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ON GENERIC SUBMANIFOLD OF SASAKIAN MANIFOLD WITH CONCURRENT VECTOR FIELD

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ABSTRACT. In the present paper, we deal with the generic submanifold admitting a Ricci soliton in Sasakian manifold endowed with concurrent vector field. Here, we find that there exists never any concurrent vector field on the invariant distribution D of generic submanifold M. Also, we provide a necessary and sufficient condition for which the invariant distribution D and anti-invariant distribution $D^{\perp}$ of M are Einstein. Finally, we give a characterization for a generic submanifold of Sasakian manifold to be a gradient Ricci soliton.

1. Introduction

The first study on semi-invariant submanifold (as a generalization both invariant and anti-invariant submanifolds) of Sasakian manifold was given by Bejancu and Papaghiuc in [5]. They obtained integrability conditions for the distributions on semi-invariant submanifold. Similar to this work, in 2011, Alegre gave some characterizations for the distributions on semi-invariant submanifold of a Lorentzian Sasakian manifold [1]. Later, Atçeken and Uddin obtained some necessary and sufficient conditions for a semi-invariant submanifold to be invariant, anti-invariant and semi-invariant product in [2]. For more details ([11], [17], [25]).

The concept of a Ricci soliton in Riemannian geometry was defined by Hamilton as a self-similar solution of the Ricci flow in 1988 [14]. Since then, this concept has been studied in many field of the manifold theory by several mahematicians. For instance, Sharma firstly applied Ricci solitons to K-contact manifolds of the contact geometry in [23]. In addition to this study, Tripathi proved that a non-Sasakian $(k,\mu)-$manifold endowed with compact Ricci soliton is $3-$dimensional and flat (see [26]). Also, Ghosh showed that if a compact $K-$contact metric is a gradient Ricci almost soliton, it is isometric to a unit sphere S^{2n+1} in [12]. Moreover, there are many works related to Ricci solitons in 3-dimensional contact and paracontact metric manifolds in the recent years. Sharma and Ghosh showed that

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Sasakian 3−manifold as a Ricci soliton represents the Heisenberg group in [24]. Moreover, Ghosh proved that the constant curvature of a Kenmotsu 3−manifold as Ricci soliton is −1 in [13]. Later, Perktaş and Keleş proved that if a 3− dimensional normal almost paracontact metric manifold admits a Ricci soliton, then it is shrinking (for details, see [21]). Ricci solitons have been studied in some different classes of contact geometry ([3], [4], [15], [16], [18], [20]).

Let $(\tilde{M}, g)$ be a Riemannian manifold and $\tilde{S}$ be the Ricci tensor of $\tilde{M}$. On such a manifold, if the following equation is satisfied

$$(\pounds_V g)(X,Y) + 2\tilde{S}(X,Y) + 2\lambda g(X,Y) = 0, \tag{1}$$

then this manifold is said to be a Ricci soliton which is as known quasi-Einstein metric in Physics literature. Here, $\pounds_V g$ is the Lie-derivative of the metric tensor g in the direction vector field V, which is called the *potential vector field* of the Ricci soliton, λ is a constant and X, Y are arbitrary vector fields on $\tilde{M}$. A Ricci soliton is denoted by $(\tilde{M}, g, V, \lambda)$. If $\pounds_V g = 0$, then potential vector field V is called Killing. Also, if $\pounds_V g = \rho g$, then vector field V is called conformal Killing, where ρ is a function. In relation (1), if V is zero or Killing, then the Ricci soliton is called trivial and in this case, the metric g is an Einstein. So, a Ricci soliton is viewed as a generalization of Einstein metric. Also, if the potential vector field V is the gradient of a potential function $-f$ (i.e., $V = -\nabla f$), then it is called a gradient Ricci soliton. In addition, a Ricci soliton is said to have shrinking, steady or expanding depending on $\lambda < 0, \lambda = 0$ or $\lambda > 0$, respectively.

On the other hand, there has been several papers on Riemannian manifolds which admits concurrent vector field. For instance, Chen and Deshmukh showed that there do not exist steady or expanding Ricci solitons with concurrent vector fields in [9]. Also, every Ricci soliton with torqued potential field is an almost quasi-Einstein under some conditions was proved by Chen in [10]. Then, many papers have been published on this topic ([19], [22], [27], [28]).

The present paper is organized as follows. In section 2, we give some basic definitions, notations and formulas of almost contact metric manifolds. In section 3, we deal with the generic submanifolds admitting a Ricci soliton in a Sasakian manifold with concurrent vector field. In section 4, we study the generic semi-invariant product admitting a Ricci soliton of the ambient manifold. Also, we give a characterization for a generic submanifold of Sasakian manifold to be a gradient Ricci soliton.

2. Preliminaries

We shall give a brief review of several fundamental notions and formulas of submanifolds of Sasakian manifolds from [7], [5], [8] and [30], for later using.

Let (φ, ξ, η, g) be an almost contact metric structure on a $(2n+1)-$ dimensional almost contact metric manifold $\tilde{M}$ such that φ is a tensor field of type $(1,1)$, ξ is a vector field (called the characteristic vector field) of type $(0,1)$, $1-$ form η is a tensor field of type $(1,0)$ on $\tilde{M}$ and the Riemannian metric g satisfies the following relations :

$$\varphi^2 X = -X + \eta(X)\xi, \quad \eta(\xi) = 1, \quad \varphi\xi = 0, \quad \eta \circ \varphi = 0, \quad \eta(X) = g(X, \xi) \tag{2}$$

and

$$g(\varphi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \quad g(\varphi X, Y) = -g(X, \varphi Y) \tag{3}$$

for all vector fields X, Y tangent to $\tilde{M}$.

An almost contact metric manifold $(\tilde{M}, \varphi, \xi, \eta, g)$ is said to be a Sasakian manifold if and only if the following condition is satisfied

$$(\tilde{\nabla}_X \varphi)Y = g(X, Y)\xi - \eta(Y)X, \tag{4}$$

where $\tilde{\nabla}$ is the Levi-Civita connection on $\tilde{M}$ with respect to the Riemannian metric g, for any $X, Y \in \Gamma(T\tilde{M})$. From (4), for a Sasakian manifold we also have

$$\tilde{\nabla}_X \xi = -\varphi X. \tag{5}$$

Let M be isometrically immersed submanifold of Sasakian manifold $\tilde{M}$. For any $X, Y \in \Gamma(TM)$, one has

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X, Y), \tag{6}$$

where $\tilde{\nabla}$ and ∇ denote the Levi-Civita connections of $\tilde{M}$ and M, respectively. Then, the equality (6) is called the Gauss formula and h is called the second fundamental form of M. Similarly, for any $U \in \Gamma(TM)$ and $V \in \Gamma(TM^\perp)$, we have

$$\tilde{\nabla}_U V = -A_V U + \nabla^\perp_U V, \tag{7}$$

where A_V and $\nabla^\perp$ denote the shape operator and the normal connection of M in the ambient space $\tilde{M}$, respectively. It is well known that the relation between second fundamental form h and the shape operator A_V are given by

$$g(A_V X, Y) = g(h(X, Y), V) \tag{8}$$

for any $X, Y \in \Gamma(TM)$. Here, we denote by the same symbol g the Riemannian metric induced by g on $\tilde{M}$.

Now, we recall some from definitions from ([7], ([6], [9]) as follows:

A smooth vector field v on a Riemannian manifold $(\tilde{M}, g)$ is said to define concurrent if it satisfies

$$\tilde{\nabla}_X v = X,$$

where $\tilde{\nabla}$ is the Levi-Civita connection on $\tilde{M}$, for any $X \in \Gamma(T\tilde{M})$. Moreover, the best known example of Riemannian manifolds endowed with concurrent vector field is the Euclidean space with concurrent vector field given by its position vector field v with respect to origin.

Let M be a real $(2m+1)-$ dimensional submanifold of Sasakian manifold $\tilde{M}$ such that ξ is tangent to M. Then, M is called a semi-invariant submanifold of $\tilde{M}$, if there exists two differentiable distributions D and $D^\perp$ on M satisfying

i) the distribution D is invariant by φ, that is, $\varphi(D_x) = D_x$,
ii) the distribution $D^\perp$ is anti-invariant by φ, that is, $\varphi(D_x^\perp) \subset T_xM^\perp$,

for each $x \in M$.

We put $\dim\tilde{M} = m$, $\dim M = n$, $\dim D = p$ and $\dim D^\perp = q$. If $m - n = q$ is satisfied, then the submanifold M is called a generic submanifold of $\tilde{M}$.

Let M be a semi-invariant submanifold of Sasakian manifold $\tilde{M}$. By using the definition of semi-invariant submanifold, the tangent bundle and normal bundle of a semi-invariant submanifold M have the orthogonal decompositions

$$TM = D \oplus D^\perp \oplus \{\xi\}, \quad TM^\perp = \varphi(D^\perp) \oplus \mu, \quad \varphi(\mu) = \mu, \tag{9}$$

where μ is the complementary subbundle orthogonal to $\varphi(D^\perp)$ in $\Gamma(TM^\perp)$ and $\{\xi\}$ is the 1-dimensional distribution which is spanned by ξ. Also, if the distributions $D \oplus \xi$ and $D^\perp$ are totally geodesic in M, then the submanifold M is called a semi-invariant product.

Furthermore, on a semi-invariant submanifold M of Sasakian manifold $\tilde{M}$, the following properties are equivalent to each other:

i) M is a semi-invariant product;

ii) the fundamental tensor of Weingarten satisfy

$$A_{\varphi(Z)}X = 0, \tag{10}$$

iii) the second fundamental form of M satisfies

$$h(Y, \varphi(X)) = \varphi h(Y, X), \tag{11}$$

for any $X \in \Gamma(D)$, $Z \in \Gamma(D^\perp)$ and $Y \in \Gamma(TM)$.

Additionaly, a semi-invariant product is called a generic semi-invariant product if $m - n = q$ is satisfied. Then, we have $\mu = \{0\}$ in (9), thus we have the following decomposition

$$TM = D \oplus D^\perp \oplus \{\xi\}, \quad TM^\perp = \varphi(D^\perp). \tag{12}$$

For a generic semi-invariant product, from (12), we can write

$$v = v^\top + v^\perp + \varphi(v^\perp) + f\xi, \tag{13}$$

where $v \in \Gamma(T\tilde{M}), v^\top \in \Gamma(D)$ and $v^\perp \in \Gamma(D^\perp)$. Throughout this paper, we assume that the function f is a constant.

Let M be a semi-invariant submanifold of Sasakian manifold $\tilde{M}$. The semi-invariant submanifold M is called $D-$ geodesic, if it satisfies

$$h(X, Y) = 0 \tag{14}$$

for any $X, Y \in \Gamma(D)$. Similarly, for any $X, Y \in \Gamma(D^\perp)$, if the relation (14) is satisfied on M, then the semi-invariant submanifold is called $D^\perp-$ geodesic. Furthermore, for any $X \in \Gamma(D), Y \in \Gamma(D^\perp)$ if the relation (14) is satisfied on M, then the semi-invariant submanifold is called $(D, D^\perp)-$ geodesic or mixed geodesic.

On the other hand, the distribution D is said to be parallel with respect to $\tilde{\nabla}$, if it satisfies

$$\tilde{\nabla}_X Y \in \Gamma(D),$$

where $\tilde{\nabla}$ denotes a Levi-Civita connection of $\tilde{M}$, for any $X \in \Gamma(T\tilde{M})$ and $Y \in \Gamma(D)$.

3. Generic Submanifolds of Sasakian Manifolds with Concurrent Vector Field

In this section, we deal with the generic semi-invariant product and the generic submanifold in a Sasakian manifold with concurrent vector field.

We suppose that the function f in the equation (13) is not equal to zero for the following theorem.

Theorem 1. *Let M be a generic submanifold of a Sasakian manifold $\tilde{M}$ endowed with a concurrent vector field v. If the submanifold M is $D-$geodesic, then vector field $v^\top$ on the invariant distribution D is never concurrent.*

Proof. Since v is a concurrent vector field on $\tilde{M}$ and from (13), one has

$$\tilde{\nabla}_X v^\top + \tilde{\nabla}_X v^\perp + \tilde{\nabla}_X \varphi(v^\perp) + \tilde{\nabla}_X (f\xi) = X \tag{15}$$

for any $X \in \Gamma(D)$. Also, from (5), (6), (7) and (15), we have

$$\nabla_X v^\top + h(X, v^\top) + \nabla_X v^\perp + h(X, v^\perp) - A_{\varphi(v^\perp)}X + \nabla^\perp_X \varphi(v^\perp) - f\varphi(X) = X.$$

By comparing tangential and normal components of the last equation we get

$$h(X, v^\top) + h(X, v^\perp) = -\nabla^\perp_X \varphi(v^\perp)$$

and

$$\nabla_X v^\perp = X + f\varphi(X) + A_{\varphi(v^\perp)}X - \nabla_X v^\top. \tag{16}$$

Assume that $v^\top$ is a concurrent vector field on the invariant distribution D, then equation (16) reduces to

$$\nabla_X v^\perp = f\varphi(X) + A_{\varphi(v^\perp)}X. \tag{17}$$

On the other hand, for any $Y \in \Gamma(T\tilde{M})$, by putting $Y = v$ in (4), one has

$$(\tilde{\nabla}_X \varphi)v = g(X, v)\xi - \eta(v)X \tag{18}$$

where

$$(\tilde{\nabla}_X \varphi)v = \tilde{\nabla}_X \varphi(v) - \varphi(\tilde{\nabla}_X v). \tag{19}$$

From (13), (18) and (19), we get

$$\tilde{\nabla}_X \varphi(v) = g(X, v^\top)\xi - fX + \varphi(X).$$

Moreover, by virtue of (2), (3), (13) and the last equation, we have

$$\tilde{\nabla}_X \varphi(v^\top) + \tilde{\nabla}_X \varphi(v^\perp) - \tilde{\nabla}_X v^\perp = g(X, v^\top)\xi - fX + \varphi(X). \tag{20}$$

Using Gauss and Weingarten formulas in (20), it gives

$$g(X, v^\top)\xi - fX + \varphi(X) \quad = \quad \nabla_X \varphi(v^\top) + h(X, \varphi(v^\top)) - A_{\varphi(v^\perp)}X$$

$$+\nabla^{\perp}_{X}\varphi(v^{\perp}) - \nabla_X v^{\perp} - h(X, v^{\perp}).$$

From the above equation, we have

$$\nabla^{\perp}_{X}\varphi(v^{\perp}) = h(X, \varphi(v^{\top})) - h(X, v^{\perp})$$

and

$$\nabla_X \varphi(v^{\top}) - A_{\varphi(v^{\perp})}X - \nabla_X v^{\perp} = g(X, v^{\top})\xi - fX + \varphi(X). \tag{21}$$

Similarly, if we take $Y = v^{\top}$ in relation (4), we get

$$\tilde{\nabla}_X \varphi(v^{\top}) - \varphi(\tilde{\nabla}_X v^{\top}) = g(X, v^{\top})\xi.$$

Since M is $D-$geodesic, using Gauss and Weingarten formulas in the last equation, we have

$$\nabla_X \varphi(v^{\top}) = g(X, v^{\top})\xi + \varphi(X). \tag{22}$$

If we use the equalities (17), (21) and (22), we get

$$A_{\varphi(v^{\perp})}X = \frac{1}{2}f(X - \varphi(X)). \tag{23}$$

From (8) and (23), one has

$$g(h(X, Y), \varphi(v^{\perp})) = \frac{1}{2}fg(X, Y) - \frac{1}{2}fg(\varphi(X), Y) \tag{24}$$

for any $Y \in \Gamma(D)$. Interchanging the roles of X and Y in (24) we have

$$g(h(Y, X), \varphi(v^{\perp})) = \frac{1}{2}fg(Y, X) - \frac{1}{2}fg(\varphi(Y), X). \tag{25}$$

Then, making use of (24) and (25) yields

$$g(h(X, Y), \varphi(v^{\perp})) = \frac{1}{2}fg(X, Y)$$

is obtained. Since M is a generic semi-invariant, we have a contradiction. Thus, a vector field $v^{\top}$ on the invariant distribution D is never concurrent. □

Theorem 2. *Let M be a generic semi-invariant product of a Sasakian manifold $\tilde{M}$ endowed with a concurrent vector field v such that $v = v^{\top} + v^{\perp} + \varphi(v^{\perp}) + f\xi$. If the following conditions hold, then vector field $v^{\top}$ is concurrent on the invariant distribution D.*
i) The function f vanishes identically.
ii) $\nabla_X v^{\top} \in \Gamma(D)$, for any $X \in \Gamma(D)$.

Proof. Using the equalities (5), (6), (7), (10) and (15), we have

$$\nabla_X v^{\top} + h(X, v^{\top}) + \nabla_X v^{\perp} + h(X, v^{\perp}) + \nabla^{\perp}_{X}\varphi(v^{\perp}) - f\varphi(X) = X$$

for any $X \in \Gamma(D)$. From the above equation, we get

$$\nabla_X v^{\perp} + \nabla_X v^{\top} - X - f\varphi(X) = 0 \tag{26}$$

and

$$h(X, v^{\perp}) + h(X, v^{\top}) + \nabla^{\perp}_{X}\varphi(v^{\perp}) = 0.$$

On the other hand, if we use the equalities (13), (18) and (19), we have

$$\tilde{\nabla}_X\varphi(v) = g(X, v^\top)\xi - fX + \varphi(X).$$

Also, with the help of (2), (3), (13) and the last equation we find that

$$\tilde{\nabla}_X\varphi(v^\top) + \tilde{\nabla}_X\varphi(v^\perp) - \tilde{\nabla}_X v^\perp = g(X, v^\top)\xi - fX + \varphi(X). \tag{27}$$

Using Gauss and Weingarten formulas in (27) and from (10),

$$\begin{aligned} g(X, v^\top)\xi - fX + \varphi(X) \quad &= \quad \nabla_X\varphi(v^\top) + h(X, \varphi(v^\top)) + \nabla^\perp_X\varphi(v^\perp) \\ &\quad\; -\nabla_X v^\perp - h(X, v^\perp) \end{aligned}$$

is obtained. From the tangential and normal components of above equation, we have

$$\nabla_X\varphi(v^\top) - \nabla_X v^\perp = g(X, v^\top)\xi - fX + \varphi(X) \tag{28}$$

and

$$h(X, \varphi(v^\top)) + h(X, v^\perp) = -\nabla^\perp_X\varphi(v^\perp).$$

On the other hand, choosing $Y = v^\top$ in (4), one has

$$\tilde{\nabla}_X\varphi(v^\top) - \varphi(\tilde{\nabla}_X v^\top) = g(X, v^\top)\xi.$$

If we use (5), (6), (7) and (11) in last equation, we get

$$\nabla_X\varphi(v^\top) = g(X, v^\top)\xi + \varphi(\nabla_X v^\top). \tag{29}$$

From (26), (28) and (29), we have

$$\nabla_X v^\top + \varphi(\nabla_X v^\top) = (1-f)X + (1+f)\varphi(X). \tag{30}$$

Applying φ on both sides of (30), we obtain

$$\varphi(\nabla_X v^\top) - \nabla_X v^\top + \eta(\nabla_X v^\top)\xi = (1-f)\varphi(X) - (1+f)X. \tag{31}$$

Subtracting (31) from (30), one can easily see that

$$\nabla_X v^\top = X + f\varphi(X) + \frac{1}{2}\eta(\nabla_X v^\top)\xi. \tag{32}$$

Taking $f = 0$ in (32) and from (2), we have the requested result. This completes the proof. □

An example for generic submanifold of Sasakian manifold endowed with a concurrent vector field is given as follows:

Example 3. *Let C^{m+1} be a complex $(m+1)-$dimensional number space and $S^m(r)$ be an $m-$dimensional sphere with radius r. We consider an odd-dimensional unit sphere S^{2m+1} in C^{m+1}. Then, S^{2m+1} admits a Sasakian structure (φ, ξ, η, g). Let v be a position vector representing a point of S^{2m+1} in C^{m+1}. Then, v is also a concurrent vector field on S^{2m+1} and the structure vector field of S^{2m+1} is given by $\xi = Jv$, J denoting the almost complex structure of C^{m+1} which satisfies $J^2 = -I$. We consider the orthogonal projection*

$$\pi : T_x(C^m + 1) \to T_x(S^{2m+1})$$

and put $\varphi = \pi \circ J$. Hence, we have $\varphi X = JX + \eta(X)v$, for any vector field X tangent to S^{2m+1}. We consider the following immersion:

$$S^{m_1}\left(\sqrt{m_1/n}\right) \times ... \times S^{m_k}\left(\sqrt{m_1/n}\right) \to S^{n+k-1}, \quad n = \sum_{i=1}^{k} m_i.$$

We suppose that $m_1, ..., m_1$ are odd. Hence, $n + k - 1$ is also odd. Let v_i be a point of $S^{m_i}\left(\sqrt{m_i/n}\right)$ on $R^{m_i+1} = C^{(m_i+1)/2}$. $S^{m_i}\left(\sqrt{m_i/n}\right)$ is a real hypersurface of $C^{(m_i+1)/2}$ with unit normal $\sqrt{n/m_i}v_i$. Thus, $v = (v_1, ..., v_k)$ is a unit vector in $R^{n+k} = C^{(n+k)/2}$. This defines a minimal immersion of $M_{m_1,...,m_k} = \prod S^{m_i}\left(\sqrt{m_i/n}\right)$ into S^{n+k-1}. This shows that $M_{m_1,...,m_k}$ is a generic submanifold of a Sasakian manifold S^{n+k-1} endowed with a concurrent vector field v (for more details, see [29]).

4. Ricci Solitons in Generic Submanifolds

In this section, we deal with the generic submanifolds admitting a Ricci soliton in a Sasakian manifold with concurrent vector field.

Lemma 4. *Let M be a generic submanifold admitting a Ricci soliton of Sasakian manifold $\tilde{M}$ endowed with a concurrent vector field v. Then, the Ricci tensor S_D of the invariant distribution D is given by*

$$S_D(X,Y) = -\left\{(\lambda+1)g(X,Y) + g(h(X,Y), \varphi(v^\perp)) + \frac{1}{2}g(v^\perp, \nabla_X Y + \nabla_Y X)\right\},$$

where ∇ is the Levi-civita connection on M, for any $X, Y \in \Gamma(D)$.

Proof. It follows from the definition of the Lie-derivative, one has

$$(\pounds_{v^\top} g)(X,Y) = g(\nabla_X v^\top, Y) + g(\nabla_Y v^\top, X). \tag{33}$$

Using (16) in (33), then we have

$$\begin{aligned}(\pounds_{v^\top} g)(X,Y) &= 2g(X,Y) + 2g(h(X,Y), \varphi(v^\perp)) - g(\nabla_X v^\perp, Y) \\ &\quad - g(\nabla_Y v^\perp, X).\end{aligned} \tag{34}$$

On the other hand, since M is a generic submanifold, it follows

$$g(v^\perp, \nabla_X Y) = -g(\nabla_X v^\perp, Y) \tag{35}$$

and

$$g(v^\perp, \nabla_Y X) = -g(\nabla_Y v^\perp, X). \tag{36}$$

Again, since the generic submanifold M admits a Ricci soliton and using the fact that (1), one can see that

$$(\pounds_{v^\top} g)(X,Y) + 2S_D(X,Y) + 2\lambda g(X,Y) = 0. \tag{37}$$

From the equalities (34)-(37), the Ricci tensor of the invariant distribution D

$$S_D(X,Y) = -\Big\{(\lambda+1)g(X,Y) + g(h(X,Y),\varphi(v^\perp)) + \frac{1}{2}g(v^\perp, \nabla_X Y + \nabla_Y X)\Big\},$$

is found, which is desired form. □

The proof of the next lemma is similar to Lemma 4.

Lemma 5. *Let M be a generic submanifold admitting a Ricci soliton of Sasakian manifold $\tilde{M}$ endowed with a concurrent vector field v. Then, the Ricci tensor $S_{D^\perp}$ of the anti-invariant distribution $D^\perp$ is given by*

$$S_{D^\perp}(X,Y) = -\Big\{(\lambda+1)g(X,Y) + g(h(X,Y),\varphi(v^\perp)) + \frac{1}{2}g(v^\top, \nabla_X Y + \nabla_Y X)\Big\},$$

where ∇ is the Levi-civita connection on M, for any $X, Y \in \Gamma(D^\perp)$.

As a result of Lemma 4 and Lemma 5 , we have the following theorems:

Theorem 6. *Let M be a generic submanifold admitting a Ricci soliton of Sasakian manifold $\tilde{M}$ endowed with a concurrent vector field v. If the invariant distribution D is $D-$parallel, then the invariant distribution D is an Einstein.*

Theorem 7. *Let M be a generic submanifold admitting a Ricci soliton of Sasakian manifold $\tilde{M}$ endowed with a concurrent vector field v. If the anti-invariant distribution $D^\perp$ is $D^\perp-$parallel, then the anti-invariant distribution $D^\perp$ is an Einstein.*

Next, using the Theorem 2, we have the following.

Theorem 8. *Let M be a generic semi-invariant product admitting a Ricci soliton of Sasakian manifold $\tilde{M}$ endowed with a concurrent vector field v. Then, the followings are satisfied:*
i) The vector field $v^\top$ is a conformal Killing on D.
ii) The invariant distribution D is an Einstein.

Proof. From the definition of the Lie-derivative, we have

$$(\pounds_{v^\top} g)(X,Y) = g(\nabla_X v^\top, Y) + g(\nabla_Y v^\top, X). \tag{38}$$

From (3), (32) and (38), one has

$$(\pounds_{v^\top} g)(X,Y) = 2g(X,Y) \tag{39}$$

which implies (i).

On the other hand, since M is a generic semi-invariant product admitting a Ricci soliton and from (1), we get

$$(\pounds_{v^\top} g)(X,Y) + S_D(X,Y) + 2\lambda g(X,Y) = 0. \tag{40}$$

Using (38), (39), and (40), the Ricci tensor of the invariant distribution D,

$$S_D(X,Y) = -(\lambda+1)g(X,Y)$$

is found, which proves (ii). □

The next lemma gives us a characterization for a generic submanifold to be a gradient Ricci soliton.

Lemma 9. *Let M be a generic submanifold admitting a Ricci soliton of Sasakian manifold $\tilde{M}$ endowed with a concurrent vector field v. Then, we have*

$$\nabla\phi = -A_{\varphi(v^\perp)}(v^\top + v^\perp), \tag{41}$$

$$v^\top = \nabla\gamma, \tag{42}$$

where $\phi = \dfrac{1}{2}g(\varphi(v^\perp), \varphi(v^\perp))$, $\gamma = \dfrac{1}{2}g(v, v)$ and $\nabla\gamma$ denotes the gradient of the function γ.

Proof. By comparing tangential and normal components of (15), we have

$$\nabla^\perp_X \varphi(v^\perp) = -h(X, v^\top + v^\perp), \tag{43}$$

for any $X \in \Gamma(D)$. Then, using the equation (43) one has

$$\begin{aligned} X\phi &= g(\tilde{\nabla}_X \varphi(v^\perp), \varphi(v^\perp)) \\ &= g(\nabla^\perp_X \varphi(v^\perp), \varphi(v^\perp)) \\ &= g(-A_{\varphi(v^\perp)}(v^\top + v^\perp), X), \end{aligned}$$

which implies (41). Since v is a concurrent vector field on $\tilde{M}$, the equation (42) follows from

$$X\gamma = g(\tilde{\nabla}_X v, v) = g(X, v) = g(X, v^\top).$$

□

The next result is an immediate consequence of the Lemma 9.

Corollary 10. *Every Ricci soliton on the invariant distribution D of generic submanifold M of $\tilde{M}$ is a gradient Ricci soliton with the potential function $\gamma = \dfrac{1}{2}g(v, v)$.*

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