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The Hasimoto Surface According to Bishop Frame

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Abstract

In this paper, we investigate the Hasimoto surfaces in Euclidean 3- space. Firstly, we investigate the geometric properties of these surfaces in Euclidean 3-space. Especially, we obtain the curvatures of Hasimoto surface according to Bishop frame. Then we give some characterization of parameter curves obtained according to Bishop frame of Hasimoto surfaces.

Keywords: Hasimoto surfaces, Smoke ring equation, Bishop frame.

Bishop Çatısı ile İlişkilendirilmiş Hasimoto Yüzeyleri

Özet

Bu çalışmada Öklidyen 3-uzayındaki Hasimoto yüzeyleri incelenmiştir. İlk olarak, Öklidyen 3-uzayındaki Hasimoto yüzeylerinin geometrik özellikleri incelenmiştir. Özellikle Bishop çatısı ile ilişkilendirilmiş bu yüzeylerin eğrilikleri elde edilmiştir. Daha sonrasında bu yüzeylerin Bishop çatısına göre parametre eğrilerinin bazı karakterizasyonları verilmiştir.

Anahtar Kelimeler: Hasimoto yüzeyler, Duman halkası denklemi, Bishop çatısı.

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1. Introduction

In the existing literature, it can be seen that, most of classical differential geometry topics have been extended to curves and surfaces. In this process, generally, researchers used standard moving Frenet frame. Some of kinematical models were adapted on this moving frame, due to transformation matrix among derivative vectors and frame vectors. Thereafter, researchers aimed to have an alternative frame for curves and other applications. Bishop frame, which is also called alternative or parallel frame of the curves, was introduced by L. R. Bishop in 1975 by means of paralllel vector fields, [2]. Recently, many research papers related to this concept have been treated in Euclidean space [3,10].

In [5], Da Rios invoked what is now known as the localized induction approximation to derive a pair of coupled nonlinear equations guiding the time evolution of the torsion and curvature of vortex filament also called smoke ring equations. Additively, in 1972, Hasimoto [8] demonstrated that the Da Rios equations may be associated to generate the celebrated nonlinear Schrodinger (NLS) equation of soliton theory and also in this work, he considered a proximity to the selfinduced motion of a thin isolated vortex filament moving without extending in an incompressible fluid. Finally he obtain that if the position vector of vortex filament is $\sigma=\sigma(u,v)$, then the formula

$$\sigma_v = \sigma_u \wedge \sigma_{uu} \tag{1.1}$$

is hold. In [5], the Da Rios equations and their composition, the NLS equation, are derived in a purely geometric manner via a binormal motion of an inextensible curve, (see for details, [9]). While in [7], authors investiged its geometric properties and also gave some characterizations of parametric curves of Hasimoto surface in Minkowski 3-space, authors discussed on the Hasimoto surface in Euclidean 3-space, where they constructed these surfaces from its fundamental form coefficients via numerical integration of Gauss-Weingarten equations and fundamental theorem of surfaces in [1]. Also in [3], authors studied parallel surfaces of Hasimoto surfaces in Euclidean 3-space.

In that work, we move the study of Hasimoto surfaces started in [1] into the Minkowski space. First, we investigate the geometric properties of these surfaces in Euclidean 3-space. Especially, we obtain the curvatures of Hasimoto surface according to Bishop frame. Then, we give some characterization of parameter curves obtained according to Bishop frame of Hasimoto surfaces.

2. Preliminiaries

Let E^3 denote the three-dimensional Euclidean space, that is, the real vector space R^3 endowed with the Riemann metric

$$\langle\, ,\rangle = -(d\xi_0)^2 + (d\xi_1)^2 + (d\xi_2)^2$$

where (ξ_0, ξ_1, ξ_2) is rectangular coordinate system of E^3. Let u an arbitrary vector in E^3. So, the norm of u is given by $\|u\| = \sqrt{|\langle u, u\rangle|}$. The Euclidean cross product of $u = (u_1, u_2, u_3)$ and $v = (v_1, v_2, v_3) \in E^3$ is defined by

$$u \times v = e_1 \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} - e_2 \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} + e_3 \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}.$$

Let $\mathbb{III}$ be a simply-connected domain in the plane, E^2 and σ: $\mathbb{III} \to E^3$ an immersion in E^3. If σ=σ(s,t) is a parametrization of surface M in E^3, then the unit normal vector field N on M is given by

$$N = \frac{\sigma_s \times \sigma_t}{\|\sigma_s \times \sigma_t\|} \tag{2.1}$$

where $\sigma_s = \partial\sigma/\partial s$ and $\sigma_t = \partial\sigma/\partial t$, $\times$ stands for the Euclidean cross product of E^3, [6].

The metric $\langle,\rangle$ on each tangent plane of M is determined by the first fundemantel form

$$I = \langle d\sigma, d\sigma\rangle = Eds^2 + 2Fdsdt + Gdt^2$$

with differentiable coefficients

$$E = \langle \sigma_s, \sigma_s \rangle, F = \langle \sigma_s, \sigma_t \rangle, G = \langle \sigma_t, \sigma_t \rangle.$$

Since we have,

$$\det I = EG - F^2.$$

On the other hand, the second fundamental form is given by

$$II = -\langle dN, d\sigma \rangle = eds^2 + 2fdsdt + gdt^2$$

with differentiable coefficients

$$e = \langle \sigma_{ss}, N \rangle, f = \langle \sigma_{st}, N \rangle, g = \langle \sigma_{tt}, N \rangle. \tag{2.2}$$

Therefore, the Gaussian curvature K and the mean curvature H of surface $\sum$ are defined by, respectively,

$$K = \frac{eg - f^2}{EG - F^2}, \tag{2.3a}$$

$$H = \frac{eG - 2fF + gE}{2\left(EG - F^2\right)}. \tag{2.3b}$$

Let $\left(E^3, \langle , \rangle\right)$ be a time-oriented Riemann 3-manifold. Let $\Gamma : I \to \Sigma$ be a unit speed curve, i.e., $\langle \Gamma', \Gamma' \rangle = 1.$ As well knowing, the curve Γ is called as a Frenet curve in case $\langle \Gamma'', \Gamma'' \rangle \neq 0.$ Also, we know that any Frenet curve Γ in $\left(\Sigma, \langle , \rangle\right)$ admits a Frenet frame field along Γ. Here, a Frenet frame field $P = \left(t, n, b\right)$ is an orthonormal frame field along Γ such that t = Γ′(s) called the tangent vector field. Also, n and b are called, principal normal vector field, and binormal vector field of Γ, respectively. Morever, ∇ is the Levi-Civita connection of and ***P*** satisfies the following Frenet-Serret formula,

$$\nabla_{\omega'} P = P \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix}.$$

The functions κ and τ are called the first and second curvatures of the curve Γ, respectively, [6].

On the other hand, the Bishop frame or parallel transport frame is an alternative approach to defining a moving frame that is well defined even when the curve Γ has vanishing second derivative. One can state parallel transport of an orthonormal frame along a curve simply by parallel transporting each component of the frame [2]. The tangent vector and any convenient arbitrary basis for remainder of the frame are used [3,10]. The Bishop frame is expressed as;

$$\begin{bmatrix} t \\ y \\ z \end{bmatrix}_s = \begin{bmatrix} 0 & k_1 & k_2 \\ -k_1 & 0 & 0 \\ -k_2 & 0 & 0 \end{bmatrix} \begin{bmatrix} t \\ y \\ z \end{bmatrix}, \tag{2.4}$$

where the set of $\{t, y, z\}$ is called as Bishop trihedra and the functions k_1 and k_2 are the Bishop curvatures, (see for details in [2]). The relation matrix between Bishop and Frenet vector fields may be express as

$$\begin{bmatrix} t \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta(s) & -\sin\theta(s) \\ 0 & \sin\theta(s) & \cos\theta(s) \end{bmatrix} \begin{bmatrix} t \\ n \\ b \end{bmatrix}, \tag{2.5}$$

where $\theta(s) = \arctan\dfrac{k_2}{k_1}$, $\tau(s) = \theta'(s)$ and $\kappa(s) = \sqrt{k_1^2 + k_2^2}$ for all t. Here, Bishop curvatures are defined by

$$k_1 = \kappa\cos\theta(s), \tag{2.6a}$$

$$k_2 = \kappa\sin\theta(s). \tag{2.6b}$$

3. Geometric Properties of Hasimoto Surfaces According to Bishop Frame in E³

In this section devoted to obtain time derivatives of the Bishop frame {t,y,z} which is given the form

$$\begin{aligned} t_t &= \alpha y + \beta z, \\ y_t &= -\alpha t + \gamma z, \\ z_t &= -\beta t - \gamma y, \end{aligned} \tag{3.1}$$

where α, β and γ are smooth functions. For this aim, we have to find firstly {α,β,γ} non-zero smooth functions s in terms of the Bishop curvatures k_1 and k_2. Applying the compatibility conditions $t_{ts} = t_{st}$ into the systems (2.3), (3.1) yields

$$\begin{aligned} \alpha_s &= (k_1)_t - k_2\gamma, \\ \beta_s &= (k_2)_t + k_1\gamma, \\ \gamma_s &= -k_1\beta + k_2\alpha. \end{aligned} \tag{3.2}$$

Assume the velocity vector of a moving curve Γ has the decomposition

$$\sigma_t = \lambda t + \mu y + \upsilon z \tag{3.3}$$

then by considering the imposition of condition $\sigma_{st} = \sigma_{ts}$, we find the following equalities

$$\begin{aligned} \lambda_s &= k_1\mu + k_2\upsilon, \\ \alpha &= k_1\lambda + \mu_s, \\ \beta &= k_2\lambda + \upsilon_s. \end{aligned} \tag{3.4}$$

One can choose the correspondance for the surface M as $\{\lambda, \mu, \upsilon\} \to \{0, -k_2, k_1\}$. Thus, the velocity vector is given by

$$\sigma_t = \sigma_s \times \sigma_{ss},$$

$$= t \times (k_1 y + k_2 z), \tag{3.5}$$

$$= -k_2 y + k_1 z \tag{3.6}$$

which is the solution of smoke ring equation. Hence, we can rewrite $(3.4)_{2,3}$ under the correspondance as

$$\alpha = -k_2', \tag{3.7}$$

$$\beta = -k_1'. \tag{3.8}$$

Substituting the last equations into $(3.4)_1$ gives

$$\gamma = -\frac{k_1^2 + k_2^2}{2}. \tag{3.9}$$

Therefore, we proved the following theorem being an expression of the linear representation of the surface Σ:

Theorem 1 Let σ=σ(s,t) be the position vector of a curve Γ moving on surface Σ in Euclidean 3-space such that σ=σ(s,t) is a unit speed curve for all t. Then the followings are satisfied;

$$\begin{bmatrix} t \\ y \\ z \end{bmatrix}_s = \begin{bmatrix} 0 & k_1 & k_2 \\ -k_1 & 0 & 0 \\ -k_2 & 0 & 0 \end{bmatrix} \begin{bmatrix} t \\ y \\ z \end{bmatrix}, \tag{3.10a}$$

$$\begin{bmatrix} t \\ y \\ z \end{bmatrix}_t = \begin{bmatrix} 0 & -k_2^{'} & k_1^{'} \\ k_2^{'} & 0 & -\dfrac{k_1^2+k_2^2}{2} \\ -k_1^{'} & \dfrac{k_1^2+k_2^2}{2} & 0 \end{bmatrix} \begin{bmatrix} t \\ y \\ z \end{bmatrix}, \tag{3.10b}$$

where {t,y,z} is the Bishop frame field, α, β and γ are given in (3.7), (3.8) and (3.9), respectively. Here, the functions k_1 and k_2 are the Bishop curvature functions, depending only on parameter s, of the curve Γ for all t.

Proposition 2 Let σ=σ(s,t) be the position vector of a curve Γ moving on surface $\sum$ in Euclidean 3-space. Then the Gaussion curvature K and the mean curvature H of surface $\sum$ are given by

$$K = \frac{\kappa^2 - 2\tau^2}{2\kappa^2}, \tag{3.11a}$$

$$H = -\frac{3}{4}, \tag{3.11b}$$

respectively. Here, κ is the curvature function and τ is the torsion function of the curve Γ for all t.

Proof. Assume that σ=σ(s,t) is the position vector of a curve Γ moving on surface $\sum$ in Euclidean 3-space. We need to find the coefficients of first and second fundamental forms of σ=σ(s,t), so

$$I = \langle d\sigma, d\sigma \rangle$$

$$I = \langle \sigma_s ds + \sigma_t dt, \sigma_s ds + \sigma_t dt \rangle$$

$$I = ds^2 + \left(k_1^2 + k_2^2\right) dt^2.$$

So, we have the first fundamental form coefficients as

$$E = 1, F = 0, G = k_1^2 + k_2^2 = \kappa^2. \tag{3.12}$$

On the other hand, by considering $\sigma_s = t$ and (3.6) into (2.1), we get the normal vector field of surface Σ as

$$N = -\frac{k_1}{k_1^2 + k_2^2} y - \frac{k_2}{k_1^2 + k_2^2} z. \tag{3.13}$$

Considering (2.6), the above vector and the derivative equations in (3.10) into the coefficients of the second fundamental form defined in (2.2), so we get

$$e = -1, f = \tau, g = -\frac{\kappa^2}{2}. \tag{3.14}$$

Therefore, taking consider the obtained results into (2.3), we obtain the Gaussian curvature K and the mean curvature H given as in (3.11). Thus, the proof is completed.

4. Some Characterization of Parameter Curves of Hasimoto Surfaces in E^3

In this section, we would like to give new characterizations of parameter curves of Hasimoto surfaces in Euclidean 3-spaces.

Theorem 3 Assume that $\sigma=\sigma(s,t)$ is a Hasimoto surface in E^3. Then the followings are satisfied;

(i) s-parameter curves of the surface $\sigma=\sigma(s,t)$ are geodesics if and only if κ is a non-zero constant,

(ii) t-parameter curves of the surface $\sigma=\sigma(s,t)$ are geodesics.

Proof. Let $\sigma=\sigma(s,t)$ be a Hasimoto surface in E^3.

(i) Let κ be a non-zero constant, Now from (3.5), we know that

$$\sigma_{ss} = k_1 y + k_2 z. \tag{3.15}$$

Thus, one can conclude that the vectors σ_{SS}, given in the above and the normal vector, N given in (3.13) are parallel which means s-parameter curves of the surface σ=σ(s,t) are geodesics. The opposite of proof can be obtained with a direct calculation.

(ii) From (3.6), we know that

$$\sigma_{tt} = \left(-k_2 y + k_1 z\right)_t . \tag{3.16}$$

As knowing, for t-parameter curves are geodesics, the binormal component of σ_{tt} must be zero. Considering the relation matrix given (2.5), we conclude

$$\sigma_{tt} = \kappa b_t .$$

Note that the derivative of binormal component b with respect to the parameter t is $b_t = -\kappa_s t - \dfrac{\kappa_{ss} - \kappa\tau^2}{\kappa} n,$ given in the equation (13), [1]. Thus, we get

$$\sigma_{tt} = -\kappa\kappa_s t - \left(\kappa_{ss} - \kappa\tau^2\right) n.$$

So, the proof is completed.

Theorem 4 Assume that σ=σ(s,t) be a Hasimoto surface in E^3. Then the followings are satisfied;

(i) s-parameter curves of the surface σ=σ(s,t) are asymtotics if and only if $\kappa = 0$,

(ii) t-parameter curves of the surface σ=σ(s,t) are asymptotics if and only if $\kappa_{ss} = \kappa\tau^2$.

Proof. Let σ=σ(s,t) be a Hasimoto surface in E^3.

(i) By considering the relation matrix given (2.5) and (3.15) with together, we have

$$\sigma_{ss} = \kappa n$$

which means s-parameter curves of the surface σ=σ(s,t) are asimptotics if and only if κ=0.

(ii) By considering the relation matrix given (2.5) and (3.16) with together, we have

$$\sigma_{tt} = \kappa b_t.$$

Here, $b_t = -\kappa_s t - \frac{\kappa_{ss} - \kappa\tau^2}{\kappa} n$. So, the proof is completed.

Corollary 5 The parameter curves of a Hasimoto surface σ=σ(s,t) in E^3 are lines of curvature if and only if τ=0.

Proof. Let the parameter curves of the surface be lines of curvature. So, F = f = 0 which concludes τ=0, from (3.12) and (3.14), directly.

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