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The Third Isomorphism Theorem on UP-Bialgebras

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Abstract

The concept of UP-bialgebras was introduced and analyzed by Mosrijai and Iampan at the beginning of 2019. Theorem that we can look at as the First theorem on UP-biisomorphism between the UP-bialgebras is given in our forthcoming text [9]. In this article we construct a form of the third theorem on UP-biisomorphism between UP-bialgebras.

1. Introduction

The concept of UP-algebras developed by Iampan in [1]. Examining the substructures in this algebra are done for example in articles [2, 3]. This author took part in analyzing the properties of UP-algebras and their substructures, also [4]-[6]. Some forms of the isomorphism theorem between UP-algebras can be found in [2, 3, 5, 6].

The concept of bi-algebraic structures was studied by Vasantha Kandasamy in 2003 [7]. The concept of UP-bialgebras with the associated substructures and their mutual connections can be found in [8]. In the forthcoming article [9], this author offered one form the first theorem of the isomorphism between the UP-bialgebras.

In this article we expose a form of the second isomorphism theorem between UP-bialgebras.

2. Preliminaries

In this section, we will present the necessary previous concepts of UP-algebras, their substructures and UP-homomorphisms taken from texts [1, 2, 3, 8]. We will also expose their mutual relationships in the form of proclaims necessary for our intention.

2.1. UP-algebras

In this subsection we will describe some elements of UP-algebras and their substructures necessary for our intentions in this text.

Definition 2.1 ([1]). *An algebra $L = (L, \cdot, 0)$ of type $(2, 0)$ is called a UP-algebra where L is a nonempty set, $'\cdot'$ is a binary operation on L, and 0 is a fixed element of L (i.e. a nullary operation) if it satisfies the following axioms:*
(UP-1) $(\forall x, y \in L)((y \cdot z) \cdot ((x \cdot y) \cdot (x \cdot z)) = 0)$,
(UP-2) $(\forall x \in L)(0 \cdot x = x)$,
(UP-3) $(\forall x \in L)(x \cdot 0 = 0)$, *and*
(UP-4) $(\forall x, y \in L)((x \cdot y = 0 \wedge y \cdot x = 0) \Longrightarrow x = y)$.

Definition 2.2 ([1]). *A nonempty subset J of a UP-algebra $(L, \cdot, 0)$ is called*
(1) *a UP-subalgebra of L if* $(\forall x, y \in J)(x \cdot y \in J)$.
(2) *a UP-ideal of L if*
(i) $0 \in J$; *and*
(ii) $(\forall x, y, z \in L)((x \cdot (y \cdot z) \in J \wedge y \in J) \Longrightarrow x \cdot z \in J)$.

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The set $\{0\}$ is a trivial UP-subalgebra (trivial UP-ideal) of L.
In the article [6], Theorem 3.3, it has been shown that the conditions (i) and (ii) in the preceding definition are equivalent to the following conditions
(iii) $(\forall x,y \in L)((x \cdot y \in J \wedge x \in J) \Longrightarrow y \in J)$,
(iv) $(\forall x,y \in L)(y \in J \Longrightarrow x \cdot y \in J)$.

Definition 2.3 ([1])**.** *Let $(L,\cdot,0_L)$ and $(M,\cdot^{'},0_M)$ be two UP-algebras. A mapping $f : L \longrightarrow M$ is called a UP-homomorphism if*

$$(\forall x,y \in L)(f(x\cdot y) = f(x) \cdot^{'} f(y)).$$

A UP-homomorphism $f : L \longrightarrow M$ is called
(3) a UP-epimorphism if f is surjective,
(4) a UP-monomorphism if f is injective, and
(5) a UP-isomorphism if f is bijective.

Let f be a mapping form UP-algebra L to UP-algebra M, and let A and B be nonempty subsets of L and of M, respectively. The set $f(A) = \{f(x)|x \in A\}$ is called the image of A under f. In particular, $f(L)$ which denoted by $Im(f)$ is called the image of f. The dually set $f^{-1}(B) = \{x \in L|f(x) \in B\}$ is called the inverse image of B under f. Especially, the set $Ker(f) = f^{-1}(\{0_M\}) = \{x \in L : f(x) = 0_M\}$ is called the kernel of f.

A relation of congruence on UP-algebras is introduced in [1] by Definition 3.1 and Proposition 3.5 on this way: If J is a UP-ideal of a UP-algebra L, then the relation $\sim_J$ defined by

$$(\forall x,y \in L)(x \sim_J y \Longleftrightarrow (x \cdot y \in J \wedge y \cdot x \in J))$$

is a UP-congruence on L. Further on, any relation of congruence on UP-algebras has this form according to the claim (1) of Theorem 3.6 and the claim (1) of Theorem 3.7 in [1]. In particular, if $f : L \longrightarrow M$ is a UP-homomorphism between UP-algebras, then the relation $\sim_f$ determined by $Ker(f)$ is a UP-congruence in L. The factor-set $L/\sim_J = \{[x]_{\sim_J} : x \in L\}$ is a UP-algebra according to the claim (4) of Theorem 3.7 in [1]. We also use the following notion $L/J = \{[x]_J : x \in L\}$ to denote this factor algebra.

2.2. UP-bialgebras

The concept of UP-bialgebras and some their substructures were introduced and analyzed by Mosrijai and Iampan in the recently published work [8]. In this subsection, taking into account their determinations, we describe the concept of UP-bialgebras and some notions connected with them. So, in this subsection, we will repeat the concept of UP-bialgebras and the notions of UP-bisubalgebras and UP-biideals of UP-bialgebras, and will expose some results related to substructures of such algebras.

Definition 2.4 ([8], Definition 3.1)**.** *An algebra $L = (L,\cdot,*,0)$ of type $(2,2,0)$ is called a UP-bialgebra where L is a nonempty set, $\cdot$ and $*$ two are binary internal operations on L, and 0 is a fixed element of L if there exist two distinct proper subsets L_1 and L_2 of L with respect to $\cdot$ and $*$, respectively, such that*
(UPB-1) $L = L_1 \cup L_2$;
(UPB-2) $(L_1,\cdot,0)$ is a UP-algebra, and
(UPB-3) $(L_2,*,0)$ is a UP-algebra.
We will denote the UP-bialgebra by $L = L_1 \uplus L_2$. In case of $L_1 \cap L_2 = \{0\}$, we call L zero disjoint.

Definition 2.5 ([8], Definition 3.7)**.** *A nonempty subset J of a UP-bialgebra $L = L_1 \uplus L_2$ is called a UP-biideal (UP-bisubalgebra) of L if there exist subsets J_1 of L_1 and J_2 of L_2 with respect to $\cdot$ and $*$, respectively, such that*
(6) $J_1 \neq J_2$ and $J = J_1 \cup J_2$;
(7) $(J_1,\cdot,0)$ is a UP-ideal (UP-subalebra) of $(L_1,\cdot,0)$, and
(8) $(J_2,*,0)$ is a UP-ideal (UP-subalgebra) of $(L_2,*,0)$.
In case of $J_1 \cap L_2 = \{0\} = L_1 \cap J_2$, we call S zero disjoint.

The important relationship between these notions is the following:

Proposition 2.6 ([9])**.** *If $J \supset \{0\}$ is a UP-subalgebra (resp., UP-ideal) of UP-algebra L_1 (of UP-algebra L_2, respectively), such that $\{0\} \neq J$, then on J can be seen as a zero disjoint UP-bisubgebra (resp., UP-biideal) of UP-bialgebra $L = L_1 \uplus L_2$.*

2.3. UP-bihomomorphisms

Let $f : L \longrightarrow M$ be a function from a set L to a set M and $C \subseteq L$. Then the restriction of f to C is the function $f_{[C]} : C \longrightarrow M$.

Definition 2.7 ([8], Definition 4.1)**.** *Let $L = L_1 \uplus L_2$ be a UP-bialgebra with two binary operations $\cdot$ and $*$, and let $M = M_1 \uplus M_2$ be a UP-bialgebra with two binary operations $\cdot^{'}$ and $*^{'}$. A mapping f form $L = L_1 \uplus L_2$ to $M = M_1 \uplus M_2$ is called a UP-bihomomorphism if it satisfies the following properties:*
(9) $f_{[L_1]} : L_1 \longrightarrow M_1$ is a UP-homomorphism, and
(10) $f_{[L_2]} : L_2 \longrightarrow M_2$ is a UP-homomorphism.
We say that these restrictions are natural restrictions. A UP-bihomomorphism $f : L \longrightarrow M$ is called
- a UP-biepimorphism if the natural restriction are UP-epimorphisms,
- a UP-bimonomorphism if the natural restriction are UP-monomorphisms, and
- a UP-biisomorphism if the natural restriction are UP-isomorphisms.

Proposition 2.8 ([8]). *let $f : L_1 \uplus L_2 \longrightarrow M_1 \uplus M_2$ be a UP-bihomomorphism. Then the following statements hold:*
(11) *$f(0_L) = 0_M$, and*
(12) *$Ker(f) = \{0_L\}$ if and only if f is an injective mapping;*
(13) *if J is a UP-bisubalgebra of L, then the image $f(J)$ is a UP-bisubalgebra of B;*
(14) *if $J = J_1 \cup J_2$ is a UP-biideal of L, and J_1 and J_2 are subsets of L_1 and of L_2, respectively, with $Ker(f) \subseteq J_1 \cap J_2$, then the image $f(J)$ is a UP-biideal of M;*
(15) *if D is a UP-bisubalgebra of M, then the inverse image $f^{-1}(D)$ is a a UP-bisubalgebra of L; and*
(16) *if D is a UP-biideal of M, then the inverse image $f^{-1}(D)$ is a UP-biideal of L.*

3. The main results

In our forthcoming article [9], we formulated and proved a form of the first isomorphism theorem between UP-bialgebras. To this direction, we used the following lemma.

Lemma 3.1 ([9]). *Let $L = L_1 \uplus L_2$ and $M = M_1 \uplus M_2$ be two UP-bialgebras and let $f : L \longrightarrow M$ be a UP-bihomomorphism. Then the set $Ker(f_{[A_1]}) \cup Ker(f_{[A_2]})$ is a UP-biideal of L and $Ker(f) = Ker(f_{[L_1]}) \uplus Ker(f_{[L_2]})$ holds.*

Let $L = L_1 \uplus L_2$ be a UP-bialgebra with two binary operations $\cdot$ and $*$, and let $M = M_1 \uplus M_2$ be a UP-bialgebra with two binary operations $\cdot'$ and $*'$ and let $f : L \longrightarrow M$ be a UP-bihomomorphism. Let $\sim_1$ is the congruence on L_1 generated by the UP-ideal $Ker(f_{[L_1]})$

$$\forall x, y \in L_1)(x \sim_1 y \Longleftrightarrow (x \cdot y \in Ker(f_{[L_1]}) \wedge y \cdot x \in Ker(f_{[L_1]})))$$

and let $\sim_2$ be the congruence on L_2 generated by the UP-ideal $Ker(f_{[L_2]})$

$$(\forall x, y \in L_2)(x \sim_2 y \Longleftrightarrow (x * y \in Ker(f_{[L_2]}) \wedge y * x \in Ker(f_{[L_2]}))).$$

Then we can construct the factor-UP-algebra $L_1/\sim_1$ and the factor-UP-algebra $L_2/\sim_2$. So, $L_1/\sim_1 \uplus L_2/\sim_2$ is a UP-bialgebra with two binary operation $'\odot'$ and $'\circledast'$ defined by

$$(\forall [x]_{\sim_1}, [y]_{\sim_1} \in L_1/\sim_1))([x]_{\sim_1} \odot [y]_{\sim_1} = [x \cdot y]_{\sim_1})$$

and

$$(\forall [x]_{\sim_2}, [y]_{\sim_2} \in L_2/\sim_2))([x]_{\sim_2} \circledast [y]_{\sim_2} = [x * y]_{\sim_2}).$$

Previous analysis enables us to introduce the following determination: Let $L = L_1 \uplus L_2$ be a UP-bialgebra. For a pair $(\sim_1, \sim_2)$ the relation of congruence $\sim_1$ on L_1 and $\sim_2$ on L_2 we write $L_1 \uplus L_2/(\sim_1, \sim_2)$ instead of $L_1/\sim_1 \uplus L_2/\sim_2$. If $\pi_1 : L_1 \longrightarrow L_1/\sim_1$ and $\pi_2 : L_2 \longrightarrow L_2/\sim_2$ are canonical UP-epimorphisms, then there is a unique canonical UP-epimorphism $\pi : L_1 \uplus L_2 \longrightarrow L_1 \uplus L_2/(\sim_1, \sim_2)$ such that $\pi_{[L_1]} = \pi_1$ and $\pi_{[L_2]} = \pi_2$. Particulary, there is a unique UP-epimorphism $\pi : L_1 \uplus L_2 \longrightarrow (L_1 \uplus L_2)/(Ker(f_{[L_1]}), Ker(f_{[L_2]}))$. The first theorem of isomorphism between UP-bialgebras has the form in which for simplicity we write $A/Ker(f)$ instead of $A/(Ker(f_{[A_1]}), Ker(f_{[A_2]}))$.

Theorem 3.2 ([9]). *Let $f : L \longrightarrow M$ be a UP-bihomomorphism. Then there exists the unique UP-bihomomorphism $g : L/Ker(f) \longrightarrow M$ such that $f = g \circ \pi$. In addition, for the UPB-subalgebra $f(L)$ of M holds $L/Ker(f) \cong f(L)$.*

Let us analyze now the following situation:

Let J and K be UP-biideals of a UP-bialgebra L such that $J \subseteq K$. Then there exist UP-ideals J_1 and K_1 of the UP-algebra L_1 and there exist UP-ideals J_2 and K_2 of the UP-algebra L_2 such that $J_1 \neq J_2$ and $J = J_1 \cup J_2$, and $K_1 \neq K_2$ and $K = K_1 \cup K_2$, by Definition 2.5. If $J_1 \subseteq K_1$ and $J_2 \subseteq K_2$ hold, then K_1/J_1 is a UP-ideal of UP-algebra L_1/J_1 and K_2/J_2 is a UP-ideal of UP-algebra L_2/J_2. From here follows $L_1/K_1 \cong (L_1/J_1)/(K_1/J_1)$ according to Theorem 3.10 in [6]. We also have it $L_2/K_2 \cong (L_2/J_2)/(K_2/J_2)$ according to same theorem. So, the set $K_1/J_1 \uplus K_2/J_2$ is a UP-biideal of the UP-bialgebra $L_1/J_1 \uplus L_2/J_2$. Thus, the mapping $g_1 : L_1/J_1 \longrightarrow L_1/K_1$ has $Ker(g_1) = K_1/J_1$. Analogously, the mapping $g_2 : L_2/J_2 \longrightarrow L_2/K_2$ has $Ker(g_2) = K_2/J_2$ as core. Therefore, the homomorphism $g : L/(J_1, J_2) \longrightarrow L/(K_1, K_2)$, determined by $g_{[L_1/J_1]} = g_1$ and $g_{[L_2/J_2]} = g_2$ has the core exactly $K_1/J_1 \uplus K_2/J_2$.

The previous analysis is a motivation for the following theorem can be seen as the Third isomorphism theorem between UP-bialgebras.

Theorem 3.3. *Let $L = L_1 \uplus L_2$ be a UP-bialgebra and let $J = J_1 \uplus J_2$ and $K = K_1 \uplus K_2$ be UP-biideals such that $J_1 \subseteq K_1$ and $J_2 \subseteq K_2$. Then*

$$L/(K_1, K_2) \cong (L/(J_1, J_2))/(K_1/J_1, K_2/J_2)$$

holds.

Final Observation

The concept of UP-algebras introduced and first results on them given by Iampan 2017 [1]. This author took part in analyzing the properties of UP-algebras and their substructures, also [4, 5, 6]. Algebraic bi-strukture was analyzed by Vasantha Kandasamy in 2003 [7]. The concept of UP-bialgebras introduced and the first results ware given by Mosrijai and Iampan at the beginning of 2019 [8]. Using by the concept of UP-bihomorphisms, introduced in [8], in this article we formulated and proved the theorem (Theorem 3.3) , which can be viewed as the Third isomorphism theorem between the UP-bialgebras.
Of course, there remains an open possibility of formulating and trying to prove other forms of these two isomorphism theorems between the UP-bialgebra.

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