, respectively. Chiriţă et al. [5] studied the propagation of surface waves over an exponentially graded half-space of isotropic Kelvin-Voigt viscoelastic material by means of wave solutions with spatial and temporal finite energy. They showed that when there is just one wave solution it is found to be retrograde at the free surface, while when there is more than one viscoelastic surface wave, one is retrograde and the others are direct at the free surface.

This completes the review of the investigations related to the Rayleigh waves in a viscoelastic half-space. The following concrete conclusions can be made from the foregoing review:

i. The investigations of the Rayleigh waves and their attenuation were carried out either by replacing the real elasticity modulus of the viscoelastic materials by frequency independent (the hysteretic model) or by frequency dependent complex modulus (the Maxwell, Kelvin-Voigt or SLS models) which are obtained from the experiments;

ii. There is no any investigation regarding the investigation of the generalized Rayleigh waves in viscoelastic half-space covered with the elastic or viscoelastic layer.

These considerations led the authors to study the generalized Rayleigh wave dispersion for a viscoelastic covering half-space utilizing more realistic mathematical model [18] by using Rabotnov [19] fractional exponential operator which are already used in the papers [20-25].

Note that the study made in paper [18] in a certain sense, is the extension of the authors previous works [26-28] on dispersion of the generalized Rayleigh waves in an initially stressed elastic covered half-space to viscoelastic cases, where the constitutive relations for the covering layer and the half-space materials are described by the fractional exponential operator by Rabotnov [19]. Nevertheless, in the paper [18], dispersion of Rayleigh waves in a viscoelastic covered half-space is studied for the selected wave attenuation cases determined according to the rules described in [29, 30]. However, up to now there has not been any investigation carried out utilizing the fractional exponential operators by Rabotnov [19] studying the dispersive attenuation of viscoelastic Rayleigh waves for the selected possible dispersion curves to which the present work relates. More precisely, the main goal of the present work is the theoretical investigation of the possible dispersive attenuation of the generalized Rayleigh waves propagating in a covered half-space made of viscoelastic materials in the cases where the constitutive relations of the materials are described through the fractional exponential operator by Rabotnov [19]. Moreover, the investigations carried out in the present work also include the study of the influence of the rheological parameters of the covering layer and the half-space materials on these attenuations of the Rayleigh waves.

The investigations are carried out within the framework of the piecewise homogeneous body model. Exact equations of the linear theory of viscoelasticity are used and it is assumed that perfect interface conditions take place between the covering layer and the half-space. Numerical results and discussions on the influence of the rheological parameters of the viscoelastic materials on the attenuation of the generalized Rayleigh waves propagating in the covered half-space are established. Theoretical results obtained in this study can be used in many engineering practical problems related to wave propagation in viscoelastic layered media, as well as, in many scientific areas such as material sciences, geophysical sciences and earthquake studies and etc.

2. Governing field equations and relations

Consider a covering half-space (Fig. 1) and assume that the thickness of the covering layer is h. The positions of the points we determine with the coordinates in the Cartesian system $Ox_1x_2x_3$ of coordinates associated with the interface plane between the covering layer and half-space. We assume that the plane-strain state in the Ox_1x_2 plane occurs in the considered "covering layer + half-space" system, according to which, the component of the displacement vector in the Ox_3 axis direction is equal to zero. Moreover, we assume that the materials of the constituents of the system are isotropic, homogeneous and hereditary-viscoelastic and the near-surface (or generalized Rayleigh) waves propagate in the positive direction of Ox_1 axis in this system. Below we use the notation with upper indices (1) and (2) to indicate the belonging of the values to the covering layer and half-space respectively.

Thus, we write the governing field equations and relations for the case under consideration under plane-strain state in Ox_1x_2 plane.

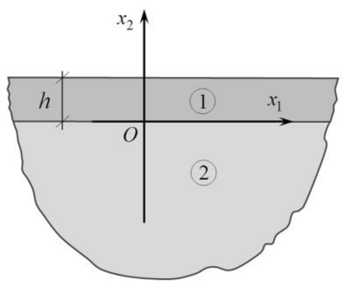

Figure 1. Geometry of the covered half-space

Equations of motion:

$$\frac{\partial \sigma_{11}^{(m)}}{\partial x_1}+\frac{\partial \sigma_{12}^{(m)}}{\partial x_2}=\rho^{(m)}\frac{\partial^2 u_1^{(m)}}{\partial t^2},$$
$$\frac{\partial \sigma_{12}^{(m)}}{\partial x_1}+\frac{\partial \sigma_{22}^{(m)}}{\partial x_2}=\rho^{(m)}\frac{\partial^2 u_2^{(m)}}{\partial t^2},\quad m=1,2. \qquad (1)$$

Constitutive relations and strain-displacement relations:

$$\sigma_{11}^{(m)}=\lambda^{(m)*}\theta^{(m)}+2\mu^{(m)*}\varepsilon_{11}^{(m)};$$
$$\sigma_{22}^{(m)}=\lambda^{(m)*}\theta^{(m)}+2\mu^{(m)*}\varepsilon_{22}^{(m)};$$
$$\sigma_{12}^{(m)}=2\mu^{(m)*}\varepsilon_{12}^{(m)};$$
$$\varepsilon_{11}^{(m)}=\frac{\partial u_1^{(m)}}{\partial x_1};\quad \varepsilon_{22}^{(m)}=\frac{\partial u_2^{(m)}}{\partial x_2}; \qquad (2)$$
$$\varepsilon_{12}^{(m)}=\frac{1}{2}\left(\frac{\partial u_1^{(m)}}{\partial x_2}+\frac{\partial u_2^{(m)}}{\partial x_1}\right);$$
$$\theta^{(m)}=\varepsilon_{11}^{(m)}+\varepsilon_{22}^{(m)};$$

where $\lambda^{(m)*}$, $\mu^{(m)*}$ are the following viscoelastic operators:

$$\begin{Bmatrix}\lambda^{(m)*}\\ \mu^{(m)*}\end{Bmatrix}\varphi(t)=\begin{Bmatrix}\lambda_0^{(m)}\\ \mu_0^{(m)}\end{Bmatrix}\varphi(t)+\int_0^t\begin{Bmatrix}\lambda_1^{(m)}(t-\tau)\\ \mu_1^{(m)}(t-\tau)\end{Bmatrix}\varphi(\tau)d\tau. \qquad (3)$$

In Eq. (3) $\lambda_0^{(m)}$, $\mu_0^{(m)}$ are the instantaneous values of Lame's constants at $t=0$, and $\lambda_1^{(m)}(t)$, $\mu_1^{(m)}(t)$ are the corresponding kernel functions describing the hereditary properties of the m-th materials of the constituents. The other notation used in the equations (1)-(3) is conventional.

According to Fig. 1, we assume that the following boundary and contact conditions on the free face plane of the covering layer and on the interface between the covering layer and half-space satisfy:

Boundary conditions:

$$\sigma_{12}^{(1)}\Big|_{x_2=h}=0,\quad \sigma_{22}^{(1)}\Big|_{x_2=h}=0, \qquad (4)$$

Contact conditions:

$$u_1^{(1)}|_{x_2=0}=u_1^{(2)}|_{x_2=0},$$
$$u_2^{(1)}|_{x_2=0}=u_2^{(2)}|_{x_2=0},$$
$$\sigma_{12}^{(1)}|_{x_2=0}=\sigma_{12}^{(2)}|_{x_2=0},$$
$$\sigma_{22}^{(1)}|_{x_2=0}=\sigma_{22}^{(2)}|_{x_2=0}. \qquad (5)$$

Moreover, the following decay conditions must be satisfied:

$$\sigma_{ij}^{(2)}\Big|_{x_2\to-\infty}\to 0,\quad u_i^{(2)}\Big|_{x_2\to-\infty}\to 0. \qquad (6)$$

This completes the consideration of the governing field equations and relations within the framework of which the present investigation is carried out.

3. Solution of the field equations and obtaining the dispersion equation

As we consider the harmonic waves propagating in Ox_1 direction, therefore we can use the factor $e^{i(kx_1-\omega t)}$ (where k is the wavenumber and ω is the circular frequency) for presentation of the components of the displacement vector and strain tensor as follows:

$$u_i^{(m)}=v_i^{(m)}(x_2)e^{i(kx_1-\omega t)},$$
$$\varepsilon_{ij}^{(m)}=\gamma_{ij}^{(m)}(x_2)e^{i(kx_1-\omega t)}, \qquad (7)$$
$$\gamma_{ij}^{(m)}=\frac{1}{2}\left(\frac{\partial v_i^{(m)}}{\partial x_j}+\frac{\partial v_j^{(m)}}{\partial x_i}\right),\quad i,j=1,2.$$

Now using the relation,

$$\int_0^t f_1(t-\tau)f_2(\tau)d\tau \approx \int_{-\infty}^t f_1(t-\tau)f_2(\tau)d\tau, \qquad (8)$$

and using the transformation $t-\tau=s$, we can do the following manipulations for the integrals in Eq. (3),

$$\begin{aligned}
&\int_{-\infty}^{t}\lambda_1^{(m)}(t-\tau)e^{-i\omega\tau}d\tau\\
&\qquad=-\int_{\infty}^{0}\lambda_1^{(m)}(s)e^{-i\omega t}e^{i\omega s}ds\\
&\qquad=e^{-i\omega t}\int_{0}^{\infty}\lambda_1^{(m)}(s)e^{i\omega s}ds\\
&\qquad=e^{-i\omega t}\left(\lambda_{1c}^{(m)}+i\lambda_{1s}^{(m)}\right),
\end{aligned} \tag{9}$$

and in a similar way,

$$\begin{aligned}
&\int_{-\infty}^{t}\mu_1^{(m)}(t-\tau)e^{-i\omega\tau}d\tau\\
&\qquad=e^{-i\omega t}\left(\mu_{1c}^{(m)}+i\mu_{1s}^{(m)}\right),
\end{aligned} \tag{10}$$

where

$$\begin{aligned}
\lambda_{1c}^{(m)}&=\int_{0}^{\infty}\lambda_1^{(m)}(s)\cos(\omega s)ds;\\
\lambda_{1s}^{(m)}&=\int_{0}^{\infty}\lambda_1^{(m)}(s)\sin(\omega s)ds;\\
\mu_{1c}^{(m)}&=\int_{0}^{\infty}\mu_1^{(m)}(s)\cos(\omega s)ds;\\
\mu_{1s}^{(m)}&=\int_{0}^{\infty}\mu_1^{(m)}(s)\sin(\omega s)ds.
\end{aligned} \tag{11}$$

Taking the relations (9)-(11) into consideration, finally we obtain the following expressions for the stresses from the equations (2) and (3):

$$\begin{aligned}
\sigma_{11}^{(m)}&=\Big[\Lambda^{(m)}(\omega)\vartheta^{(m)}(x_2)\\
&\quad+2M^{(m)}(\omega)\gamma_{11}^{(m)}(x_2)\Big]e^{i(kx_1-\omega t)},\\
\sigma_{22}^{(m)}&=\Big[\Lambda^{(m)}(\omega)\vartheta^{(m)}(x_2)\\
&\quad+2M^{(m)}(\omega)\gamma_{22}^{(m)}(x_2)\Big]e^{i(kx_1-\omega t)},\\
\sigma_{12}^{(m)}&=2M^{(m)}(\omega)\gamma_{12}^{(m)}(x_2)e^{i(kx_1-\omega t)}
\end{aligned} \tag{12}$$

where

$$\begin{aligned}
\Lambda^{(m)}(\omega)&=\lambda_0^{(m)}+\lambda_{1c}^{(m)}(\omega)+i\lambda_{1s}^{(m)}(\omega),\\
M^{(m)}(\omega)&=\mu_0^{(m)}+\mu_{1c}^{(m)}(\omega)+i\mu_{1s}^{(m)}(\omega).
\end{aligned} \tag{13}$$

In this way, instead of the Lame constants in the relations (2) and (3) we obtain the complex modulus $\Lambda^{(m)}(\omega)$, $M^{(m)}(\omega)$, where the real and imaginary parts are determined through the expressions (11) and (13). In other words, for the case under consideration the complete system of the field equations for the viscoelastic medium, can also be obtained from the corresponding ones for the purely elastic system just by replacing the elastic Lame constants $\lambda_0^{(m)}$ and $\mu_0^{(m)}$ with the complex modulus $\Lambda^{(m)}(\omega)$ and $M^{(m)}(\omega)$, respectively.

Thus, according to (7), (12) and (13), we obtain the following equations of motion in terms of the displacement amplitudes from the equation of motion (1):

$$\begin{aligned}
&M^{(m)}\frac{d^2v_1^{(m)}}{dx_2^2}+\left(\Lambda^{(m)}+M^{(m)}\right)ik\frac{dv_2^{(m)}}{dx_2}\\
&\quad+\left(-\Lambda^{(m)}-2M^{(m)}+\frac{\omega^2}{k^2}\rho^{(m)}\right)k^2v_1^{(m)}=0,\\
&\left(\Lambda^{(m)}+2M^{(m)}\right)\frac{d^2v_2^{(m)}}{dx_2^2}\\
&\qquad+\left(\Lambda^{(m)}+M^{(m)}\right)ik\frac{dv_1^{(m)}}{dx_2}\\
&\qquad+\left(-M^{(m)}+\frac{\omega^2}{k^2}\rho^{(m)}\right)k^2v_2^{(m)}=0.
\end{aligned} \tag{14}$$

After some mathematical operations, we derive the following equation for $v_2^{(m)}$:

$$\begin{aligned}
&\frac{d^4v_2^{(m)}}{d(kx_2)^4}+B_2^{(m)}\frac{d^2v_2^{(m)}}{d(kx_2)^2}+C_2^{(m)}v_2^{(m)}=0,\\
&B_2^{(m)}=b_{22}^{(m)}+b_{21}^{(m)}-c_{21}^{(m)}c_{22}^{(m)},\\
&C_2^{(m)}=b_{21}^{(m)}b_{22}^{(m)},\\
&b_{21}^{(m)}=\frac{-\Lambda^{(m)}-2M^{(m)}+\frac{\omega^2}{k^2}\rho^{(m)}}{M^{(m)}},\\
&c_{21}^{(m)}=\frac{\left(\Lambda^{(m)}+M^{(m)}\right)i}{M^{(m)}},\\
&b_{22}^{(m)}=\frac{-M^{(m)}+\frac{\omega^2}{k^2}\rho^{(m)}}{\Lambda^{(m)}+2M^{(m)}},\\
&c_{22}^{(m)}=\frac{\left(\Lambda^{(m)}+M^{(m)}\right)i}{\Lambda^{(m)}+2M^{(m)}}.
\end{aligned} \tag{15}$$

The general solution of equation (15) for the *m*-th layer can be written as:

$$
\begin{aligned}
v_2^{(1)}(x_2) = & Z_1^{(1)}\exp\left(R_1^{(1)}kx_2\right) \\
& + Z_2^{(1)}\exp\left(-R_1^{(1)}kx_2\right) \\
& + Z_3^{(1)}\exp\left(R_2^{(1)}kx_2\right) \\
& + Z_4^{(1)}\exp\left(-R_2^{(1)}kx_2\right), \qquad (16) \\
v_2^{(2)}(x_2) = & Z_1^{(2)}\exp\left(R_1^{(2)}kx_2\right) \\
& + Z_3^{(2)}\exp\left(R_2^{(2)}kx_2\right).
\end{aligned}
$$

where

$$
R_1^{(m)} = \sqrt{-\frac{B_2^{(m)}}{2} + \sqrt{\frac{\left(B_2^{(m)}\right)^2}{4} - C_2^{(m)}}}, \qquad (17)
$$

$$
R_2^{(m)} = \sqrt{-\frac{B_2^{(m)}}{2} - \sqrt{\frac{\left(B_2^{(m)}\right)^2}{4} - C_2^{(m)}}}.
$$

In a similar way we can also determine the function $v_1^{(m)}(x_2)$ from Eq. (14). Note that, as we consider the surface waves, according to the decay conditions in Eq.(6) both $\mathrm{Re}(R_1^{(2)}k) < 0$ and $\mathrm{Re}(R_2^{(2)}k) < 0$ inequalities must be satisfied simultaneously.

Finally, using the expressions (16) and Eq. (7) and (2) we obtain the following dispersion equation from the boundary and contact conditions (4) and (5):

$$
\det\left\|\alpha_{ij}\right\| = 0, \quad i; j = 1, 2, \dots, 6. \qquad (18)
$$

The explicit expressions of the components of the matrix (α_{ij}) are given in Appendix A through the expressions $(A1)$.

This completes the consideration to the solution to the field equations and obtaining the dispersion equation (18).

4. Numerical results and discussions

4.1 The selection of the viscoelastic operators and the determination of the dimensionless rheological parameters

Solving the dispersion equation (18) requires given values of $\lambda_{1c}^{(m)}$, $\lambda_{1s}^{(m)}$, $\mu_{1c}^{(m)}$ and $\mu_{1s}^{(m)}$ determined through the expressions (11) by the kernel functions $\lambda_1^{(m)}$, and $\mu_1^{(m)}$ of the viscoelastic operators. We recall that these operators (3) describe the viscoelastic properties of the m−th material. Consequently, for determination of the quantities $\lambda_{1c}^{(m)}$, $\lambda_{1s}^{(m)}$, $\mu_{1c}^{(m)}$ and $\mu_{1s}^{(m)}$ it is necessary to have explicit expressions for the kernel functions $\lambda_1^{(m)}(t)$ and $\mu_1^{(m)}(t)$. Here we describe the viscoelasticity of the materials of the constituents through the fractional exponential operator by Rabotnov [19], i.e. we assume that

$$
\begin{aligned}
\lambda^{(m)*}\varphi(t) = \lambda_0^{(m)} & \left[\varphi(t) - \frac{\beta_0^{(m)}}{(1+\nu_0^{(m)})}\right. \\
& \left. R_{\alpha^{(m)}}^{(m)*}\left(-\frac{3\beta_0^{(m)}}{2(1+\nu_0^{(m)})} - \beta_\infty^{(m)}\right)\varphi(t)\right], \\
\mu^{(m)*}\varphi(t) = \mu_0^{(m)} & \left[\varphi(t) - \frac{3\beta_0^{(m)}}{2(1+\nu_0^{(m)})}\right. \\
& \left. R_{\alpha^{(m)}}^{(m)*}\left(-\frac{3\beta_0^{(m)}}{2(1+\nu_0^{(m)})} - \beta_\infty^{(m)}\right)\varphi(t)\right],
\end{aligned} \qquad (19)
$$

where

$$
\begin{aligned}
& R_{\alpha^{(m)}}^{(m)*}\left(x^{(m)}\right)\varphi(t) = \\
& \qquad \int_0^t R_{\alpha^{(m)}}^{(m)}\left(x^{(m)}, t-\tau\right)\varphi(\tau)d\tau, \\
& R_{\alpha^{(m)}}^{(m)}\left(x^{(m)}, t\right)\varphi(t) = \\
& \qquad t^{-\alpha^{(m)}}\sum_{n=0}^{\infty}\frac{(x^{(m)})^n t^{n(1-\alpha^{(m)})}}{\Gamma((1+n)(1-\alpha^{(m)}))}.
\end{aligned} \qquad (20)
$$

Here $\Gamma(x)$ is the Gamma function and the constants $\alpha^{(m)}$, $\beta_0^{(m)}$ and $\beta_\infty^{(m)}$ are the rheological parameters of the m−th viscoelastic material. The mechanical meanings of these rheological parameters are explained in the papers by Akbarov [20] and Akbarov and Kepceler [22].

Thus, using the relations (11) and (20) we obtain:

$$\lambda_c^{(m)} = \lambda_0^{(m)}\left[1 - \frac{1}{(1+\nu_0^{(m)})}\left(d^{(m)} + \frac{3\beta_0^{(m)}}{2(1+\nu_0^{(m)})}\right)^{-1} R_{\alpha^{(m)}c}^{(m)}(-\beta_{01}^{(m)} - \beta_\infty^{(m)}, \omega)\right],$$

$$\lambda_s^{(m)} = \lambda_0^{(m)}\frac{1}{(1+\nu_0^{(m)})}\left(d^{(m)} + \frac{3\beta_0^{(m)}}{2(1+\nu_0^{(m)})}\right)^{-1} R_{\alpha^{(m)}s}^{(m)}(-\beta_{01}^{(m)} - \beta_\infty^{(m)}, \omega),$$

$$\mu_c^{(m)} = \mu_0^{(m)}\left[1 - \frac{3}{2(1+\nu_0^{(m)})}\left(d^{(m)} + \frac{3\beta_0^{(m)}}{2(1+\nu_0^{(m)})}\right)^{-1} R_{\alpha^{(m)}c}^{(m)}(-\beta_{01}^{(m)} - \beta_\infty^{(m)}, \omega)\right],$$

$$\mu_s^{(m)} = -\mu_0^{(m)}\frac{3}{2(1+\nu_0^{(m)})}\left(d^{(m)} + \frac{3\beta_0^{(m)}}{2(1+\nu_0^{(m)})}\right)^{-1} R_{\alpha^{(m)}s}^{(m)}(-\beta_{01}^{(m)} - \beta_\infty^{(m)}, \omega), \tag{21}$$

where

$$R_{\alpha^{(m)}c}^{(m)} = \frac{(\xi^{(m)})^2 + \xi^{(m)}\sin\frac{\pi\alpha^{(m)}}{2}}{(\xi^{(m)})^2 + 2\xi^{(m)}\sin\frac{\pi\alpha^{(m)}}{2} + 1},$$

$$R_{\alpha^{(m)}s}^{(m)} = \frac{\xi^{(m)}\cos\frac{\pi\alpha^{(m)}}{2}}{(\xi^{(m)})^2 + 2\xi^{(m)}\sin\frac{\pi\alpha^{(m)}}{2} + 1}, \tag{22}$$

$$\xi^{(m)} = \left(Q^{(m)}\Omega\right)^{\alpha^{(m)}-1}, \quad \Omega = k_1 h\frac{c}{c_2^{(1)}}. \tag{23}$$

At the same time, as in the papers [20, 22] introducing the following dimensionless rheological parameters:

$$d^{(m)} = \frac{\beta_\infty^{(m)}}{\beta_0^{(m)}},\ Q^{(m)} = \frac{\beta_\infty^{(m)}}{R\left(\beta_0^{(m)} + \beta_\infty^{(m)}\right)^{\frac{1}{1-\alpha^{(m)}}}}, \tag{24}$$

the following expressions can be obtained for the long-term values of the mechanical constants $\lambda_c^{(m)}$, $\lambda_s^{(m)}$, $\mu_c^{(m)}$ and $\mu_s^{(m)}$ which enter into the equation (15):

$$\lambda_\infty^{(m)} = \lim_{t\to\infty}(\lambda^{(m)*}\cdot 1) = \lambda_0^{(m)}\left[1 + \frac{(1-2\nu_0^{(m)})}{2\nu_0^{(m)}(1+\nu_0^{(m)})}\frac{1}{(3/(2(1+\nu_0^{(m)})) + d^{(m)}}\right],$$

$$\mu_\infty^{(m)} = \lim_{t\to\infty}(\mu^{(m)*}\cdot 1) = \mu_0^{(m)}\left[1 + 1 - \frac{3}{2(1+\nu_0^{(m)})}\frac{1}{(3/(2(1+\nu_0^{(m)})) + d^{(m)}}\right]. \tag{25}$$

The dimensionless rheological parameter $d^{(m)}$ in (24) characterizes the long-term value of the elastic constants, the parameter $Q^{(m)}$ characterizes the creep time, and the rheological parameter $\alpha^{(m)}$ characterizes the form of the creep (or relaxation) function of the m-th viscoelastic material at the beginning region of deformation. Note that the case where $\alpha^{(m)} = 0$ corresponds to the 'standard viscoelastic body' model (or the model by Kelvin). Consequently, according to the above expressions, the effect of the viscoelasticity of the m-th material on the attenuation curves will be estimated through these three dimensionless rheological parameters.

This completes the selection of the viscoelastic operators and dimensionless viscoelastic operators.

4.2 Algorithm for determination of the attenuation curves

As we consider the time harmonic wave propagation in a viscoelastic material, therefore it must be assumed that the wave number k is a complex one and can be presented as:

$$k = k_1 + ik_2 = k_1(1+i\beta), \quad \beta = \frac{k_2}{k_1}. \tag{26}$$

Here, the imaginary part k_2 of the wave number k (or parameter β which is called the coefficient of the attenuation) defines the attenuation of the wave amplitude under consideration. Note that we determine the phase velocity of the studied waves through the expression:

$$c = \omega / k_1. \tag{27}$$

Considering the relations (22)-(24) and according to the known physico-mechanical considerations, it can be predicted that in the case where $\left(Q^{(m)}\Omega\right) \ll 1$, the behavior of the viscoelastic system must be very close to the corresponding purely elastic case with long-term values of the elastic constants at $t=\infty$. Also, according to the physico-mechanical considerations, it can be predicted that in the case $\left(Q^{(m)}\Omega\right) \gg 1$, its behavior must be very close to the corresponding purely elastic system with instantaneous values of the elastic constants at $t=0$. Thus, according to the statements, it can be predicted that increasing the values of the parameters $Q^{(m)}$ and $d^{(m)}$ correspond to decreasing the viscous part of the viscoelastic deformations in the constituents. Consequently, by decreasing the values of the rheological parameters $Q^{(m)}$ and $d^{(m)}$ we increase the effect of the material viscosity on the dispersion curves.

Regarding the solution of the dispersion equation (18), since the values of the determinant obtained in (18) are complex, therefore the dispersion equation can be reduced to the following form

$$\left| \det\left\|\alpha_{ij}\right\| \right| = 0, \qquad (28)$$

where $\left|\det\left\|\alpha_{ij}\right\|\right|$ means the modulus of the complex number $\det\left\|\alpha_{ij}\right\|$. Consequently for construction of the attenuation or dispersion curves for the selected parameters of the problem it is necessary to solve numerically the equation (28).

For more clarity of the features of the solution procedure to the dispersion equation related to the viscoelastic case we first recall the features for the purely elastic case:

(a) The dispersion equation contains only two unknowns: C and k_1h, where for each possible selected value of k_1h the values of the velocity C are determined through the solution to this equation;

(b) This solution procedure is carried out by employing the well-known numerical methods such as bi-section method which is based on the sign change of the dispersion determinant.

However in the viscoelastic case the above-noted features (a) and (b) change to the following ones:

(c) The dispersion equation contains three unknowns: C, k_1h and β;

(d) The sign of the dispersion determinant does not change.

Consequently, according to the feature (c), in the viscoelastic case, the values of two unknowns must be given in advance to determine the values of the remained third one from the dispersion equation. If the selected two unknowns are k_1h and β, then we can determine the wave propagation velocity C as a result of the solution to the dispersion equation. Note that this approach was already made in the papers by Akbarov and Negin [28], Akbarov and Kepceler [22], Akbarov et al. [23, 24] under which the wave attenuation coefficient β was determined according to the expressions given in the references Ewing et al. [29] and Kolsky [30]. However, if the selected two unknowns are C and k_1h, then we can determine the attenuation coefficient β as a result of the solution to the dispersion equation, which is made in the present paper and the attenuation curves are determined. The latter approach was also made in the paper by Barshinger and Rose [7] and Kocal and Akbarov [25].

According to the feature (d), as in the viscoelastic case $\left|\det\left\|\alpha_{ij}\right\|\right| \geq 0$, we cannot employ the aforementioned algorithm based on for example the bi-section method. Therefore, in the viscoelastic case we use the algorithm which is based on direct calculation of the values of the moduli of the dispersion determinant $\det\left\|\alpha_{ij}\right\|$ and the sought roots are determined from the criterion $\left|\det\left\|\alpha_{ij}\right\|\right| \leq 10^{-9}$.

Thus, in the present paper we investigate the attenuation of the generalized Rayleigh waves within the scope of the foregoing algorithm. It should be noted that (see for instance, the paper by Sharma [31]) there is no general method for finding the complex roots of the transcendental secular equations. At the same time, it is known that the functional iteration method detailed by Sharma [31] can be applied for determination of the complex roots of an analytical function. Namely, this method is employed for solution of the complex roots of the corresponding secular equations in the papers Sharma [32], Sharma and Othman [33], Kumar and Parter [34], Sharma and Kumar [35], Sharma et al. [36] and others listed therein. However, under application the functional iteration method the secular equation is reduced to the corresponding algebraic equations within the scope of certain assumptions and the application of this method requires the successful selection of the initial iteration. The aforementioned requirements are the disadvantages of the functional iteration method. Nevertheless, this method allows simultaneously determine the real and imaginary parts of the complex roots of the secular equations, which is the advantage of this method.

However, the algorithm used in the present paper and detailed above allows us to determine only the real or only the imaginary parts of the complex roots of the secular equation, which is the disadvantage of that. At the same time, the application of the present algorithm does not require to reduce the secular equation to the corresponding algebraic equation and the successful selection of the initial iteration, which are the advantages of the present algorithm.

4.3 Concrete numerical results and their discussions

Now we consider numerical results related to the attenuation curves which are obtained within the scope of the following assumptions $\nu_0^{(1)} = \nu_0^{(2)} = 0.3,\ \rho^{(1)} = \rho^{(2)}$ and $c_2^{(2)} / c_2^{(1)} = \sqrt{\mu_0^{(2)} / \mu_0^{(1)}}$ in the cases where $\mu_0^{(2)} / \mu_0^{(1)} = 2$. We suppose that the viscoelasticity properties of the covering layer are the half-space are the same, i.e. we suppose that $Q^{(1)} = Q^{(2)}(= Q)$, $d^{(1)} = d^{(2)}(= d)$, $\alpha^{(1)} = \alpha^{(2)}(= \alpha)$. Furthermore, throughout the numerical investigation carried out in the present paper it is assumed that $\alpha = 0.5$.

Note that the numerical results detailed in the paper Akbarov and Negin [28], as well as many other ones which are not given here, show that the dispersion curves obtained for the corresponding purely elastic cases with instantaneous and long-term values of the elastic constants can be taken as the lower and upper limit cases for the dispersion curves obtained for the considered viscoelastic case. This statement allows us namely to select admissible dispersion curves and corresponding wave propagation velocity c for the viscoelastic case. Then using these curves we can find the corresponding attenuation coefficient β from the solution of the dispersion equation (28) for each fixed value of the dimensionless wavenumber.

Thus, first we construct dispersion curves related to the purely elastic case with instantaneous and long-term values of the elastic constants of the materials of the covering layer and of the half-space. These dispersion curves are illustrated in Fig. 2 with dashed lines. In this way, according to the discussions made above, after construction of the dispersion curves related to the purely elastic cases, now we can select the admissible dispersion curves related to the different viscoelastic cases. For example, we can take the dispersion curves shown in Fig. 2 with solid lines which are numbered as 1 to 5 from the dispersion curve constructed at $t = \infty$ the dispersion curve constructed at $t = 0$.

After the above preparation, now we choose values for the dimensionless wavenumber $k_1 h$ and the wave propagation velocity $c / c_2^{(1)}$ according to the admissible dispersion curves indicated in Fig. 2. For example, if we take the dispersion curve indicated by number 1 in Fig. 2 as an admissible dispersion curve, then for the given wavenumber $k_1 h$ the values of wave propagation velocities $c/c_2^{(1)}$ are determined from this curve. After this determination, finally, we calculate the attenuation coefficient β from the dispersion equation (28) given the values for the rheological parameters Q, d and α we construct the attenuation curves which will be discussed below. In other words, first, the wave propagation velocity $c/c_2^{(1)}$ is chosen from the corresponding admissible dispersion curves (Fig. 2) for the selected value of wavenumber k_1h and then the unknown attenuation coefficient β is determined numerically from the solution to the dispersion equation (28).

Thus, by employing the above solution procedure we found the attenuation curves given in Figs. 3, 4, 5, 6 and 7 which are constructed under various values of the parameter Q for the dispersion curves indicated by numbers 1, 2, 3, 4 and 5 in Fig. 2, respectively, in the case where $d^{(1)} = d^{(2)} = 25$. Note that for more illustration of the influence of the rheological parameter d on the attenuation curves, these curves are also constructed for the cases where $d^{(1)} = d^{(2)} = 5$ and $d^{(1)} = d^{(2)} = 50$. However, the admissible dispersion curves and the attenuation curves related to these cases are not given here for reducing of the paper volume.

The concrete conclusions followed from the foregoing numerical results are given in the next section.

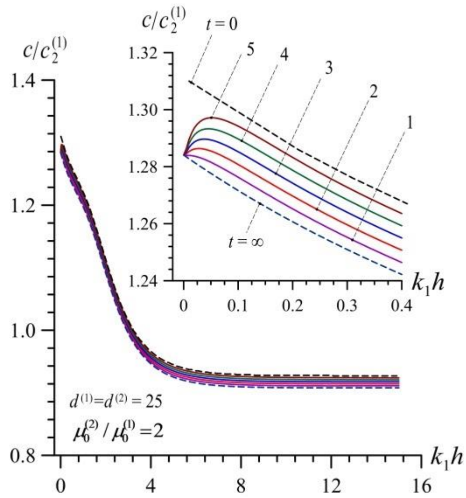

Figure 2. Selected dispersion curves for the case where $d^{(1)} = d^{(2)} = 25$

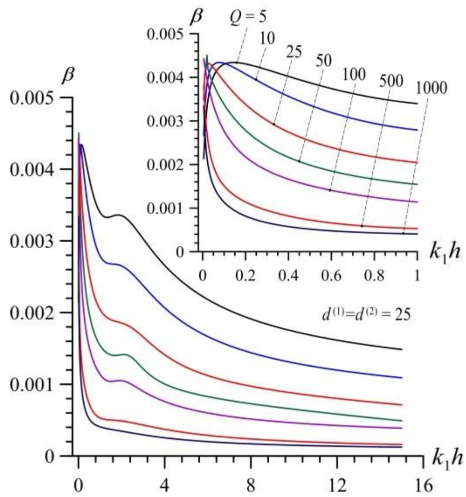

Figure 3. Attenuation of the curve indicated by number 1 in Fig. 2 for various values of the rheological parameter Q

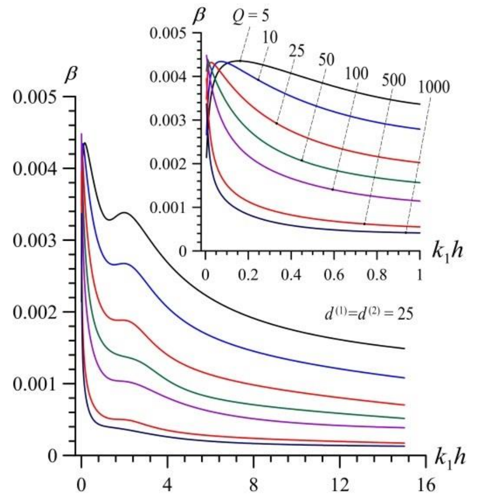

Figure 4. Attenuation of the curve indicated by number 2 in Fig. 2 for various values of the rheological parameter Q

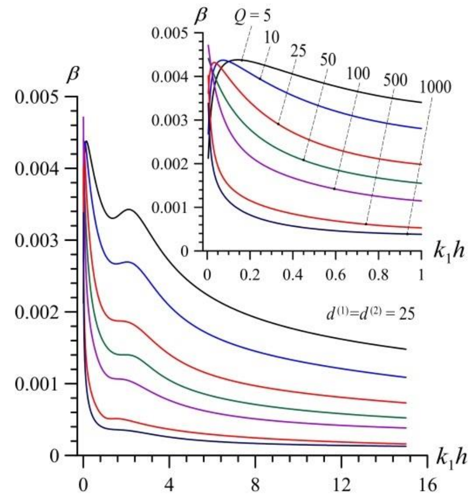

Figure 6. Attenuation of the curve indicated by number 4 in Fig. 2 for various values of the rheological parameter Q

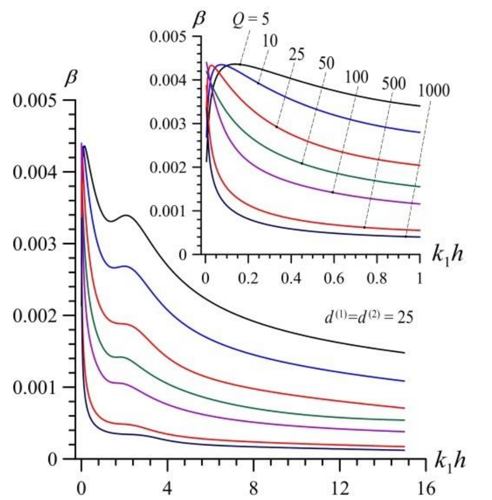

Figure 5. Attenuation of the curve indicated by number 3 in Fig. 2 for various values of the rheological parameter Q

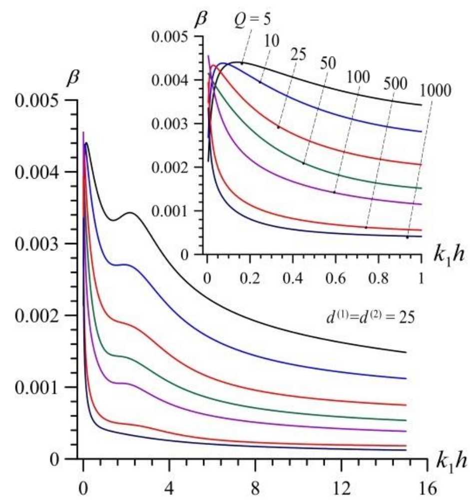

Figure 7. Attenuation of the curve indicated by number 5 in Fig. 2 for various values of the rheological parameter Q

5. Conclusions

We proposed an approach for determination of the attenuation coefficient of the generalized Rayleigh waves propagating in a covered half-space made of viscoelastic materials. The approach is based on selection of the admissible dispersion curves of the generalized Rayleigh wave which can propagate in the viscoelastic covered half-space under consideration. The investigations are made within the scope of the exact equations of motion of the theory of linear viscoelasticity. The constitutive relations of the viscoelastic materials of the both covering layer and the half-space are described through the fractional exponential operators by Rabotnov and three dimensionless rheological parameters are introduced and through these parameters the influence of the viscosity of the covered half-space materials on the attenuation curves is studied. The numerical results related to these curves are presented. According to analyses of these results, the following concrete conclusions can be drawn:

- An increase in the values of the rheological parameters d and Q causes a decrease in the values of the attenuation coefficient;
- Considerable values of the attenuation coefficient are obtained for the low wavenumber cases;
- The decreasing rate of the attenuation coefficient increase with the rheological parameter Q;
- After a certain value of the dimensionless wavenumber (denote it by $(k_1h)^*$) the values of the attenuation coefficient decrease monotonically with the k_1h ;
- The value of the $(k_1h)^*$ depends on the rheological parameters d and Q;
- The influence of the “distance” from the selected dispersion curves from the limit ones on the attenuation coefficient is insignificant.

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