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On generalized Γ-hyperideals in ordered Γ-semihypergroups

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Abstract

In this article, we deal with ordered generalized Γ-hyperideals in ordered Γ-semihypergroups. In particular, we study (m,n)-regular ordered Γ-semihypergroups in terms of ordered (m,n)-Γ-hyperideals. Moreover, we obtain some ideal theoretic results in ordered Γ-semihypergroups.

1. Introduction

A semigroup is an algebraic structure together with a nonempty set and an associative binary operation. The systematic study of semigroups started in the early 20th century. Semigroups are important in different areas of Mathematics. The concept of hyperstructures was introduced in 1934 as a suitable generalization of classical algebraic structures by Marty [1]. He obtained various results on hypergroups and applied them in different areas, for instance, in algebraic rational fractions, functions, and noncommutative groups. Thereafter, many research papers have been published on this subject and has been studied recently by many algebraists such as: Prenowitz, Corsini, Jantosciak, Leoreanu, Heideri, Davvaz, Hila, Gutan, Griffiths and Halzen.

It is a well known fact that, in a semigroup, the composition of two elements is an element, while in a semihypergroup, the composition of two elements is a nonempty set. In fact, semihypergroups are the simplest algebraic hyperstructures with the properties of closure and associativity. They are very important in certain applications. Around the 1940s, the general notions of the theory and some applications in Geometry, Physics and Chemistry were studied. Various classical notions of semigroups have been extended to semihypergroups and Γ-semihypergroups and a lot of results on ordered Γ-semihypergroups are obtained by many algebraists all over the world.

The monograph on application of hyperstructures to various area of study has been written by Corsini et al. [2]. Prenowitz et al. investigated its applications in Geometry [3]. Davvaz et al. wrote a book beginning with some basic notions related to ring theory and algebraic hyperstructures [4]. Various types of hyperrings are introduced and discussed in this book. For application in Chemistry and Physics, we refer [5]-[12]. It describes various types of hyperstructures: e-hyperstructures and transposition hypergroups. Heideri et al. studied ordered hyperstructures [11]. For semihypergroups, we refer [6, 7, 8]. Hila and Davvaz studied quasi-hyperideals of ordered semihypergroups [13]. Corsini also studied hypergroup theory [14]- [15]. The notion of a Γ-hyperideal of a Γ-semihypergroup was introduced by Anvariyeh et al. [16]. Hila et al. studied the structure of Γ-semihypergroups [17]. Recently, Basar et.al. obtained various types of hyperideals in ordered semihypergroups, ordered LA-Γ-semigroups and LA-Γ-semihypergroups [18]- [20].

In the second part of this paper, we recollect some basic definitions and then, we define the concepts of (m,n)-Γ-hyperideal (resp. generalized (m,n)-Γ-hyperideal) and (m,n)-regular ordered Γ-semihypergroup, where m,n are non-negative integers. In the third part of this paper, we study ordered generalized Γ-hyperideals in ordered Γ-semihypergroups. In particular, we study (m,n)-regular ordered Γ-semihypergroups in terms of ordered (m,n)-Γ-hyperideals and obtain some ideal theoretic results in ordered Γ-semihypergroups.

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2. Basic definitions

Let H be a nonempty set, then the mapping $\circ : H \times H \to H$ is called a hyperoperation or a join operation on H, where $P^{\star}(H) = P(H) \setminus \{0\}$ is the set of all nonempty subsets of H. Let A and B be two nonempty sets. Then, a hypergroupoid $(S,\circ)$ is called a Γ-semihypergroups if for every $x,y,z \in S$ and $\alpha,\beta \in \Gamma$,

$$x\circ\alpha\circ(y\circ\beta\circ z) = (x\circ\alpha\circ y)\circ\beta\circ z,$$

i.e.,

$$\bigcup_{u\in y\circ\alpha\circ z} x\circ\alpha\circ u = \bigcup_{v\in x\circ\alpha\circ y} v\circ\beta\circ z.$$

A Γ-semihypergroup $(S,\circ)$ together with a partial order " $\leq$ " on S that is compatible with Γ-semihypergroup operation such that for all $x,y,z \in S$, we have

$$x \leq y \Rightarrow z\circ\alpha\circ x \leq z\circ\beta\circ y \text{ and } \mathrm{x}\circ\alpha\circ\mathrm{z} \leq \mathrm{y}\circ\beta\circ\mathrm{z},$$

is called an ordered Γ-semihypergroup. For subsets A,B of an ordered Γ-semihypergroup S, the product set $A\circ\Gamma\circ B$ of the pair (A,B) relative to S is defined as below:

$$A\circ\Gamma\circ B = \{a\circ\gamma\circ b : a\in A, b\in B, \gamma\in\Gamma\},$$

and for $A \subseteq S$, the product set $A\circ\Gamma\circ A$ relative to S is defined as $A^2 = A\circ\Gamma\circ A$.

For $M \subseteq S$, $(M] = \{s \in S \mid s \leq m \text{ for some } m \in M\}$. Also, we write $(s]$ instead of $(\{s\}]$ for $s \in S$.

Let $A \subseteq S$. Then, for a non-negative integer m, the power of A is defined by $A^m = A\circ\Gamma\circ A\circ\Gamma\circ A\circ\Gamma\circ A\cdots$, where A occurs m times. Note that the power vanishes if $m = 0$. So, $A^0\circ\Gamma\circ S = S = S\circ\Gamma\circ A^0$.

In what follows, we denote ordered Γ-semihypergroup $(S,\circ,\Gamma,\leq)$ by S unless otherwise specified.

Suppose S is an ordered Γ-semihypergroup and I is a nonempty subset of S. Then, I is called an ordered right (resp. left) Γ-hyperideal of S if

(i) $I\circ\Gamma\circ S \subseteq I (resp.\ S\circ\Gamma\circ I \subseteq I)$,
(ii) $a \in I, b \leq a$ for $b \in S \Rightarrow b \in I$.

We now define the concepts of (m,n)-Γ-hyperideal (resp. generalized (m,n)-Γ-hyperideal) and (m,n)-regular ordered Γ-semihypergroup, where m,n are non-negative integers.

Definition 2.1. Suppose B is a sub-Γ-semihypergroup (resp. nonempty subset) of an ordered Γ-semihypergroup S. Then, B is called an (m,n)-Γ-hyperideal (resp. generalized (m,n)-Γ-hyperideal) of S, where m,n are non-negative integers if (i) $B^m\circ\Gamma\circ S\circ\Gamma\circ B^n \subseteq B$, and (ii) for $b \in B$, $s \in S$, $s \leq b \Rightarrow s \in B$.

Note that in the above Definition 2.1, if we set $m = n = 1$, then B is called a (generalized) bi-Γ-hyperideal of S.

Definition 2.2. Suppose $(S,\Gamma,\circ,\leq)$ is an ordered Γ-semihypergroup and m,n are non-negative integers. Then, S is called (m,n)-regular if for any $s \in S$, there exists $x \in S$ such that $s \leq s^m\circ\gamma_1\circ x\circ\gamma_2\circ s^n$ for $\gamma_1,\gamma_2 \in \Gamma$. Equivalently: $(S,\Gamma,\circ,\leq)$ is (m,n)-regular if $s \in (s^m\circ\Gamma\circ S\circ\Gamma\circ s^n]$ for all $s \in S$.

3. Ordered (m,n)-Γ-hyperideals

In this part, some classical notions of semigroups and semihypergroups have been extended to ordered Γ-semihypergroups and some results on generalized ordeded (m,n)-Γ-hyperideals and (m,n)-regular ordered Γ-semihypergroups are obtained. The results concern with ordered Γ-semihypergroup theory which represent the most general algebraic context in which these results are studied. We begin with the following:

Lemma 3.1. Suppose $(S,\Gamma,\circ,\leq)$ is an ordered Γ-semihypergroup and $s \in S$. Let m,n be non-negative integers. Then, the intersection of all ordered (generalized) (m,n)-Γ-hyperideals of S containing s, denoted by $[s]_{m,n}$, is an ordered (generalized) (m,n)-Γ-hyperideal of S containing s.

Proof. Let $\{A_i : i \in I\}$ be the set of all ordered (generalized) (m,n)-Γ-hyperideals of S containing s. Obviously, $\bigcap_{i\in I} A_i$ is a sub-Γ-semihypergroup of S containing s. Let $j \in I$. As $\bigcap_{i\in I} A_i \subseteq A_j$, we have

$$(\bigcap_{i\in I} A_i)^m\circ\Gamma\circ S\circ\Gamma\circ(\bigcap_{i\in I} A_i)^n \subseteq A_j^m\circ\Gamma\circ S\circ\Gamma\circ A_j^n \subseteq A_j.$$

Therefore,

$$(\bigcap_{i\in I} A_i)^m\circ\Gamma\circ S\circ\Gamma\circ(\bigcap_{i\in I} A_i)^n \subseteq \bigcap_{i\in I} A_i.$$

Let $a \in \bigcap_{i\in I} A_i$ and $b \in S$ so that $b \leq a$. Therefore, $b \in \bigcap_{i\in I} A_i$.
Hence, $\bigcap_{i\in I} A_i$ is an ordered (generalized) (m,n)-Γ-hyperideal of S containing s.

Theorem 3.2. Suppose $(S,\Gamma,\circ,\leq)$ is an ordered Γ-semihypergroup and $s \in S$. Then, we have the following:

(i) $[s]_{m,n} = (\bigcup_{i=1}^{m+n} s^i \cup s^m\circ\Gamma\circ S\circ\Gamma\circ s^n]$ for any positive integers m,n.
(ii) $[s]_{m,0} = (\bigcup_{i=1}^{m} s^i \cup s^m\circ\Gamma\circ S]$ for any positive integer m.
(iii) $[s]_{0,n} = (\bigcup_{i=1}^{n} s^i \cup s^n]$ for any positive integer n.

Proof. (i) $(\bigcup_{i=1}^{m+n} s^i \cup s^m \circ \Gamma \circ S \circ \Gamma \circ s^n] \neq \emptyset$. Let $a,b \in (\bigcup_{i=1}^{m+n} s^i \cup s^m \circ \Gamma \circ S \circ \Gamma \circ s^n]$ be such that $a \leq x$ and $b \leq y$ for some $x,y \in (\bigcup_{i=1}^{m+n} s^i \cup s^m \circ \Gamma \circ S \circ \Gamma \circ s^n]$. If $x,y \in s^m \circ \Gamma \circ S \circ \Gamma \circ s^n$ or $x \in \bigcup_{i=1}^{m+n} s^i$, $y \in s^m \circ \Gamma \circ S \circ \Gamma \circ s^n$ or $x \in s^m \circ \Gamma \circ S \circ \Gamma \circ s^n$, $y \in \bigcup_{i=1}^{m+n} s^i$, then, $x \circ \gamma \circ y \subseteq s^m \circ \Gamma \circ S \circ \Gamma \circ s^n$, and therefore, $x \circ \gamma \circ y \subseteq \bigcup_{i=1}^{m+n} s^i \cup s^m \circ \Gamma \circ S \circ \Gamma \circ s^n$ for $\gamma \in \Gamma$. It follows that $a \circ \gamma \circ b \subseteq (\bigcup_{i=1}^{m+n} s^i \cup s^m \circ \Gamma \circ S \circ \Gamma \circ s^n]$. Let $x,y \in \bigcup_{i=1}^{m+n} s^i$. Then, $x = s^p, y = s^q$ for some $1 \leq p,q \leq m+n$. Now, two cases arise: If $1 \leq p+q \leq m+n$, then, $x \circ \gamma \circ y \subseteq \bigcup_{i=1}^{m+n} s^i$. If $m+n < p+q$, then, $x \circ \gamma \circ y \subseteq s^m \circ \Gamma \circ S \circ \Gamma \circ s^n$. So, $x \circ \gamma \circ y \subseteq (\bigcup_{i=1}^{m+n} s^i \cup s^m \circ \Gamma \circ S \circ \Gamma \circ s^n]$. This implies that $(\bigcup_{i=1}^{m+n} s^i \cup s^m \circ \Gamma \circ S \circ \Gamma \circ s^n]$ is a sub-Γ-semihypergroup of S. Moreover, we have

$$\begin{aligned}
(\bigcup_{i=1}^{m+n} s^i \cup s^m \circ \Gamma \circ S]^m \circ \Gamma \circ S &= (\bigcup_{i=1}^{m+n} s^i \cup s^m \circ \Gamma \circ S]^{m-1} \circ \Gamma \circ (\bigcup_{i=1}^{m+n} s^i \cup s^m \circ \Gamma \circ S] \circ \Gamma \circ S \\
&\subseteq (\bigcup_{i=1}^{m+n} s^i \cup s^m \circ \Gamma \circ S]^{m-1} \circ \Gamma \circ (\bigcup_{i=1}^{m+n} s^i \circ \Gamma \circ S \cup s^m \circ \Gamma \circ S \circ \Gamma \circ S] \\
&\subseteq (\bigcup_{i=1}^{m+n} s^i \cup s^m \circ \Gamma \circ S]^{m-1} \circ \Gamma \circ (s \circ \Gamma \circ S] \\
&= (\bigcup_{i=1}^{m+n} s^i \cup s^m \circ \Gamma \circ S]^{m-2} \circ \Gamma \circ (\bigcup_{i=1}^{m+n} s^i \cup s^m \circ \Gamma \circ S] \circ \Gamma \circ (s \circ \Gamma \circ S] \\
&\subseteq (\bigcup_{i=1}^{m+n} s^i \cup s^m \circ \Gamma \circ S]^{m-2} \circ \Gamma \circ (\bigcup_{i=1}^{m+n} s^i \cup s^m \circ \Gamma \circ S \circ \Gamma \circ (s \circ \Gamma \circ S] \\
&\subseteq (\bigcup_{i=1}^{m+n} s^i \cup s^m \circ \Gamma \circ S]^{m-2} \circ \Gamma \circ (s^2 \circ \Gamma \circ S] \\
&\;\;\vdots \\
&\subseteq (s^m \circ \Gamma \circ S].
\end{aligned}$$

In a similar fashion, $S \circ \Gamma \circ (\bigcup_{i=1}^{m+n} s^i \cup s^m \circ \Gamma \circ S \circ \Gamma \circ s^n]^n \subseteq (S \circ \Gamma \circ s^n]$. Therefore, $(\bigcup_{i=1}^{m+n} s^i \cup s^m \circ \Gamma \circ S \circ \Gamma \circ s^n]^m \circ \Gamma \circ S \circ \Gamma \circ (\bigcup_{i=1}^{m+n} s^i \cup s^m \circ \Gamma \circ S \circ \Gamma \circ s^n]^n \subseteq (s^m \circ \Gamma \circ S \circ \Gamma \circ s^n] \subseteq (\bigcup_{i=1}^{m+n} s^i \cup s^m \circ \Gamma \circ S \circ \Gamma \circ s^n]$. So, $(\bigcup_{i=1}^{m+n} s^i \cup s^m \circ \Gamma \circ S \circ \Gamma \circ s^n]$ is an (m,n)-Γ-hyperideal of S containing s; hence, $[s]_{m,n} \subseteq (\bigcup_{i=1}^{m+n} s^i \cup s^m \circ \Gamma \circ S \circ \Gamma \circ s^n]$. For the reverse inclusion, suppose $a \in (\bigcup_{i=1}^{m+n} s^i \cup s^m \circ \Gamma \circ S \circ \Gamma \circ s^n]$ is such that $a \leq t$ for some $t \in (\bigcup_{i=1}^{m+n} s^i \cup s^m \circ \Gamma \circ S \circ \Gamma \circ s^n]$. If $t = s^j$ for some $1 \leq j \leq m+n$, then, $t \in [s]_{m,n}$, therefore, $a \in [s]_{m,n}$. If $t \in s^m \circ \Gamma \circ S \circ \Gamma \circ s^n$, by

$$s^m \circ \Gamma \circ S \circ \Gamma \circ s^n \subseteq ([s]_{m,n})^m \circ \Gamma \circ S \circ \Gamma \circ ([s]_{m,n})^n \subseteq [s]_{m,n},$$

then, $t \in [s]_{m,n}$; hence, $a \in [s]_{m,n}$. This implies that $(\bigcup_{i=1}^{m+n} s^i \cup s^m \circ \Gamma \circ S \circ \Gamma \circ s^n] \subseteq [s]_{m,n}$.
Hence, $[s]_{m,n} = (\bigcup_{i=1}^{m+n} s^i \cup s^m \circ \Gamma \circ S \circ \Gamma \circ s^n]$.
(ii) and (iii) can be proved in a similar fashion.

Lemma 3.3. Suppose $(S,\Gamma,\circ,\leq)$ is an ordered Γ-semihypergroup and $s \in S$. Suppose m,n are non-negative integers. Then, we have the following:

(i) $([s]_{m,0})^m \circ \Gamma \circ S \subseteq (s^m \circ \Gamma \circ S]$.
(ii) $S \circ \Gamma \circ ([s]_{0,n})^n \subseteq (S \circ \Gamma \circ s^n]$.
(iii) $([s]_{m,n})^m \circ \Gamma \circ S \circ \Gamma \circ ([s]_{m,n})^n \subseteq (s^m \circ \Gamma \circ S \circ \Gamma \circ s^n]$.

Proof. (i) Using Theorem 3.2, we have

$$\begin{aligned}
([s]_{m,0})^m \circ \Gamma \circ S &= (\bigcup_{i=1}^{m+n} s^i \cup s^m \circ \Gamma \circ S]^m \circ \Gamma \circ S \\
&= (\bigcup_{i=1}^{m+n} s^i \cup s^m \circ \Gamma \circ S]^{m-1} \circ \Gamma \circ (\bigcup_{i=1}^{m+n} s^i \cup s^m \circ \Gamma \circ S] \circ \Gamma \circ S \\
&\subseteq (\bigcup_{i=1}^{m+n} s^i \cup s^m \circ \Gamma \circ S]^{m-1} \circ \Gamma \circ (\bigcup_{i=1}^{m+n} s^i \circ \Gamma \circ S \cup s^m \circ \Gamma \circ S \circ \Gamma \circ S] \\
&\subseteq (\bigcup_{i=1}^{m+n} s^i \cup s^m \circ \Gamma \circ S]^{m-1} \circ \Gamma \circ (s \circ \Gamma \circ S] \\
&\;\;\vdots \\
&\subseteq (s^m \circ \Gamma \circ S].
\end{aligned}$$

Hence, $([s]_{m,0})^m \circ \Gamma \circ S \subseteq (s^m \circ \Gamma \circ S]$.
(ii) can be proved similarly as (i).

(iii) Applying Theorem 3.2, we have

$$\begin{aligned}
([s]_{m,n})^m \circ \Gamma \circ S &= (\bigcup_{i=1}^{m+n} s^i \cup s^m \circ \Gamma \circ S \circ \Gamma \circ s^n]^m \circ \Gamma \circ S \\
&= (\bigcup_{i=1}^{m+n} s^i \cup s^m \circ \Gamma \circ S \circ \Gamma \circ s^n]^{m-1} \circ \Gamma \circ (\bigcup_{i=1}^{m+n} s^i \cup s^m \circ \Gamma \circ S \circ \Gamma \circ s^n] \circ \Gamma \circ S \\
&\subseteq (\bigcup_{i=1}^{m+n} s^i \cup s^m \circ \Gamma \circ S \circ \Gamma \circ s^n]^{m-1} \circ \Gamma \circ (\bigcup_{i=1}^{m+n} s^i \circ \Gamma \circ S \cup s^m \circ \Gamma \circ S \circ \Gamma \circ s^n \circ \Gamma \circ S] \\
&= (\bigcup_{i=1}^{m+n} s^i \cup s^m \circ \Gamma \circ S \circ \Gamma \circ s^n]^{m-1} \circ \Gamma \circ (s \circ \Gamma \circ S] \\
&\ \ . \\
&\ \ . \\
&\ \ . \\
&= (s^m \circ \Gamma \circ S].
\end{aligned}$$

Therefore, $([s]_{m,n})^m \circ \Gamma \circ S \subseteq (s^m \circ \Gamma \circ S]$. In a similar fashion, $S \circ \Gamma \circ ([s]_{m,n})^n \subseteq (S \circ \Gamma \circ s^n]$. So,

$$\begin{aligned}
([s]_{m,n})^m \circ \Gamma \circ S \circ \Gamma \circ ([s]_{m,n})^n &\subseteq (s^m \circ \Gamma \circ S] \circ \Gamma \circ ([s]_{m,n})^n \\
&\subseteq (s^m \circ \Gamma \circ (S \circ \Gamma \circ ([s]_{m,n})^n)] \\
&\subseteq (s^m \circ \Gamma \circ (S \circ \Gamma \circ s^n]] \\
&\subseteq (s^m \circ \Gamma \circ S \circ \Gamma \circ s^n].
\end{aligned}$$

Hence, (iii) holds.

Theorem 3.4. Suppose $(S, \Gamma, \circ, \leq)$ is an ordered Γ-semihypergroup and m, n are non-negative integers. Let $\mathscr{R}_{(m,0)}$ and $\mathscr{L}_{(0,n)}$ be the set of all ordered $(m,0)$-Γ-hyperideals and the set of all ordered $(0,n)$-Γ-hyperideals of S, respectively. Then,

(i) S is $(m,0)$-regular if and only if for all $R \in \mathscr{R}_{(m,0)}, R = (R^m \circ \Gamma \circ S]$.
(ii) S is $(0,n)$-regular if and only if for all $L \in \mathscr{R}_{(0,n)}, L = (S \circ \Gamma \circ L^n]$.

Proof.(i) Suppose S is $(m,0)$-regular. Then,

$$\text{for all } s \in S, s \in (s^m \circ \Gamma \circ S]. \tag{3.1}$$

Suppose $R \in \mathscr{R}_{(m,0)}$. As $R^m \circ \Gamma \circ S \subseteq R$, and $R = (R]$, we have $(R^m \circ \Gamma \circ S] \subseteq R$. If $s \in R$, by Equation (3.1), we obtain $s \in (s^m \circ \Gamma \circ S] \subseteq (R^m \circ \Gamma \circ S]$, therefore, $R \subseteq (R^m \circ \Gamma \circ S]$. So, $(R^m \circ \Gamma \circ S] = R$.
Conversely, suppose

$$\text{for all } R \in \mathscr{R}_{(m,0)}, R = (R^m \circ \Gamma \circ S]. \tag{3.2}$$

Suppose $s \in S$. Therefore, $[s]_{m,0} \in \mathscr{R}_{(m,0)}$. By Equation (3.2), we obtain

$$[s]_{m,o} = (([s]_{m,0})^m \circ \Gamma \circ S].$$

Applying Lemma 3.3, we obtain

$$[s]_{m,o} \subseteq (s^m \circ \Gamma \circ S].$$

Therefore, $s \in (s^m \circ \Gamma \circ S]$.
Hence, S is $(m,0)$-regular.
(ii) It can be proved analogously.

Theorem 3.5. Suppose $(S, \Gamma, \circ, \leq)$ is an ordered Γ-semihypergroup and m, n are non-negative integers. Suppose $\mathscr{A}_{(m,n)}$ is the set of all ordered (m,n)-Γ-hyperideals of S. Then,

$$S \textit{ is } (m,n)-\textit{regular} \Longleftrightarrow \text{ for all } A \in \mathscr{A}_{(m,n)}, A = (A^m \circ \Gamma \circ S \circ \Gamma \circ A^n]. \tag{3.3}$$

Proof. Consider the following four conditions:
Case (i): $m = 0$ and $n = 0$. Then, Equation (3.3) implies
S is (0, 0)-regular $\Longleftrightarrow$ for all $A \in \mathscr{A}_{(0,0)}, A = S$ because $\mathscr{A}_{(0,0)} = \{S\}$ and S is (0, 0)-regular.
Case (ii): $m = 0$ and $n \neq 0$. Therefore, Equation (3.3) implies
S is (0, n)-regular$\Longleftrightarrow$ for all $A \in \mathscr{A}_{(0,n)}, A = (S \circ \Gamma \circ A^n]$. This follows by Theorem 3.4(ii).
Case (iii): $m \neq 0$ and $n = 0$. This can be proved applying Theorem 3.4(i).
Case (iv): $m \neq 0$ and $n \neq 0$. Suppose S is (m, n)-regular. Therefore,

$$\text{for all } s \in S, s \in (s^m \circ \Gamma \circ S \circ \Gamma \circ s^n]. \tag{3.4}$$

Let $A \in \mathscr{A}_{(m,n)}$. As $A^m \circ \Gamma \circ S \circ \Gamma \circ A^n \subseteq A$ and $A = (A]$, we obtain $(A^m \circ \Gamma \circ S \circ \Gamma \circ A^n] \subseteq A$. Suppose $s \in A$. Applying Equation (3.4), $s \in (s^m \circ \Gamma \circ S \circ \Gamma \circ s^n] \subseteq (A^m \circ \Gamma \circ S \circ \Gamma \circ A^n]$. Therefore, $A \subseteq (A^m \circ \Gamma \circ S \circ \Gamma \circ A^n]$. Hence, $A = (A^m \circ \Gamma \circ S \circ \Gamma \circ A^n]$.
Conversely, suppose $A = (A^m \circ \Gamma \circ S \circ \Gamma \circ A^n]$ for all $A \in \mathscr{A}_{(m,n)}$. Suppose $s \in S$. As $[s]_{m,n} \in \mathscr{A}_{(m,n)}$, we have

$$[s]_{m,n} = (([s]_{m,n})^m \circ \Gamma \circ S \circ \Gamma \circ ([s]_{m,n})^n].$$

Applying Lemma 3.3(iii), we obtain $[s]_{m,n} \subseteq (s^m \circ \Gamma \circ S \circ \Gamma \circ s^n]$, therefore, $s \in (s^m \circ \Gamma \circ S \circ \Gamma \circ s^n]$.
Hence, S is (m,n)-regular.

Theorem 3.6. Suppose $(S,\Gamma,\circ,\leq)$ is an ordered Γ-semihypergroup and m,n are non-negative integers. Suppose $\mathscr{R}_{(m,0)}$ and $\mathscr{L}_{(0,n)}$ is the set of all $(m,0)$-Γ-hyperideals and $(0,n)$-Γ-hyperideals of S, respectively. Then,

$$S \text{ is } (m,n)-\text{regular ordered } \Gamma-\text{semihypergroup} \iff \text{for all } \mathrm{R}\in\mathscr{R}_{(\mathrm{m},0)}, \text{ for all } \mathrm{L}\in\mathscr{L}_{(0,\mathrm{n})}, \quad R\cap L=(R^m\circ\Gamma\circ L\cap R\circ\Gamma\circ L^n]. \tag{3.5}$$

Proof. Consider the following four cases:
Case (i): $m=0$ and $n=0$. Therefore, Equation (3.5) implies
S is (0, 0)-regular $\iff$ for all $\mathrm{R}\in\mathscr{R}_{(0,0)}$ for all $\mathrm{L}\in\mathscr{L}_{(0,0)}, \mathrm{R}\cap\mathrm{L}=(\mathrm{L}\cap\mathrm{R}]$ because $\mathscr{R}_{(0,0)}=\mathscr{L}_{(0,0)}=\{S\}$ and S is $(0,0)$-regular.
Case (ii): $m=0$ and $n=0$. Therefore, Equation (3.5) implies
S is $(0,n)$-regular $\iff$ for all $\mathrm{R}\in\mathscr{R}_{(0,\mathrm{n})}$ for all $\mathrm{L}\in\mathscr{L}_{(0,\mathrm{n})}, \mathrm{R}\cap\mathrm{L}=(\mathrm{L}\cap\mathrm{R}\circ\Gamma\circ\mathrm{L}^{\mathrm{n}}]$. Suppose S is $(0,n)$-regular. Suppose $R\in\mathscr{R}_{(0,0)}$ and $L\in\mathscr{L}_{(0,n)}$. By Theorem 3.4(ii), $L=(S\circ\Gamma\circ L^n]$. As $R\in\mathscr{R}_{(0,0)}$, we have $R=S$, therefore, $R\cap L=L$. Therefore,

$$(L\cap R\circ\Gamma\circ L^n]=(L\cap S\circ\Gamma\circ L^n]=((S\circ\Gamma\circ L^n]\cap S\circ\Gamma\circ L^n]=(S\circ\Gamma\circ L^n]=L=R\cap L.$$

Conversely, suppose

$$\text{for all } \mathrm{R}\in\mathscr{R}_{(0,0)}, \text{ for all } \mathrm{L}\in\mathscr{L}_{(0,\mathrm{n})}, \mathrm{R}\cap\mathrm{L}=(\mathrm{L}\cap\mathrm{R}\circ\Gamma\circ\mathrm{L}^{\mathrm{n}}]. \tag{3.6}$$

If $R\in\mathscr{R}_{(0,0)}$, then $R=S$. If $L\in\mathscr{L}_{(0,n)}$, $S\circ\Gamma\circ L^n\subseteq L$ and $L=(L]$. Therefore, Equation (3.6) implies

$$\text{for all } \mathrm{L}\in\mathscr{L}_{(0,\mathrm{n})}, \mathrm{L}=(\mathrm{S}\circ\Gamma\circ\mathrm{L}^{\mathrm{n}}].$$

Applying Theorem 3.4(ii), S is $(0,n)$-regular.
Case (iii): $m\neq 0$ and $n=0$. This can be proved as before.
Case (iv): $m\neq 0$ and $n\neq 0$. Suppose that S is (m,n)-regular. Suppose $R\in\mathscr{R}_{(m,0)}$ and $L\in\mathscr{L}_{(0,n)}$. To prove that $R\cap L\subseteq(R^m\circ\Gamma\circ L]\cap(R\circ\Gamma\circ L^n]$, suppose $s\in R\cap L$. We have

$$s\in(s^m\circ\Gamma\circ S\circ\Gamma\circ s^n]\subseteq(s^m\circ\Gamma\circ L]\subseteq(R^m\circ\Gamma\circ L] \text{ and } \mathrm{s}\in(\mathrm{s}^{\mathrm{m}}\circ\Gamma\circ\mathrm{S}\circ\Gamma\circ\mathrm{s}^{\mathrm{n}}]\subseteq(\mathrm{R}\circ\Gamma\circ\mathrm{s}^{\mathrm{n}}]\subseteq(\mathrm{R}\circ\Gamma\circ\mathrm{L}^{\mathrm{n}}].$$

Hence, $R\cap L\subseteq(R^m\circ\Gamma\circ L]\cap(R\circ\Gamma\circ L^n]$. As

$$(R^m\circ\Gamma\circ L]\subseteq(R^m\circ\Gamma\circ S]\subseteq(R]=R \text{ and } (\mathrm{R}\circ\Gamma\circ\mathrm{L}^{\mathrm{n}}]\subseteq(\mathrm{S}\circ\Gamma\circ\mathrm{L}^{\mathrm{n}}]\subseteq(\mathrm{L}]=\mathrm{L}.$$

This implies that $(R^m\circ\Gamma\circ L]\cap(R\circ\Gamma\circ L^n]\subseteq R\cap L$, therefore, $R\cap L=(R^m\circ\Gamma\circ L]\cap(R\circ\Gamma\circ L^n]$.
Conversely, suppose

$$\text{for all } \mathrm{R}\in\mathscr{R}_{(\mathrm{m},0)}, \text{ for all } \mathrm{L}\in\mathscr{L}_{(0,\mathrm{n})}, \mathrm{R}\cap\mathrm{L}=(\mathrm{R}^{\mathrm{m}}\circ\Gamma\circ\mathrm{L}\cap\mathrm{R}\circ\Gamma\circ\mathrm{L}^{\mathrm{n}}]. \tag{3.7}$$

Suppose $R=[s]_{m,0}$ and $L=S$. Applying Equation (3.7), we obtain $[s]_{m,0}\subseteq(([s]_{m,0})^m\circ\Gamma\circ S]$. Applying Lemma 3.3, we obtain

$$[s]_{m,0}\subseteq(s^m\circ\Gamma\circ S]. \tag{3.8}$$

In a similar fashion, we obtain

$$[s]_{0,n}\subseteq(S\circ\Gamma\circ s^n]. \tag{3.9}$$

As $R^m\subseteq R$ and $L^n\subseteq L$, by Equation (3.7), we have

$$\text{for all } \mathrm{R}\in\mathscr{R}_{(\mathrm{m},0)}, \text{ for all } \mathrm{L}\in\mathscr{L}_{(0,\mathrm{n})}, \mathrm{R}\cap\mathrm{L}\subseteq(\mathrm{R}\circ\Gamma\circ\mathrm{L}].$$

As $(s^m\circ\Gamma\circ S]\in\mathscr{R}_{(m,0)}$ and $(S\circ\Gamma\circ s^n]\in\mathscr{L}_{(0,n)}$, we obtain

$$(s^m\circ\Gamma\circ S]\cap(S\circ\Gamma\circ s^n]\subseteq((s^m\circ\Gamma\circ S]\circ\Gamma\circ(S\circ\Gamma\circ s^n]]\subseteq(s^m\circ\Gamma\circ S\circ\Gamma\circ s^n].$$

Applying Equations (3.8) and (3.9), we obtain

$$[s]_{m,0}\cap[s]_{0,n}\subseteq(s^m\circ\Gamma\circ S\circ\Gamma\circ s^n].$$

Hence, S is (m,n)-regular.

4. Conclusion

In this paper, we introduced the concepts of (m,n)-Γ-hyperideal (resp. generalized (m,n)-Γ-hyperideal) and (m,n)-regular ordered Γ-semihypergroup, where m,n are non-negative integers and studied some properties of (m,n)-Γ-hyperideals in ordered Γ-semihypergroups. In particular, we studied (m,n)-regular ordered Γ-semihypergroups. We proved that if $(S,\Gamma,\circ,\leq)$ is an ordered Γ-semihypergroup, where m,n are non-negative integers and if $\mathscr{A}_{(m,n)}$ is the set of all ordered (m,n)-Γ-hyperideals of S. Then, S *is* $(m,n)-$*regular* $\iff$ for all $\mathrm{A}\in\mathscr{A}_{(\mathrm{m},\mathrm{n})}, \mathrm{A}=(\mathrm{A}^{\mathrm{m}}\circ\Gamma\circ\mathrm{S}\circ\Gamma\circ\mathrm{A}^{\mathrm{n}}]$. We also proved that if $(S,\Gamma,\circ,\leq)$ is an ordered Γ-semihypergroup, where m,n are non-negative integers; and if $\mathscr{R}_{(m,0)}$, $\mathscr{L}_{(0,n)}$ is the set of all $(m,0)$-Γ-hyperideals and $(0,n)$-Γ-hyperideals of S, respectively. Then, S *is* $(m,n)-$regular ordered $\Gamma-$semihypergroup $\iff$ for all $\mathrm{R}\in\mathscr{R}_{(\mathrm{m},0)}$, for all $\mathrm{L}\in\mathscr{L}_{(0,\mathrm{n})}, \mathrm{R}\cap\mathrm{L}=(\mathrm{R}^{\mathrm{m}}\circ\Gamma\circ\mathrm{L}\cap\mathrm{R}\circ\Gamma\circ\mathrm{L}^{\mathrm{n}}]$. The results of this article can also be applied on semihypergroups and on ordered semihypergroups by some suitable modifications. We hope that this work will provide the basis for further study on ordered Γ-semihypergroups.

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References

[1] F. Marty, *Sur une generalization de la notion de groupe*, 8th Congress Math. Scandenaves, Stockholm, (1934), 45–49.
[2] P. Corsini, V. Leoreanu, *Applications of Hyperstructure Theory*, Advances in Mathematics, Kluwer Academic Publishers, Dordrecht, 2003.
[3] W. Prenowitz, J. Jantosciak, *Join Geometries*, Springer-Verlag, UTM., 1979.
[4] B. Davvaz, V. Leoreanu-Fotea, *Hyperring Theory and Applications*, International Academic Press, USA, 2007.
[5] B. Davvaz, A. Dehghan Nezhad, *Chemical examples in hypergroups*, Ratio Mathematica-Numero, **14** (2003), 71–74.
[6] B. Davvaz, *Some results on congruences in semihypergroups*, Bull. Malays. Math. Sci. Soc., **23**(2) (2000), 53—58.
[7] B. Davvaz, *Polygroup Theory and Related Systems*, World Scientific Publishing Co. Pvt. Ltd., Hackensack, 2013.
[8] B. Davvaz, N. S. Poursalavati, *Semihypergroups and S-hypersystems*, Pure Math. Appl., **11** (2000), 43—49.
[9] C. Gutan, *Simplifiable Semihypergroups*, in: Algebraic Hyperstructures and Applications, Xanthi, World Scientific, 1990.
[10] D. Griffiths, *Introduction to Elementary Particles*, John Wiley & Sons, 1987.
[11] D. Heidari, B. Davvaz. *On ordered hyperstructures*, Politehn. Univ. Bucharest Sci. Bull. Ser., A Appl. Math. Phys., **73**(2) (2011), 85—96.
[12] F. Halzen, A. Martin, Quarks & Leptons: *An Introductory Course in Modern Particle Physics*, John Wiley & Sons, 1984.
[13] K. Hila, B. Davvaz, K. Naka, *On quasi-hyperideals in semihypergroups*, Comm. Algebra, **39** (2011), 4183-–4194.
[14] P. Corsini, *Prolegomena of hypergroup theory*, second ed., Aviani Editore, Tricesimo, 1993.
[15] P. Corsini, Sur les semi-hypergroupes, Atti Soc. Pelorit. Sci. Fis. Math. Nat., **26**(4) (1980), 363—372.
[16] S. M. Anvariyeh, S. Mirvakili, B. Davvaz, *On* Γ*-hyperideals in* Γ*-semihypergroups*, Carpathian J. Math., **26**(1) (2010), 11-23.
[17] K. Hila, B. Davvaz, J. Dine, *Study on the structure of* Γ*-semihypergroups*, Commun. Algebra, **40**(8) (2012), 2932-2948.
[18] A. Basar, M. Y. Abbasi, S. A. Khan, *An introduction of theory of involutions and their weakly prime hyperideals*, J. Indian Math. Soc., **86**(3-4) (2019), 1-11.
[19] A. Basar, *Application of (m, n)-* Γ *-hyperideals in characterization of LA-* Γ *-semihypergroups*, Discuss. Math. Gen. Algebra Appl., **39**(1)(2019), 135-147.
[20] A. Basar, *A note on (m, n)-*Γ*-ideals of ordered LA-*Γ*–semigroups*, Konuralp J. Math., **7**(1) (2019), 107-111.