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Schur-Convexity for a Class of Completely Symmetric Function Dual

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Abstract

By using the decision theorem and properties of the Schur-convex function, the Schur-geometric convex function and the Schur-harmonic function, the Schur- convexity, Schur-geometric convexity and Schur-harmonic convexity of a class of complete symmetric functions are studied. As applications, some symmetric function inequalities are established.

Keywords: Schur-convexity; Schur-geometric convexity; Schur-harmonic convexity; completely symmetric function; dual form.
2010 MSC: 05E05, 26B25.

1. Introduction

Let us begin with some basic definitions and notation that will be needed in this paper. Throughout this paper, we denote by $\mathbb{N}$ and $\mathbb{R}$, the set of positive integers and real numbers, respectively. Denote

$$\mathbb{R}^n := \{\boldsymbol{x} = (x_1, x_2, \cdots, x_n) : x_i \in \mathbb{R}, i = 1, 2, \ldots, n\},$$

$$\mathbb{R}^n_+ := \{\boldsymbol{x} = (x_1, x_2, \ldots, x_n) : x_i > 0, i = 1, 2, \ldots, n\}$$

and

$$\mathbb{R}^n_- := \{\boldsymbol{x} = (x_1, x_2, \ldots, x_n) : x_i < 0, i = 1, 2, \ldots, n\},$$

where $n \in \mathbb{N}$. In particular, we simply use the notations $\mathbb{R}$ and $\mathbb{R}_+$ instead of $\mathbb{R}^1$ and $\mathbb{R}^1_+$, respectively.

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During the past more than two decades, many authors are dedicated to the hot topic of inequality research area on the Schur-convexity, Schur-geometric and Schur-harmonic convexity of various symmetric functions; see, e.g., [7]-[25] and references therein.

The family of complete symmetric functions is an important class of symmetric functions.

For $\boldsymbol{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, the complete symmetric function $c_n(\boldsymbol{x}, r)$ is defined by

$$c_n(\boldsymbol{x}, r) = \sum_{i_1+i_2+\cdots+i_n=r} x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n}, \tag{1}$$

where $c_0(\boldsymbol{x}, r) = 1,\ r \in \{1, 2, \ldots, n\},\ \ i_1, i_2, \ldots, i_n$ are non-negative integers.

Guan [11] discussed the Schur-convexity of $c_n(\boldsymbol{x}, r)$ and proved the following proposition.

Proposition 1. $c_n(\boldsymbol{x}, r)$ is increasing and Schur-convex on $\mathbb{R}^n_+$.

Subsequently, Chu et al. [8] prove the following proposition.

Proposition 2. $c_n(\boldsymbol{x}, r)$ is Schur-geometrically convex and Schur-harmonically convex on $\mathbb{R}^n_+$.

The dual form of the complete symmetric function $c_n(\boldsymbol{x}, r)$ is defined by

$$c_n^*(\boldsymbol{x}, r) = \prod_{i_1+i_2+\cdots+i_n=r} \sum_{j=1}^{n} i_j x_j, \tag{2}$$

where $c_0^*(\boldsymbol{x}, r) = 1,\ r \in \{1, 2, \ldots, n\},\ \ i_1, i_2, \ldots, i_n$ are non-negative integers.

Zhang and Shi [24] established the following two propositions.

Proposition 3. For $r = 1, 2, \ldots, n$, $c_n^*(\boldsymbol{x}, r)$ is increasing and Schur-concave on $\mathbb{R}^n_+$.

Proposition 4. For $r = 1, 2, \ldots, n$, $c_n^*(\boldsymbol{x}, r)$ is Schur-geometrically convex and Schur-harmonically convex on $\mathbb{R}^n_+$.

Notice that

$$c_n^*(-\boldsymbol{x}, r) = (-1)^r c_n^*(\boldsymbol{x}, r).$$

It is not difficult to verify the following proposition.

Proposition 5. If r is even integer (or odd integer, respectively), then $c_n^*(\boldsymbol{x}, r)$ is decreasing and Schur-concave (or increasing and Schur-convex, respectively) on $\mathbb{R}^n_-$.

In 2014, Sun et al. [12] studied the Schur-convexity, Schur-geometric and harmonic convexities of the following composite function of $c_n(\boldsymbol{x}, r)$

$$c_n\left(\frac{\boldsymbol{x}}{1-\boldsymbol{x}}, r\right) = \sum_{i_1+i_2+\cdots+i_n=r} \prod_{j=1}^{n} \left(\frac{x_j}{1-x_j}\right)^{i_j}. \tag{3}$$

Using Lemmas 1, 2 and 3 in second section, they proved the following Theorems A, B and C, respectively.

Theorem A. *For* $\boldsymbol{x} = (x_1, x_2, \ldots, x_n) \in (0,1)^n \cup (1, +\infty)^n$ *and* $r \in \mathbb{N}$,

(*i*) $c_n\left(\frac{\boldsymbol{x}}{1-\boldsymbol{x}}, r\right)$ *is increasing and Schur-convex on* $(0,1)^n$*;*

(ii) *if* r *is even integer* (*or odd integer, respectively*), *then* $c_n\left(\frac{\boldsymbol{x}}{1-\boldsymbol{x}},r\right)$ *is Schur-convex* (*or Schur-concave, respectively*) *on* $(1,+\infty)^n$, *and is decreasing* (*or increasing, respectively*).

Theorem B. *For* $\boldsymbol{x}=(x_1,x_2,\ldots,x_n)\in(0,1)^n\cup(1,+\infty)^n$ *and* $r\in\mathbb{N}$,

(i) $c_n\left(\frac{\boldsymbol{x}}{1-\boldsymbol{x}},r\right)$ *is Schur-geometrically convex on* $(0,1)^n$*;*

(ii) *if* r *is even integer* (*or odd integer, respectively*), *then* $c_n\left(\frac{\boldsymbol{x}}{1-\boldsymbol{x}},r\right)$ *is Schur-geometrically convex* (*or Schur-geometrically concave, respectively*) *on* $(1,+\infty)^n$.

Theorem C. *For* $\boldsymbol{x}=(x_1,x_2,\ldots,x_n)\in(0,1)^n\cup(1,+\infty)^n$ *and* $r\in\mathbb{N}$,

(i) $c_n\left(\frac{\boldsymbol{x}}{1-\boldsymbol{x}},r\right)$ *is Schur-harmonically convex on* $(0,1)^n$*;*

(ii) *if* r *is even integer* (*or odd integer, respectively*), *then* $c_n\left(\frac{\boldsymbol{x}}{1-\boldsymbol{x}},r\right)$ *is Schur-harmonically convex* (*or Schur-harmonically concave, respectively*) *on* $(1,+\infty)^n$.

In 2016, Shi et al. [25] used the properties of Schur-convex, Schur-geometrically convex and Schur-harmonically convex functions respectively to give simple proofs of Theorems A, B and C.

In [25], Shi et al. also further considered the Schur-convexity of $c_n(\boldsymbol{x},r)$ on $\mathbb{R}^n_-$, which established the following proposition.

Proposition 6. If r is even integer(or odd integer, respectively), then $c_n(\boldsymbol{x},r)$ is decreasing and Schur-convex (or increasing and Schur-concave, respectively) on $\mathbb{R}^n_-$.

The dual form of the function $c_n\left(\frac{\boldsymbol{x}}{1-\boldsymbol{x}},r\right)$ is defined by

$$c_n^*\left(\frac{\boldsymbol{x}}{1-\boldsymbol{x}},r\right)=\prod_{i_1+i_2+\cdots+i_n=r}\sum_{j=1}^{n}i_j\left(\frac{x_j}{1-x_j}\right). \tag{4}$$

A function associated with this function is

$$c_n^*\left(\frac{\boldsymbol{x}}{\boldsymbol{x}-1},r\right)=\prod_{i_1+i_2+\cdots+i_n=r}\sum_{j=1}^{n}i_j\left(\frac{x_j}{x_j-1}\right). \tag{5}$$

This paper we will study the Schur-convexity, Schur-geometric and Schur-harmonic convexiies of Symmetric functions $c_n^*\left(\frac{\boldsymbol{x}}{\boldsymbol{x}-1},r\right)$ and $c_n^*\left(\frac{\boldsymbol{x}}{1-\boldsymbol{x}},r\right)$.

Our main results will be established as follows:

Theorem 1. *For* $r\in\mathbb{N}$, $c_n^*\left(\frac{\boldsymbol{x}}{\boldsymbol{x}-1},r\right)$ *is Schur-convex, Schur-geometrically convex and Schur-harmonically convex on* $(1,+\infty)^n$.

Theorem 2. *For* $\boldsymbol{x}=(x_1,x_2,\ldots,x_n)\in\mathbb{R}^n_+\cup\mathbb{R}^n_-$ *and* $r\in\mathbb{N}$,

(i) $c_n^*\left(\frac{\boldsymbol{x}}{1-\boldsymbol{x}},r\right)$ *is increasing on* $\mathbb{R}^n_+$ *and Schur-convex on* $[\frac{1}{2},1)^n$*;*

(ii) *if* r *is even integer* (*or odd integer, respectively*), *then* $c_n^*\left(\frac{\boldsymbol{x}}{1-\boldsymbol{x}},r\right)$ *is Schur-convex* (*or Schur-concave, respectively*) *on* $(1,+\infty)^n$*;*

(iii) *if* r *is even integer* (*or odd integer, respectively*), *then* $c_n^*\left(\frac{\boldsymbol{x}}{1-\boldsymbol{x}},r\right)$ *is decreasing and Schur-concave* (*or increasing and Schur-convex, respectively*) *on* $\mathbb{R}^n_-$.

Theorem 3. *For $\boldsymbol{x} = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n_+$ and $r \in \mathbb{N}$,*

(*i*) $c_n^*\left(\frac{\boldsymbol{x}}{1-\boldsymbol{x}}, r\right)$ *is Schur-geometrically convex on* $(0,1)^n$*;*

(*ii*) *if r is even integer (or odd integer, respectively), then* $c_n^*\left(\frac{\boldsymbol{x}}{1-\boldsymbol{x}}, r\right)$ *is Schur-geometrically convex (or Schur-geometrically concave, respectively) on* $(1,+\infty)^n$.

Theorem 4. *For $\boldsymbol{x} = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n_+ \cup \mathbb{R}^n_-$ and $r \in \mathbb{N}$,*

(*i*) $c_n^*\left(\frac{\boldsymbol{x}}{1-\boldsymbol{x}}, r\right)$ *is Schur-harmonically convex on* $(0,1)^n$*;*

(*ii*) *if r is even integer (or odd integer, respectively), then* $c_n^*\left(\frac{\boldsymbol{x}}{1-\boldsymbol{x}}, r\right)$ *is Schur-harmonically convex (or Schur-harmonically concave, respectively) on* $(1,+\infty)^n$.

2. Preliminaries

For convenience, we first recall some known definitions and results.

Definition 1. [1, 2] For $\boldsymbol{x} = (x_1, x_2, \dots, x_n)$ and $\boldsymbol{y} = (y_1, y_2, \dots, y_n) \in \mathbb{R}^n$,

(*i*) $\boldsymbol{x} \geq \boldsymbol{y}$ means $x_i \geq y_i$ for all $i = 1, 2, \dots, n$.

(*ii*) Let $\Omega \subset \mathbb{R}^n$, φ: $\Omega \to \mathbb{R}$ is said to be increasing if $\boldsymbol{x} \geq \boldsymbol{y}$ implies $\varphi(\boldsymbol{x}) \geq \varphi(\boldsymbol{y})$. φ is said to be decreasing if and only if $-\varphi$ is increasing.

Definition 2. [1, 2] For $\boldsymbol{x} = (x_1, x_2, \dots, x_n)$ and $\boldsymbol{y} = (y_1, y_2, \dots, y_n) \in \mathbb{R}^n$,

(*i*) $\boldsymbol{x}$ is said to be majorized by $\boldsymbol{y}$ (in symbols $\boldsymbol{x} \prec \boldsymbol{y}$) if $\sum_{i=1}^k x_{[i]} \leq \sum_{i=1}^k y_{[i]}$ for $k = 1, 2, \dots, n-1$ and $\sum_{i=1}^n x_i = \sum_{i=1}^n y_i$, where $x_{[1]} \geq x_{[2]} \geq \cdots \geq x_{[n]}$ and $y_{[1]} \geq y_{[2]} \geq \cdots \geq y_{[n]}$ are rearrangements of $\boldsymbol{x}$ and $\boldsymbol{y}$ in a descending order.

(*ii*) Let $\Omega \subset \mathbb{R}^n$, φ: $\Omega \to \mathbb{R}$ is said to be a Schur-convex function on Ω if $\boldsymbol{x} \prec \boldsymbol{y}$ on Ω implies $\varphi(\boldsymbol{x}) \leq \varphi(\boldsymbol{y})$. φ is said to be a Schur-concave function on Ω if and only if $-\varphi$ is Schur-convex function on Ω.

Definition 3. [1, 2] Let $\boldsymbol{x} = (x_1, x_2, \dots, x_n)$ and $\boldsymbol{y} = (y_1, y_2, \dots, y_n) \in \mathbb{R}^n$.

(*i*) A set $\Omega \subset \mathbb{R}^n$ is said to be a convex set if $\boldsymbol{x}, \boldsymbol{y} \in \Omega, 0 \leq \alpha \leq 1$, implies $\alpha\boldsymbol{x} + (1-\alpha)\boldsymbol{y} = (\alpha x_1 + (1-\alpha)y_1, \alpha x_2 + (1-\alpha)y_2, \dots, \alpha x_n + (1-\alpha)y_n) \in \Omega$.

(*ii*) Let $\Omega \subset \mathbb{R}^n$ be convex set. A function φ: $\Omega \to \mathbb{R}$ is said to be a convex function on Ω if

$$\varphi(\alpha\boldsymbol{x} + (1-\alpha)\boldsymbol{y}) \leq \alpha\varphi(\boldsymbol{x}) + (1-\alpha)\varphi(\boldsymbol{y})$$

for all $\boldsymbol{x}, \boldsymbol{y} \in \Omega$, and all $\alpha \in [0,1]$. φ is said to be a concave function on Ω if and only if $-\varphi$ is convex function on Ω.

Definition 4. [1, 2]

(*i*) A set $\Omega \subset \mathbb{R}^n$ is called a symmetric set, if $\boldsymbol{x} \in \Omega$ implies $\boldsymbol{x}P \in \Omega$ for every $n \times n$ permutation matrix P.

(*ii*) A function $\varphi : \Omega \to \mathbb{R}$ is called symmetric if for every permutation matrix P, $\varphi(\boldsymbol{x}P) = \varphi(\boldsymbol{x})$ for all $\boldsymbol{x} \in \Omega$.

Lemma 1. (*Schur-convex function decision theorem*)[1, 2] *Let $\Omega \subset \mathbb{R}^n$ be symmetric and have a nonempty interior convex set. Ω° is the interior of Ω. $\varphi : \Omega \to \mathbb{R}$ is continuous on Ω and differentiable in Ω°. Then φ is the $Schur - convex$ (or $Schur - concave,\ respectively$) $function$ if and only if φ is symmetric on Ω and*

$$(x_1 - x_2)\left(\frac{\partial \varphi}{\partial x_1} - \frac{\partial \varphi}{\partial x_2}\right) \geq 0 (or \leq 0,\ respectively) \tag{6}$$

holds for any $\boldsymbol{x} \in \Omega^\circ$.

The first systematical study of the functions preserving the ordering of majorization was made by Issai Schur in 1923. In Schur's honor, such functions are said to be "Schur-convex". It can be used extensively in analytic inequalities, combinatorial optimization, quantum physics, information theory, and other related fields. See [1].

Definition 5. [3] Let $\boldsymbol{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n_+$ and $\boldsymbol{y} = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n_+$.

(*i*) A set $\Omega \subset \mathbb{R}^n_+$ is called a geometrically convex set if $(x_1^\alpha y_1^\beta, x_2^\alpha y_2^\beta, \ldots, x_n^\alpha y_n^\beta) \in \Omega$ for all $\boldsymbol{x}$,$\boldsymbol{y} \in \Omega$ and α,$\beta \in [0, 1]$ such that $\alpha + \beta = 1$.

(*ii*) Let $\Omega \subset \mathbb{R}^n_+$. The function φ: $\Omega \to \mathbb{R}_+$ is said to be Schur-geometrically convex function on Ω if $(\log x_1, \log x_2, \ldots, \log x_n) \prec (\log y_1, \log y_2, \ldots,$
$\log y_n)$ on Ω implies $\varphi(\boldsymbol{x}) \leq \varphi(\boldsymbol{y})$. The function φ is said to be a Schur-geometrically concave function on Ω if and only if $-\varphi$ is Schur-geometrically convex function on Ω.

We can obtain the following result immediately from Definitions 5.

Proposition 7. Let $\Omega \subset \mathbb{R}^n_+$ be a set, and let $\log \Omega = \{(\log x_1, \log x_2, \ldots, \log x_n) : (x_1, x_2, \ldots, x_n) \in \Omega\}$. Then $\varphi : \Omega \to \mathbb{R}_+$ is a Schur-geometrically convex (or Schur-geometrically concave, respectively) function on Ω if and only if $\varphi(e^{x_1}, e^{x_2}, \ldots, e^{x_n})$ is a Schur-convex (or Schur-concave, respectively) function on $\log \Omega$.

Lemma 2. (*Schur-geometrically convex function decision theorem*)[3] *Let $\Omega \subset \mathbb{R}^n_+$ be a symmetric and geometrically convex set with a nonempty interior Ω°. Let $\varphi : \Omega \to \mathbb{R}_+$ be continuous on Ω and differentiable in Ω°. If φ is symmetric on Ω and*

$$(\log x_1 - \log x_2)\left(x_1 \frac{\partial \varphi}{\partial x_1} - x_2 \frac{\partial \varphi}{\partial x_2}\right) \geq 0 \quad (or \leq 0, respectively) \tag{7}$$

holds for any $\boldsymbol{x} = (x_1, x_2, \ldots, x_n) \in \Omega^\circ$, then φ is a Schur-geometrically convex (or Schur-geometrically concave, respectively) function.

The Schur-geometric convexity was proposed by Zhang [3] in 2004, and was investigated by Chu et al. [4], Guan [5], Sun et al. [6], and so on. We also note that some authors use the term "Schur multiplicative convexity".

In 2009, Chu ([7], [8], [9]) introduced the notion of Schur-harmonically convex function, and some interesting inequalities were obtained.

Definition 6. [7] Let $\Omega \subset \mathbb{R}^n_+$ or $\Omega \subset \mathbb{R}^n_-$.

(*i*) A set Ω is said to be harmonically convex if $\dfrac{\boldsymbol{xy}}{\lambda \boldsymbol{x} + (1-\lambda)\boldsymbol{y}} \in \Omega$ for every $\boldsymbol{x}, \boldsymbol{y} \in \Omega$ and $\lambda \in [0, 1]$, where $\boldsymbol{xy} = \sum_{i=1}^n x_i y_i$ and $\dfrac{1}{\boldsymbol{x}} = (\dfrac{1}{x_1}, \dfrac{1}{x_2}, \ldots, \dfrac{1}{x_n})$.

(ii) A function $\varphi : \Omega \to \mathbb{R}_+$ is said to be Schur-harmonically convex on Ω if $\frac{1}{\boldsymbol{x}} \prec \frac{1}{\boldsymbol{y}}$ implies $\varphi(\boldsymbol{x}) \leq \varphi(\boldsymbol{y})$. A function φ is said to be a Schur-harmonically concave function on Ω if and only if $-\varphi$ is a Schur-harmonically convex function.

By Definitions 6, the following is obvious.

Proposition 8. Let $\Omega \subset \mathbb{R}^n_+$ be a set, and let $\frac{1}{\Omega} = \{(\frac{1}{x_1}, \frac{1}{x_2}, \dots, \frac{1}{x_n}) :$ $(x_1, x_2, \dots, x_n) \in \Omega\}$. Then $\varphi : \Omega \to \mathbb{R}_+$ is a Schur-harmonically convex (or Schur-harmonically concave, respectively) function on Ω if and only if $\varphi(\frac{1}{x_1}, \frac{1}{x_2}, \dots, \frac{1}{x_n})$ is a Schur-convex (or Schur-concave, respectively) function on $\frac{1}{\Omega}$.

Lemma 3. (*Schur-harmonically convex function decision theorem)[7] Let $\Omega \subset \mathbb{R}^n_+$ or $\Omega \subset \mathbb{R}^n_-$ be a symmetric and harmonically convex set with inner points and let $\varphi : \Omega \to \mathbb{R}$ be a continuously symmetric function which is differentiable on Ω°. Then φ is Schur-harmonically convex (or Schur-harmonically concave, respectively) on Ω if and only if*

$$(x_1 - x_2)\left(x_1^2 \frac{\partial \varphi(\boldsymbol{x})}{\partial x_1} - x_2^2 \frac{\partial \varphi(\boldsymbol{x})}{\partial x_2}\right) \geq 0 \quad (or \leq 0, respectively), \quad \boldsymbol{x} \in \Omega^\circ. \tag{8}$$

Remark 1. We extend the definition and determination theorem of Schur-harmonically convex function established by Chu as follows:

(i) The set $\Omega \subset \mathbb{R}^n_+$ is extended to $\Omega \subset \mathbb{R}^n_+$ or $\Omega \subset \mathbb{R}^n_-$;
(ii) The function $\varphi : \Omega \to \mathbb{R}$ must not be a positive function.

Lemma 4. *([1], [2]) Let the set $\mathcal{A}, \mathcal{B} \subset \mathbb{R}$, $\varphi : \mathcal{B}^n \to \mathbb{R}$, $f : \mathcal{A} \to \mathcal{B}$ and $\psi(x_1, x_2, \dots, x_n) = \varphi(f(x_1), f(x_2), \dots, f(x_n)) :$ $\mathcal{A}^n \to \mathbb{R}$.*

(i) If f is convex and φ is increasing and Schur-convex, then ψ is Schur-convex;
(ii) If f is convex and φ is decreasing and Schur-concave, then ψ is Schur-concave.

Lemma 5. *[3, 26] Let the set $\Omega \subset \mathbb{R}^n_+$. The function $\varphi : \Omega \to \mathbb{R}_+$ is differentiable.*

(i) If φ is increasing and Schur-convex or Schur-geometrically convex, then φ is Schur-harmonically convex.
(ii) If φ is decreasing and Schur-geometrically concave, then φ is Schur-harmonically concave.

Lemma 6. *[1] Let $\boldsymbol{x} = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n_+, n \geq 2, 0 < r \leq s$. Then*

$$\left(\frac{x_1^r}{\sum_{j=1}^n x_j^r}, \frac{x_2^r}{\sum_{j=1}^n x_j^r}, \dots, \frac{x_n^r}{\sum_{j=1}^n x_j^r}\right) \prec \left(\frac{x_1^s}{\sum_{j=1}^n x_j^s}, \frac{x_2^s}{\sum_{j=1}^n x_j^s}, \dots, \frac{x_n^s}{\sum_{j=1}^n x_j^s}\right). \tag{9}$$

Lemma 7. *[1] Let $\boldsymbol{x} = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n_+, n \geq 2, \sum_{i=1}^n x_i = s > 0, c \geq s$. Then*

$$\left(\frac{c - x_1}{nc - s}, \frac{c - x_2}{nc - s}, \dots, \frac{c - x_n}{nc - s}\right) \prec \left(\frac{x_1}{s}, \frac{x_2}{s}, \dots, \frac{x_n}{s}\right). \tag{10}$$

3. Proofs of main results

Proof of Theorem 1:

for $r=1$ and $r=2$, it is easy to prove that $c_n^*\left(\frac{\boldsymbol{x}}{\boldsymbol{x}-1},r\right)$ is Schur-convex on $(1,+\infty)^n$.

Now consider the case of $r\geq 3$. By the symmetry of $c_n^*\left(\frac{\boldsymbol{x}}{\boldsymbol{x}-1},r\right)$, without loss of generality, we can set $x_1>x_2$.

$$c_n^*\left(\frac{\boldsymbol{x}}{\boldsymbol{x}-1},r\right)=\prod_{\substack{i_1+i_2+\cdots+i_n=r\\ i_1\neq 0,i_2=0}}\sum_{j=1}^{n}\frac{i_jx_j}{x_j-1}\times\prod_{\substack{i_1+i_2+\cdots+i_n=r\\ i_1=0,i_2\neq 0}}\sum_{j=1}^{n}\frac{i_jx_j}{x_j-1}$$
$$\times\prod_{\substack{i_1+i_2+\cdots+i_n=r\\ i_1\neq 0,i_2\neq 0}}\sum_{j=1}^{n}\frac{i_jx_j}{x_j-1}\times\prod_{\substack{i_1+i_2+\cdots+i_n=r\\ i_1=0,i_2=0}}\sum_{j=1}^{n}\frac{i_jx_j}{x_j-1}.$$

Then

$$\frac{\partial c_n^*\left(\frac{\boldsymbol{x}}{\boldsymbol{x}-1},r\right)}{\partial x_1}=c_n^*\left(\frac{\boldsymbol{x}}{\boldsymbol{x}-1},r\right)$$
$$\times\left(\sum_{\substack{i_1+i_2+\cdots+i_n=r\\ i_1\neq 0,i_2=0}}\frac{-i_1}{(x_1-1)^2\sum_{j=1}^{n}\frac{i_jx_j}{x_j-1}}+\sum_{\substack{i_1+i_2+\cdots+i_n=r\\ i_1\neq 0,i_2\neq 0}}\frac{-i_1}{(x_1-1)^2\sum_{j=1}^{n}\frac{i_jx_j}{x_j-1}}\right)$$
$$=c_n^*\left(\frac{\boldsymbol{x}}{\boldsymbol{x}-1},r\right)\Big(\sum_{\substack{k+k_3+\ldots+k_n=r\\ k\neq 0}}\frac{-k}{(x_1-1)^2(\frac{kx_1}{x_1-1}+\sum_{j=3}^{n}\frac{k_jx_j}{x_j-1})}$$
$$+\sum_{\substack{k+m+i_3+\cdots+i_n=r\\ k\neq 0,m\neq 0}}\frac{-k}{(x_1-1)^2(\frac{kx_1}{x_1-1}+\frac{mx_2}{x_2-1}+\sum_{j=3}^{n}\frac{k_jx_j}{x_j-1})}\Big). \tag{11}$$

By the same arguments,

$$\frac{\partial c_n^*\left(\frac{\boldsymbol{x}}{\boldsymbol{x}-1},r\right)}{\partial x_2}=c_n^*\left(\frac{\boldsymbol{x}}{\boldsymbol{x}-1},r\right)\Big(\sum_{\substack{k+k_3+\cdots+k_n=r\\ k\neq 0}}\frac{-k}{(x_2-1)^2(\frac{kx_2}{x_2-1}+\sum_{j=3}^{n}\frac{k_jx_j}{x_j-1})}$$
$$+\sum_{\substack{k+m+i_3+\cdots+i_n=r\\ k\neq 0,m\neq 0}}\frac{-k}{(x_2-1)^2(\frac{kx_2}{x_2-1}+\frac{mx_1}{x_1-1}+\sum_{j=3}^{n}\frac{k_jx_j}{x_j-1})}\Big), \tag{12}$$

then

$$\frac{\partial c_n^*\left(\frac{\boldsymbol{x}}{\boldsymbol{x}-1},r\right)}{\partial x_1}-\frac{\partial c_n^*\left(\frac{\boldsymbol{x}}{\boldsymbol{x}-1},r\right)}{\partial x_2}=c_n^*\left(\frac{\boldsymbol{x}}{\boldsymbol{x}-1},r\right)(A_1+A_2),$$

where

$$A_1 = \sum_{\substack{k+k_3+\cdots+k_n=r\\ k\neq 0}} \left(\frac{-k}{(x_1-1)^2(\frac{kx_1}{x_1-1}+\sum\limits_{j=3}^{n}\frac{k_jx_j}{x_j-1})} - \frac{-k}{(x_2-1)^2(\frac{kx_2}{x_2-1}+\sum\limits_{j=3}^{n}\frac{k_jx_j}{x_j-1})} \right)$$

$$= k \sum_{\substack{k+k_3+\cdots+k_n=r\\ k\neq 0}} \frac{k(x_1+x_2-1)(x_1-x_2)+(x_1-x_2)(2-x_1-x_2)\sum\limits_{j=3}^{n}\frac{k_jx_j}{x_j-1}}{(x_1-1)^2(\frac{kx_1}{x_1-1}+\sum\limits_{j=3}^{n}\frac{k_jx_j}{x_j-1})(x_2-1)^2(\frac{kx_2}{x_2-1}+\sum\limits_{j=3}^{n}\frac{k_jx_j}{x_j-1})}$$

and

$$A_2 = \sum_{\substack{k+m+i_3+\cdots+i_n=r\\ k\neq 0, m\neq 0}} \left(\frac{-k}{(x_1-1)^2(\frac{kx_1}{x_1-1}+\frac{mx_2}{x_2-1}+\sum\limits_{j=3}^{n}\frac{k_jx_j}{x_j-1})} - \frac{-k}{(x_2-1)^2(\frac{kx_2}{x_2-1}+\frac{mx_1}{x_1-1}+\sum\limits_{j=3}^{n}\frac{k_jx_j}{x_j-1})} \right)$$

$$= k \sum_{\substack{k+m+i_3+\cdots+i_n=r\\ k\neq 0, m\neq 0}} \frac{\lambda_1}{(x_1-1)^2(\frac{kx_1}{x_1-1}+\frac{mx_2}{x_2-1}+\sum\limits_{j=3}^{n}\frac{k_jx_j}{x_j-1})(x_2-1)^2(\frac{kx_2}{x_2-1}+\frac{mx_1}{x_1-1}+\sum\limits_{j=3}^{n}\frac{k_jx_j}{x_j-1})},$$

where

$$\lambda_1 = k(x_1+x_2-1)(x_1-x_2) + \left(\frac{(1-x_2)^2mx_1}{1-x_1} - \frac{(1-x_1)^2mx_2}{1-x_2}\right) + (x_1-x_2)(x_1+x_2-2)\sum_{j=3}^{n}\frac{k_jx_j}{x_j-1}.$$

Let $f(t) = \frac{(1-t)^3}{mt}$. Then $f'(t) = -\frac{m(1+2t)(1-t)^2}{m^2t^2} \le 0$, this means that $f(t)$ is descending on $\mathbb{R}_+$. So that $\frac{(1-x_1)^3}{mx_1} \le \frac{(1-x_2)^3}{mx_2}$, namely $\frac{(1-x_2)^2mx_1}{1-x_1} - \frac{(1-x_1)^2mx_2}{1-x_2} \ge 0$. It is easy to see that $A_1 \ge 0$ and $A_2 \ge 0$ for $\boldsymbol{x} \in (1,+\infty)^n$, so

$$\frac{\partial c_n^*\left(\frac{\boldsymbol{x}}{\boldsymbol{x}-1}, r\right)}{\partial x_1} - \frac{\partial c_n^*\left(\frac{\boldsymbol{x}}{\boldsymbol{x}-1}, r\right)}{\partial x_2} \ge 0,$$

by Lemma 1, it follows that $c_n^*\left(\frac{\boldsymbol{x}}{1-\boldsymbol{x}}, r\right)$ is Schur-convex on $(1,+\infty)^n$.

From (11) and (12), it follows that

$$x_1\frac{\partial c_n^*\left(\frac{\boldsymbol{x}}{\boldsymbol{x}-1}, r\right)}{\partial x_1} - x_2\frac{\partial c_n^*\left(\frac{\boldsymbol{x}}{\boldsymbol{x}-1}, r\right)}{\partial x_2} = c_n^*\left(\frac{\boldsymbol{x}}{\boldsymbol{x}-1}, r\right)(B_1+B_2),$$

where

$$B_1 = \sum_{\substack{k+k_3+\cdots+k_n=r\\ k\neq 0}} \left(\frac{-kx_1}{(x_1-1)^2(\frac{kx_1}{x_1-1}+\sum\limits_{j=3}^{n}\frac{k_jx_j}{x_j-1})} - \frac{-kx_2}{(x_2-1)^2(\frac{kx_2}{x_2-1}+\sum\limits_{j=3}^{n}\frac{k_jx_j}{x_j-1})} \right)$$

$$= k \sum_{\substack{k+k_3+\cdots+k_n=r\\ k\neq 0}} \frac{kx_1x_2(x_1-x_2)+(x_1-x_2)(x_1x_2-1)\sum\limits_{j=3}^{n}\frac{k_jx_j}{x_j-1}}{(x_1-1)^2(\frac{kx_1}{x_1-1}+\sum\limits_{j=3}^{n}\frac{k_jx_j}{x_j-1})(x_2-1)^2(\frac{kx_2}{x_2-1}+\sum\limits_{j=3}^{n}\frac{k_jx_j}{x_j-1})}$$

and

$$B_2 = \sum_{\substack{k+m+i_3+\cdots+i_n=r\\ k\neq 0, m\neq 0}} \left(\frac{-kx_1}{(x_1-1)^2(\frac{kx_1}{x_1-1}+\frac{mx_2}{x_2-1}+\sum_{j=3}^{n}\frac{k_jx_j}{x_j-1})} - \frac{-kx_2}{(x_2-1)^2(\frac{kx_2}{x_2-1}+\frac{mx_1}{x_1-1}+\sum_{j=3}^{n}\frac{k_jx_j}{x_j-1})}\right)$$
$$= k\sum_{\substack{k+m+i_3+\cdots+i_n=r\\ k\neq 0, m\neq 0}} \frac{\lambda_2}{(x_1-1)^2(\frac{kx_1}{x_1-1}+\frac{mx_2}{x_2-1}+\sum_{j=3}^{n}\frac{k_jx_j}{x_j-1})(x_2-1)^2(\frac{kx_2}{x_2-1}+\frac{mx_1}{x_1-1}+\sum_{j=3}^{n}\frac{k_jx_j}{x_j-1})},$$

where

$$\lambda_2 = kx_1x_2(x_1-x_2) + \left(\frac{(x_1-1)^2mx_2^2}{x_2-1} - \frac{(x_2-1)^2mx_1^2}{x_1-1}\right) + (x_1-x_2)(x_1x_2-1)\sum_{j=3}^{n}\frac{k_jx_j}{x_j-1}.$$

Let $g(t) = \frac{(t-1)^3}{mt^2}$. Then $g'(t) = \frac{mt(t+2)(t-1)^2}{m^2t^4} \geq 0$, this means that $g(t)$ is increasing on $\mathbb{R}_+$. So that $\frac{(x_1-1)^3}{mx_1^2} \geq \frac{(x_2-1)^3}{mx_2^2}$, namely $\frac{(x_1-1)^2mx_2^2}{x_2-1} - \frac{(x_2-1)^2mx_1^2}{x_1-1} \geq 0$. It is easy to see that $B_1 \geq 0$ and $B_2 \geq 0$ for $\boldsymbol{x} \in (1,+\infty)^n$, so

$$x_1\frac{\partial c_n^*\left(\frac{\boldsymbol{x}}{\boldsymbol{x}-1}, r\right)}{\partial x_1} - x_2\frac{\partial c_n^*\left(\frac{\boldsymbol{x}}{\boldsymbol{x}-1}, r\right)}{\partial x_2} \geq 0,$$

by Lemma 2, it follows that $c_n^*\left(\frac{\boldsymbol{x}}{\boldsymbol{x}-1}, r\right)$ is Schur-geometrically convex on $(1,+\infty)^n$.

From (11) and (12), it follows that

$$x_1^2\frac{\partial c_n^*\left(\frac{\boldsymbol{x}}{\boldsymbol{x}-1}, r\right)}{\partial x_1} - x_2^2\frac{\partial c_n^*\left(\frac{\boldsymbol{x}}{\boldsymbol{x}-1}, r\right)}{\partial x_2} = c_n^*\left(\frac{\boldsymbol{x}}{\boldsymbol{x}-1}, r\right)(C_1+C_2),$$

where

$$C_1 = \sum_{\substack{k+k_3+\cdots+k_n=r\\ k\neq 0}} \left(\frac{-kx_1^2}{(x_1-1)^2(\frac{kx_1}{x_1-1}+\sum_{j=3}^{n}\frac{k_jx_j}{x_j-1})} - \frac{-kx_2^2}{(x_2-1)^2(\frac{kx_2}{x_2-1}+\sum_{j=3}^{n}\frac{k_jx_j}{x_j-1})}\right)$$
$$= k\sum_{\substack{k+k_3+\cdots+k_n=r\\ k\neq 0}} \frac{kx_1x_2(x_1-x_2) + (x_1-x_2)(2x_1x_2-x_1-x_2)\sum_{j=3}^{n}\frac{k_jx_j}{x_j-1}}{(x_1-1)^2(\frac{kx_1}{x_1}+\sum_{j=3}^{n}\frac{k_jx_j}{x_j-1})(x_2-1)^2(\frac{kx_2}{x_2-1}+\sum_{j=3}^{n}\frac{k_jx_j}{x_j-1})}$$

and

$$C_2 = \sum_{\substack{k+m+i_3+\cdots+i_n=r\\ k\neq 0, m\neq 0}} \left(\frac{-kx_1^2}{(x_1-1)^2(\frac{kx_1}{x_1-1}+\frac{mx_2}{x_2-1}+\sum_{j=3}^{n}\frac{k_jx_j}{x_j-1})} - \frac{-kx_2^2}{(x_2-1)^2(\frac{kx_2}{x_2-1}+\frac{mx_1}{x_1-1}+\sum_{j=3}^{n}\frac{k_jx_j}{x_j-1})}\right)$$
$$= k\sum_{\substack{k+m+i_3+\cdots+i_n=r\\ k\neq 0, m\neq 0}} \frac{\lambda_3}{(x_1-1)^2(\frac{kx_1}{x_1-1}+\frac{mx_2}{x_2-1}+\sum_{j=3}^{n}\frac{k_jx_j}{x_j-1})(x_2-1)^2(\frac{kx_2}{x_2-1}+\frac{mx_1}{x_1-1}+\sum_{j=3}^{n}\frac{k_jx_j}{x_j-1})},$$

where

$$\lambda_3 = kx_1x_2(x_1-x_2) + \left(\frac{(x_1-1)^2mx_2^3}{x_2-1} - \frac{(x_2-1)^2mx_1^3}{x_1-1}\right) + (x_1-x_2)(2x_1x_2-x_1-x_2)\sum_{j=3}^{n}\frac{k_jx_j}{x_j-1}.$$

Let $h(t)=\frac{(t-1)^3}{mt^3}$. Then $h'(t)=\frac{3mt^2(t-1)^2}{m^2t^6}\geq 0$, this means that $h(t)$ is increasing on $\mathbb{R}$. So that $\frac{(x_1-1)^3}{mx_1^3}\geq\frac{(x_2-1)^3}{mx_2^3}$, namely $\frac{(x_1-1)^2mx_2^3}{x_2-1}-\frac{(x_2-1)^2mx_1^3}{x_1-1}\geq 0$. It is easy to see that $C_1\geq 0$ and $C_2\geq 0$ for $\boldsymbol{x}\in(1,+\infty)^n$, so

$$x_1^2\frac{\partial c_n^*\left(\frac{\boldsymbol{x}}{\boldsymbol{x}-1},r\right)}{\partial x_1}-x_2^2\frac{\partial c_n^*\left(\frac{\boldsymbol{x}}{\boldsymbol{x}-1},r\right)}{\partial x_2}\geq 0,$$

by Lemma 3, it follows that $c_n^*\left(\frac{\boldsymbol{x}}{\boldsymbol{x}-1},r\right)$ is Schur-harmonically convex on $(1,+\infty)^n$.

The proof of Theorem 1 is completed. □

Proof of Theorem 2:

(i) Let $p(t)=\frac{t}{1-t}$. Then

$$p'(t)=\frac{1}{(1-t)^2},\quad p''(t)=\frac{2}{(1-t)^3}. \tag{13}$$

From Proposition 4, we know that $c_n^*(\boldsymbol{x},r)$ is increasing on $\mathbb{R}_+^n$, but $p(t)$ is increasing on $\mathbb{R}$, therefore, the function $c_n^*\left(\frac{\boldsymbol{x}}{1-\boldsymbol{x}},r\right)$ is increasing on $\mathbb{R}_+^n$.

For the case of $r=1$ and $r=2$, it is easy to prove that $c_n^*\left(\frac{\boldsymbol{x}}{1-\boldsymbol{x}},r\right)$ is Schur-convex on $[\frac{1}{2},1)^n$.

Now consider the case of $r\geq 3$. By the symmetry of $c_n^*\left(\frac{\boldsymbol{x}}{1-\boldsymbol{x}},r\right)$, without loss of generality, we can set $x_1>x_2$.

$$\begin{aligned}c_n^*\left(\frac{\boldsymbol{x}}{1-\boldsymbol{x}},r\right)=&\prod_{\substack{i_1+i_2+\cdots+i_n=r\\ i_1\neq 0,i_2=0}}\sum_{j=1}^n\frac{i_jx_j}{1-x_j}\times\prod_{\substack{i_1+i_2+\cdots+i_n=r\\ i_1=0,i_2\neq 0}}\sum_{j=1}^n\frac{i_jx_j}{1-x_j}\\ &\times\prod_{\substack{i_1+i_2+\cdots+i_n=r\\ i_1\neq 0,i_2\neq 0}}\sum_{j=1}^n\frac{i_jx_j}{1-x_j}\times\prod_{\substack{i_1+i_2+\cdots+i_n=r\\ i_1=0,i_2=0}}\sum_{j=1}^n\frac{i_jx_j}{1-x_j}.\end{aligned}$$

Then

$$\begin{aligned}&\frac{\partial c_n^*\left(\frac{\boldsymbol{x}}{1-\boldsymbol{x}},r\right)}{\partial x_1}=c_n^*\left(\frac{\boldsymbol{x}}{1-\boldsymbol{x}},r\right)\\ &\times\left(\sum_{\substack{i_1+i_2+\cdots+i_n=r\\ i_1\neq 0,i_2=0}}\frac{i_1}{(1-x_1)^2\sum\limits_{j=1}^n\frac{i_jx_j}{1-x_j}}+\sum_{\substack{i_1+i_2+\cdots+i_n=r\\ i_1\neq 0,i_2\neq 0}}\frac{i_1}{(1-x_1)^2\sum\limits_{j=1}^n\frac{i_jx_j}{1-x_j}}\right)\\ &=c_n^*\left(\frac{\boldsymbol{x}}{1-\boldsymbol{x}},r\right)\Bigg(\sum_{\substack{k+k_3+\cdots+k_n=r\\ k\neq 0}}\frac{k}{(1-x_1)^2(\frac{kx_1}{1-x_1}+\sum\limits_{j=3}^n\frac{k_jx_j}{1-x_j})}\\ &+\sum_{\substack{k+m+i_3+\cdots+i_n=r\\ k\neq 0,m\neq 0}}\frac{k}{(1-x_1)^2(\frac{kx_1}{1-x_1}+\frac{mx_2}{1-x_2}+\sum\limits_{j=3}^n\frac{k_jx_j}{1-x_j})}\Bigg).\end{aligned} \tag{14}$$

By the same arguments,

$$\frac{\partial c_n^*\left(\frac{\boldsymbol{x}}{1-\boldsymbol{x}},r\right)}{\partial x_2}=c_n^*\left(\frac{\boldsymbol{x}}{1-\boldsymbol{x}},r\right)\Big(\sum_{\substack{k+k_3+\cdots+k_n=r\\k\neq 0}}\frac{k}{(1-x_2)^2(\frac{kx_2}{1-x_2}+\sum\limits_{j=3}^{n}\frac{k_jx_j}{1-x_j})}+\sum_{\substack{k+m+i_3+\cdots+i_n=r\\k\neq 0,m\neq 0}}\frac{k}{(1-x_2)^2(\frac{kx_2}{1-x_2}+\frac{mx_1}{1-x_1}+\sum\limits_{j=3}^{n}\frac{k_jx_j}{1-x_j})}\Big), \tag{15}$$

$$\frac{\partial c_n^*\left(\frac{\boldsymbol{x}}{1-\boldsymbol{x}},r\right)}{\partial x_1}-\frac{\partial c_n^*\left(\frac{\boldsymbol{x}}{1-\boldsymbol{x}},r\right)}{\partial x_2}=c_n^*\left(\frac{\boldsymbol{x}}{1-\boldsymbol{x}},r\right)(D_1+D_2),$$

where

$$\begin{aligned}D_1&=\sum_{\substack{k+k_3+\cdots+k_n=r\\k\neq 0}}\left(\frac{k}{(1-x_1)^2(\frac{k}{1-x_1}+\sum\limits_{j=3}^{n}\frac{k_jx_j}{1-x_j})}-\frac{k}{(1-x_2)^2(\frac{kx_2}{1-x_2}+\sum\limits_{j=3}^{n}\frac{k_jx_j}{1-x_j})}\right)\\&=k\sum_{\substack{k+k_3+\cdots+k_n=r\\k\neq 0}}\frac{k(x_1+x_2-1)(x_1-x_2)+(x_1-x_2)(2-x_1-x_2)\sum\limits_{j=3}^{n}\frac{k_jx_j}{1-x_j}}{(1-x_1)^2(\frac{kx_1}{1-x_1}+\sum\limits_{j=3}^{n}\frac{k_jx_j}{1-x_j})(1-x_2)^2(\frac{kx_2}{1-x_2}+\sum\limits_{j=3}^{n}\frac{k_jx_j}{1-x_j})}\end{aligned} \tag{16}$$

and

$$\begin{aligned}D_2&=\sum_{\substack{k+m+i_3+\cdots+i_n=r\\k\neq 0,m\neq 0}}\Big(\frac{k}{(1-x_1)^2(\frac{kx_1}{1-x_1}+\frac{mx_2}{1-x_2}+\sum\limits_{j=3}^{n}\frac{k_jx_j}{1-x_j})}-\frac{k}{(1-x_2)^2(\frac{kx_2}{1-x_2}+\frac{mx_1}{1-x_1}+\sum\limits_{j=3}^{n}\frac{k_jx_j}{1-x_j})}\Big)\\&=k\sum_{\substack{k+m+i_3+\cdots+i_n=r\\k\neq 0,m\neq 0}}\frac{\delta_1}{(1-x_1)^2(\frac{kx_1}{1-x_1}+\frac{mx_2}{1-x_2}+\sum\limits_{j=3}^{n}\frac{k_jx_j}{1-x_j})(1-x_2)^2(\frac{kx_2}{1-x_2}+\frac{mx_1}{1-x_1}+\sum\limits_{j=3}^{n}\frac{k_jx_j}{1-x_j})}\end{aligned}$$

where

$$\delta_1=k(x_1+x_2-1)(x_1-x_2)+\left(\frac{(1-x_2)^2mx_1}{1-x_1}-\frac{(1-x_1)^2mx_2}{1-x_2}\right)+(x_1-x_2)(2-x_1-x_2)\sum_{j=3}^{n}\frac{k_jx_j}{1-x_j}.$$

Let $q(t)=\frac{(1-t)^3}{mt}$. Then $q'(t)=-\frac{m(1+2t)(1-t)^2}{m^2t^2}\leq 0$, this means that $q(t)$ is descending on $\mathbb{R}_+$. So that $\frac{(1-x_1)^3}{mx_1}\leq\frac{(1-x_2)^3}{mx_2}$, namely $\frac{(1-x_2)^2mx_1}{1-x_1}-\frac{(1-x_1)^2mx_2}{1-x_2}\geq 0$. It is easy to see that $D_1\geq 0$ and $D_2\geq 0$ for $\boldsymbol{x}\in[\frac{1}{2},1)^n$, so

$$\frac{\partial c_n^*\left(\frac{\boldsymbol{x}}{1-\boldsymbol{x}},r\right)}{\partial x_1}-\frac{\partial c_n^*\left(\frac{\boldsymbol{x}}{1-\boldsymbol{x}},r\right)}{\partial x_2}\geq 0,$$

by Lemma 1, it follows that $c_n^*\left(\frac{\boldsymbol{x}}{1-\boldsymbol{x}},r\right)$ is Schur-convex on $[\frac{1}{2},1)^n$.

(ii)

Notice that

$$c_n^*\left(\frac{\boldsymbol{x}}{\boldsymbol{x}-1},r\right)=(-1)^rc_n^*\left(\frac{\boldsymbol{x}}{1-\boldsymbol{x}},r\right), \tag{17}$$

combining with the Schur-convexity of $c_n^*\left(\frac{\boldsymbol{x}}{\boldsymbol{x}-1}, r\right)$ on $(1,+\infty)^n$ (see Theorem 1), we can prove (ii) in Theorem 2.

(iii) For $t<0$, from (13), we have $p(t)<0$, $p'(t)>0$ and $p''(t)>0$, this means that $p(t)$ is an increasing convex function with a negative value for $t<0$.

By Proposition 6, we know that if r is an even integer, then $c_n^*(\boldsymbol{x},r)$ is decreasing and Schur-concave on $\mathbb{R}_-^n$, from Lemma 5 (ii), it follows that $c_n^*\left(\frac{\boldsymbol{x}}{1-\boldsymbol{x}}, r\right)$ is decreasing and Schur-concave on $\mathbb{R}_-^n$.

From Proposition 6, we know that if r is an odd integer, then $c_n^*(\boldsymbol{x},r)$ is increasing and Schur-convex on $\mathbb{R}_-^n$, by Lemma 5 (i), it follows that $c_n^*\left(\frac{\boldsymbol{x}}{1-\boldsymbol{x}}, r\right)$ is increasing and Schur-convex on $\mathbb{R}_-^n$.

The proof of Theorem 2 is completed. □

Proof of Theorem 3:

For $r=1$ and $r=2$, it is easy to prove that $c_n^*\left(\frac{\boldsymbol{x}}{1-\boldsymbol{x}}, r\right)$ is Schur-geometrically convex on $(0,1)^n$.

Now consider the case of $r\ge 3$. By the symmetry of $c_n^*\left(\frac{\boldsymbol{x}}{1-\boldsymbol{x}}, r\right)$, without loss of generality, we can set $x_1>x_2$.

From (14) and (15), it follows that

$$x_1\frac{\partial c_n^*\left(\frac{\boldsymbol{x}}{1-\boldsymbol{x}}, r\right)}{\partial x_1}-x_2\frac{\partial c_n^*\left(\frac{\boldsymbol{x}}{1-\boldsymbol{x}}, r\right)}{\partial x_2}=c_n^*\left(\frac{\boldsymbol{x}}{1-\boldsymbol{x}}, r\right)(E_1+E_2),$$

where

$$\begin{aligned}E_1&=\sum_{\substack{k+k_3+\cdots+k_n=r\\k\neq 0}}\Big(\frac{kx_1}{(1-x_1)^2(\frac{kx_1}{1-x_1}+\sum\limits_{j=3}^n\frac{k_jx_j}{1-x_j})}-\frac{kx_2}{(1-x_2)^2(\frac{kx_2}{1-x_2}+\sum\limits_{j=3}^n\frac{k_jx_j}{1-x_j})}\Big)\\&=k\sum_{\substack{k+k_3+\cdots+k_n=r\\k\neq 0}}\frac{kx_1x_2(x_1-x_2)+(x_1-x_2)(1-x_1x_2)\sum\limits_{j=3}^n\frac{k_jx_j}{1-x_j}}{(1-x_1)^2(\frac{kx_1}{1-x_1}+\sum\limits_{j=3}^n\frac{k_jx_j}{1-x_j})(1-x_2)^2(\frac{kx_2}{1-x_2}+\sum\limits_{j=3}^n\frac{k_jx_j}{1-x_j})}\end{aligned}$$

and

$$\begin{aligned}E_2&=\sum_{\substack{k+m+i_3+\cdots+i_n=r\\k\neq 0,m\neq 0}}\Big(\frac{kx_1}{(1-x_1)^2(\frac{kx_1}{1-x_1}+\frac{mx_2}{1-x_2}+\sum\limits_{j=3}^n\frac{k_jx_j}{1-x_j})}-\frac{kx_2}{(1-x_2)^2(\frac{kx_2}{1-x_2}+\frac{mx_1}{1-x_1}+\sum\limits_{j=3}^n\frac{k_jx_j}{1-x_j})}\Big)\\&=k\sum_{\substack{k+m+i_3+\cdots+i_n=r\\k\neq 0,m\neq 0}}\frac{\delta_2}{(1-x_1)^2(\frac{kx_1}{1-x_1}+\frac{mx_2}{1-x_2}+\sum\limits_{j=3}^n\frac{k_jx_j}{1-x_j})(1-x_2)^2(\frac{kx_2}{1-x_2}+\frac{mx_1}{1-x_1}+\sum\limits_{j=3}^n\frac{k_jx_j}{1-x_j})}\end{aligned}$$

where

$$\delta_2=kx_1x_2(x_1-x_2)+\left(\frac{(1-x_2)^2mx_1^2}{1-x_1}-\frac{(1-x_1)^2mx_2^2}{1-x_2}\right)+(x_1-x_2)(1-x_1x_2)\sum_{j=3}^n\frac{k_jx_j}{1-x_j}.$$

Let $s(t)=\frac{(1-t)^3}{t^2}$. Then $s'(t)=-\frac{t(2+t)(1-t)^2}{t^4}\le 0$, this means that $s(t)$ is decreasing on $\mathbb{R}_+$, so $\frac{(1-x_1)^3}{x_1^2}\le\frac{(1-x_2)^3}{x_2^2}$, namely, $\frac{(1-x_2)^2mx_1^2}{1-x_1}-\frac{(1-x_1)^2mx_2^2}{1-x_2}\ge 0$. It is easy to see that $E_1\ge 0$ and $E_2\ge 0$ for $\boldsymbol{x}\in(0,1)^n\cup(1,+\infty)^n$, so

$$x_1\frac{\partial c_n^*\left(\frac{\boldsymbol{x}}{1-\boldsymbol{x}}, r\right)}{\partial x_1}-x_2\frac{\partial c_n^*\left(\frac{\boldsymbol{x}}{1-\boldsymbol{x}}, r\right)}{\partial x_2}\ge 0,$$

By Lemma 3, it follows that $c_n^*\left(\frac{\boldsymbol{x}}{1-\boldsymbol{x}}, r\right)$ is Schur-geometrically convex on $(0,1)^n$.

(ii) From (17) and combining with the Schur-geometrically convexity of $c_n^*\left(\frac{\boldsymbol{x}}{\boldsymbol{x}-1}, r\right)$ on $(1,+\infty)^n$(see Theorem 1), we can prove (ii) in Theorem 3.

The proof of Theorem 3 is completed. □

Proof of Theorem 4:

For $r=1$ and $r=2$, it is easy to prove that $c_n^*\left(\frac{\boldsymbol{x}}{1-\boldsymbol{x}}, r\right)$ is Schur-harmonically convex on $(0,1)^n$.

Now consider the case of $r \geq 3$. By the symmetry of $c_n^*\left(\frac{\boldsymbol{x}}{1-\boldsymbol{x}}, r\right)$, without loss of generality, we can set $x_1 > x_2$.

From (14) and (15), we have

$$x_1^2 \frac{\partial c_n^*\left(\frac{\boldsymbol{x}}{1-\boldsymbol{x}}, r\right)}{\partial x_1} - x_2^2 \frac{\partial c_n^*\left(\frac{\boldsymbol{x}}{1-\boldsymbol{x}}, r\right)}{\partial x_2} = c_n^*\left(\frac{\boldsymbol{x}}{1-\boldsymbol{x}}, r\right)(F_1+F_2),$$

where

$$\begin{aligned} F_1 &= \sum_{\substack{k+k_3+\cdots+k_n=r \\ k\neq 0}} \left(\frac{kx_1^2}{(1-x_1)^2(\frac{kx_1}{1-x_1}+\sum_{j=3}^n \frac{k_jx_j}{1-x_j})} - \frac{kx_2^2}{(1-x_2)^2(\frac{kx_2}{1-x_2}+\sum_{j=3}^n \frac{k_jx_j}{1-x_j})}\right) \\ &= k \sum_{\substack{k+k_3+\cdots+k_n=r \\ k\neq 0}} \frac{kx_1x_2(x_1-x_2)+(x_1-x_2)(x_1+x_2-2x_1x_2)\sum_{j=3}^n \frac{k_jx_j}{1-x_j}}{(1-x_1)^2(\frac{kx_1}{1-x_1}+\sum_{j=3}^n \frac{k_jx_j}{1-x_j})(1-x_2)^2(\frac{kx_2}{1-x_2}+\sum_{j=3}^n \frac{k_jx_j}{1-x_j})} \end{aligned}$$

and

$$\begin{aligned} F_2 &= \sum_{\substack{k+m+i_3+\cdots+i_n=r \\ k\neq 0, m\neq 0}} \left(\frac{kx_1^2}{(1-x_1)^2(\frac{kx_1}{1-x_1}+\frac{mx_2}{1-x_2}+\sum_{j=3}^n \frac{k_jx_j}{1-x_j})} - \frac{kx_2^2}{(1-x_2)^2(\frac{kx_2}{1-x_2}+\frac{mx_1}{1-x_1}+\sum_{j=3}^n \frac{k_jx_j}{1-x_j})}\right) \\ &= k \sum_{\substack{k+m+i_3+\cdots+i_n=r \\ k\neq 0, m\neq 0}} \frac{\delta_3}{(1-x_1)^2(\frac{kx_1}{1-x_1}+\frac{mx_2}{1-x_2}+\sum_{j=3}^n \frac{k_jx_j}{1-x_j})(1-x_2)^2(\frac{kx_2}{1-x_2}+\frac{mx_1}{1-x_1}+\sum_{j=3}^n \frac{k_jx_j}{1-x_j})} \end{aligned}$$

where

$$\delta_3 = kx_1x_2(x_1-x_2) + \left(\frac{(1-x_2)^2mx_1^3}{1-x_1} - \frac{(1-x_1)^2mx_2^3}{1-x_2}\right) + (x_1-x_2)(x_1+x_2-2x_1x_2)\sum_{j=3}^n \frac{k_jx_j}{1-x_j}.$$

Let $v(t) = \frac{(1-t)^3}{mt^3}$. Then $v'(t) = -\frac{3mt^2(1-t)^2}{m^2t^6} \leq 0$ this means that $v(t)$ is decreasing on $\mathbb{R}$, so $\frac{(1-x_1)^3}{mx_1^3} \leq \frac{(1-x_2)^3}{mx_2^3}$, namely, $\frac{(1-x_2)^2mx_1^3}{1-x_1} - \frac{(1-x_1)^2mx_2^3}{1-x_2} \geq 0$. It is easy to see that $F_1 \geq 0$ and $F_2 \geq 0$ for $\boldsymbol{x} \in (0,1)^n$, and then

$$x_1^2 \frac{\partial c_n^*\left(\frac{\boldsymbol{x}}{1-\boldsymbol{x}}, r\right)}{\partial x_1} - x_2^2 \frac{\partial c_n^*\left(\frac{\boldsymbol{x}}{1-\boldsymbol{x}}, r\right)}{\partial x_2} \geq 0,$$

By Lemma 3, it follows that $c_n^*\left(\frac{\boldsymbol{x}}{1-\boldsymbol{x}}, r\right)$ is Schur-harmonically convex on $(0,1)^n$.

From Theorem 2, we know that $c_n^*\left(\frac{\boldsymbol{x}}{1-\boldsymbol{x}}, r\right)$ is Schur-geometrically convex on $(0,1)^n$, so that according to Lemma 5, it follows that $c_n^*\left(\frac{\boldsymbol{x}}{1-\boldsymbol{x}}, r\right)$ is Schur-harmonically convex on $(0,1)^n$.

(ii) From (17) and combining with the Schur-harmonically convexity of $c_n^*\left(\frac{\boldsymbol{x}}{\boldsymbol{x}-1}, r\right)$ on $(1,+\infty)^n$ (see Theorem 1), we can prove (ii) in Theorem 4.

The proof of Theorem 4 is completed. □

Here, a question arises naturally.

Question 1. For $\boldsymbol{x} \in \left(0, \frac{1}{2}\right)^n$, what is the Schur-convexity of $c_n^*\left(\frac{\boldsymbol{x}}{1-\boldsymbol{x}}, r\right)$?

4. Applications

It is not difficult to prove the following result by applying Theorem 2 and the majorizing relation

$$(A_n(\boldsymbol{x}), A_n(\boldsymbol{x}), \ldots, A_n(\boldsymbol{x})) \prec (x_1, x_2, \ldots, x_n)\,.$$

Theorem 5. *If $\boldsymbol{x} = (x_1, x_2, \ldots, x_n) \in [\frac{1}{2}, 1)^n$ and $r \in \mathbb{N}$, or r is even integer and $\boldsymbol{x} \in (1,+\infty)^n$ or r is odd integer and $\boldsymbol{x} \in \mathbb{R}_-^n$, then*

$$c_n^*\left(\frac{\boldsymbol{x}}{1-\boldsymbol{x}}, r\right) \geq \left(\frac{rA_n(\boldsymbol{x})}{1-A_n(\boldsymbol{x})}\right)^{\binom{n+r-1}{r}}, \tag{18}$$

where $A_n(\boldsymbol{x}) = \frac{1}{n}\sum_{i=1}^n x_i$ and $\binom{n+r-1}{r} = \frac{(n+r-1)!}{r!((n+r-1)-r)!}$.

If r is odd and $\boldsymbol{x} \in (1,+\infty)^n$, or r is even integer and $\boldsymbol{x} \in \mathbb{R}_-^n$, then the inequality (18) *is reversed.*

By Theorem 3 and the majorizing relation

$$(\log G_n(\boldsymbol{x}), \log G_n(\boldsymbol{x}), \ldots, \log G_n(\boldsymbol{x})) \prec (\log x_1, \log x_2, \ldots, \log x_n)\,,$$

we can establish the following theorem.

Theorem 6. *If $\boldsymbol{x} = (x_1, x_2, \ldots, x_n) \in (0,1)^n$ and $r \in \mathbb{N}$ or r is even integer $\boldsymbol{x} \in (1,+\infty)^n$, then*

$$c_n^*\left(\frac{\boldsymbol{x}}{1-\boldsymbol{x}}, r\right) \geq \left(\frac{rG_n(\boldsymbol{x})}{1-G_n(\boldsymbol{x})}\right)^{\binom{n+r-1}{r}}, \tag{19}$$

where $G_n(\boldsymbol{x}) = \sqrt[n]{\prod_{i=1}^n x_i}$ and $\binom{n+r-1}{r} = \frac{(n+r-1)!}{r!((n+r-1)-r)!}$.

If r is odd integer and $\boldsymbol{x} \in (1,+\infty)^n$, then the inequality (19) *is reversed.*

By using Theorem 4 and the majorizing relation

$$\left(\frac{1}{H_n(\boldsymbol{x})}, \frac{1}{H_n(\boldsymbol{x})}, \ldots, \frac{1}{H_n(\boldsymbol{x})}\right) \prec \left(\frac{1}{x_1}, \frac{1}{x_2}, \ldots, \frac{1}{x_n}\right),$$

we obtain the following theorem.

Theorem 7. *If* $\boldsymbol{x} = (x_1, x_2, \dots, x_n) \in (0,1)^n$ *and* $r \in \mathbb{N}$, *or* r *is even integer and* $\boldsymbol{x} \in (1, +\infty)^n$, *then*

$$c_n^*\left(\frac{\boldsymbol{x}}{1-\boldsymbol{x}}, r\right) \geq \left(\frac{rH_n(\boldsymbol{x})}{1-H_n(\boldsymbol{x})}\right)^{\binom{n+r-1}{r}}, \tag{20}$$

where $H_n(\boldsymbol{x}) = \frac{n}{\sum_{i=1}^n x_i^{-1}}$ *and* $\binom{n+r-1}{r} = \frac{(n+r-1)!}{r!((n+r-1)-r)!}$.

If r *is odd and* $\boldsymbol{x} \in (1, +\infty^n$, *then the inequality* (20) *is reversed.*

By applying Theorem 2 and Lemma 6, it is not difficult to show the following theorem.

Theorem 8. *If* $\boldsymbol{x} = (x_1, x_2, \dots, x_n) \in \mathbb{R}_+^n, n \geq 2$ *and* $k \in \mathbb{N}$, $0 < r \leq s$, *then*

$$\prod_{i_1+i_2+\dots+i_n=k} \sum_{j=1}^n \frac{i_j x_j^r}{\sum_{j=1}^n x_j^r - x_j^r} \leq \prod_{i_1+i_2+\dots+i_n=k} \sum_{j=1}^n \frac{i_j x_j^s}{\sum_{j=1}^n x_j^s - x_j^s}. \tag{21}$$

By Theorem 2 and Lemma 7, we establish the following theorem.

Theorem 9. *Let* $\boldsymbol{x} = (x_1, x_2, \dots, x_n) \in \mathbb{R}_+^n, n \geq 2, \sum_{i=1}^n x_i = s > 0, c \geq s$. *Then*

$$\prod_{i_1+i_2+\dots+i_n=k} \sum_{j=1}^n \frac{i_j(c-x_j)}{(n-1)c-(s-x_i)} \leq \prod_{i_1+i_2+\dots+i_n=k} \sum_{j=1}^n \frac{i_j x_j}{s-x_j}. \tag{22}$$

Discovering and judging Schur convexity of various symmetric functions is an important subject in the study of the majorization theory. In recent years, many domestic scholars have made a lot of achievements in this field (see monographs [27, 28]).

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