Araştırma Makalesi / Research Article

Some Remarks on Positive Real Functions and Their Circuit Applications

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Abstract
In this paper, a boundary version of the Schwarz lemma has been considered for driving point impedance functions at $s=0$ point of the imaginary axis. Accordingly, under $Z(0)=0$ condition, the modulus of the derivative of the $Z(s)$ function has been considered from below. Here, $Z(a)$, c_1 and c_2 coefficients of the Taylor expansion of the $Z(s)=b+c_1(s-a)^p+\ldots$ function have been used in the obtained inequalities. The sharpness of these inequalities has also been proved. Using the obtained driving point impedance functions in the proposed theorems, corresponding LC circuits have been synthesized and related figures have been presented.

Keywords: Extremal function, Schwarz lemma, Positive real function, Taylor coefficients.

Pozitif Reel Fonksiyonlar ve Devre Uygulamaları Üzerine Bazı Sonuçlar

Öz

Bu çalışmada, Schwarz lemmasının bir sınır versiyonu, süren nokta empedans fonksiyonları için sanal eksen üzerindeki $s=0$ noktasında değerlendirilmiştir. Buna göre, $Z(0)=0$ koşulu altında, $Z(s)$ fonksiyonunun türevinin modülü aşağıdan değerlendirilmiştir. Burada, elde edilen eşitsizliklerde, $Z(s)=b+c_1(s-a)^p+\ldots$ fonksiyonunun Taylor açılımındaki $Z(a)$, c_1 ve c_2 katsayıları kullanılmıştır. Aynı zamanda, bu eşitsizliklerin keskinliği ispatlanmıştır. Önerilen teoremlerde elde edilen empedans fonksiyonları kullanılarak bunlara karşılık gelen LC devreleri sentezlenmiş ve bu devrelerle ilgili figürler sunulmuştur.

Anahtar kelimeler: Kesin fonksiyon, Schwarz lemması, Pozitif gerçel fonksiyon, Taylor katsayıları.

1. Introduction

In electrical engineering, positive real function (PRFs) are used as driving point impedance function (DPIF), $Z(s)$, which depend on complex frequency parameter, s. DPIFs are mainly used in network synthesis [1], however they are also utilized in control systems [2]. DPIFs are generally utilized for representation of spectral characteristics of circuits containing inductor, capacitor or resistors. A typical DPIF is defined as in the following structure [3]:

$$Z(s)=\frac{a_n s^n+a_{n-1}s^{n-1}+\cdots+a_0}{b_m s^m+b_{m-1}s^{m-1}+\cdots+b_0} \tag{1}$$

where s is the complex frequency parameter, and $\{a_0,\ldots,a_n,b_0,\ldots,b_m\}$ are real and non-negative

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Geliş Tarihi: 05/12/2018 Kabul Tarihi: 20/05/2019

coefficients. In electrical engineering, it is essential for DPIFs to be positive real since this is required for them to be realizable in terms of circuit synthesis. Therefore, any DPIF must satisfy the conditions of PRFs. It is possible to synthesize lossless positive real DPIFs using Cauer synthesis method where it is mainly used for design of lossless electric circuits [4].

A physically realizable DPIF satisfies the properties of the PRFs which are given as follows [5], [6]:

1. $Z(s)$ is analytic in $\Re s \geq 0$.
2. $Z(s)$ has no singularities other than poles on $s=0$.
3. $Z(\overline{s}) = \overline{Z(s)}$ (i.e. real on the real axis).
4. $\Re Z(s) \geq 0$, $s \geq 0$.

In this study, it is aimed to carry out a boundary analysis for the derivative of the DPIFs. This also makes it possible to observe what kind of circuits will be obtained by performing boundary analysis of DPIFs. Another goal of the presented study is to find the answer of the question “what can be said about $Z_0(s)$ when it is considered at the boundary?” Motivating by this question, we investigate the Schwarz lemma at the boundary for DPIFs. In addition, the derivative of DPIFs are frequently encountered in electrical engineering, e.g. they are mainly used in network synthesis [7,8], in control systems [9], signal processing [10] and in microwave engineering [11]. We present a theorem by evaluating the derivative at the origin and assuming $Z(0)=0$ so that a lower boundary is obtained for $|Z'(0)|$.

2. Preliminary Considerations

One of the main tool of complex functions theory is Schwarz Lemma. Its present statement has been written by Constantin Caratheodory. Schwarz Lemma is an important results which gives the estimates about the values of holomorphic functions defined from the unit disc into itself [12]. Before applying Schwarz lemma, firstly, we will exploit the following map.

Let

$$f(z) = \frac{Z(s)-\beta}{Z(s)+\beta}, z = \frac{s-\alpha}{s+\alpha}, \tag{2}$$

where $Z(s) = \beta + c_1(s-\alpha) + c_2(s-\alpha)^2 + \ldots$ and α, β are positive real numbers [1, 6]. Consider the product

$$B(z) = \prod_{i=1}^{n} \frac{z-z_i}{1-\overline{z_i}z}. \tag{3}$$

The function $B(z)$ is called a finite Blaschke product, where $z_1, z_2, \ldots, z_n \in U$.

Let

$$p(z) = \frac{f(z)}{\prod_{i=1}^{n} \frac{z-z_i}{1-\overline{z_i}z}}, z_i = \frac{s_i-\alpha}{s_i+\alpha}, i = 1,2,..,n. \tag{4}$$

Here, $s_1, \ldots, s_n$ are points in right half plane and $z_1, \ldots, z_n$ are zeros of $f(z)$. Also, $p(z)$ is an analytic function in U, $p(0)=0$ and $|p(z)|<1$ for $|z|<1$. So, $p(z)$ satisfy the conditions of the Schwarz lemma.

Therefore, from the Schwarz lemma, we obtain

$$p(z)=\frac{Z(\alpha\frac{1+z}{1-z})-\beta}{Z(\alpha\frac{1+z}{1-z})+\beta}\frac{1}{\prod_{i=1}^{n}\frac{z-z_i}{1-\overline{z_i}z}}=\frac{c_1\frac{2\alpha z}{(1-z)}+c_2\frac{2^2\alpha^2z^2}{(1-z)^2}+\ldots}{2\beta+c_1\frac{2\alpha z}{(1-z)}+c_2\frac{2^2\alpha^2z^2}{(1-z)^2}+\ldots}\frac{1}{\prod_{i=1}^{n}\frac{z-z_i}{1-\overline{z_i}z}}, \tag{5}$$

$$\frac{p(z)}{z}=\frac{c_1\frac{2\alpha}{(1-z)}+c_2\frac{2^2\alpha^2z}{(1-z)^2}+\ldots}{2\beta+c_1\frac{2\alpha z}{(1-z)}+c_2\frac{2^2\alpha^2z^2}{(1-z)^2}+\ldots}\frac{1}{\prod_{i=1}^{n}\frac{z-z_i}{1-\overline{z_i}z}} \tag{6}$$

and

$$|p'(0)|=\frac{\alpha}{\beta}|c_1|\frac{1}{\prod_{i=1}^{n}|z_i|}\leq 1. \tag{7}$$

Since $|c_1|=|Z'(\alpha)|$ and $|z_i|=\left|\frac{s_i-\alpha}{s_i+\alpha}\right|$, we take

$$|Z'(\alpha)|\leq\frac{\beta}{\alpha}\prod_{i=1}^{n}\left|\frac{s_i-\alpha}{s_i+\alpha}\right|. \tag{8}$$

This result is sharp and the extremal function is

$$Z(s)=\beta\frac{1-\frac{s-\alpha}{s+\alpha}\prod_{i=1}^{n}\frac{\frac{s-\alpha}{s+\alpha}-\frac{s_i-\alpha}{s_i+\alpha}}{1-\overline{\frac{s_i-\alpha}{s_i+\alpha}}\frac{s-\alpha}{s+\alpha}}}{1+\frac{s-\alpha}{s+\alpha}\prod_{i=1}^{n}\frac{\frac{s-\alpha}{s+\alpha}-\frac{s_i-\alpha}{s_i+\alpha}}{1-\overline{\frac{s_i-\alpha}{s_i+\alpha}}\frac{s-\alpha}{s+\alpha}}}, \tag{9}$$

where $s_1,\ldots,s_n$ are positive real numbers.

Since the area of applicability of Schwarz Lemma is quite wide, there exist many studies about it. Some of these studies is called the boundary version of Schwarz Lemma. An important result of Schwarz lemma was given by Osserman [12]. Also, it’s still a hot topic in the mathematics literature [13-15].

3. Main Results

In this section, a boundary version of the Schwarz lemma has been considered for driving point impedance functions at $s=0$ point. Accordingly, under $Z(0)=0$ condition, the modulus of the derivative of the $Z(s)$ function has been considered from below. Here, $Z(a)$, c_1 and c_2 coefficients of the Taylor expansion of the $Z(s) = b + c_1(s - a)^p + \ldots$ function have been used in the obtained inequalities. We also investigate some obtained inequalities in terms of sharpness analysis.

Theorem 3.1 *Let* $Z(s) = \beta + c_1(s-\alpha) + c_2(s-\alpha)^2 + \ldots$ *be a positive real function that is also analytic at the point* $s = 0$ *of the imaginary axis with* $Z(0) = 0$. *Assume that* $s_1, \ldots, s_n$ *are points in the s-plane with* $Z(s_i) = \beta$, $i = 1, 2, \ldots, n$. *Then*

$$|Z'(0)| \geq \frac{\beta}{\alpha}\left(1 + \sum_{i=1}^{n} \frac{\Re s_i}{|s_i|^2} + \frac{2\left(\beta\left(\prod_{i=1}^{n}\left|\frac{s_i-1}{s_i+1}\right|\right) - \alpha|c_1|\right)^2}{\beta^2\left(\prod_{i=1}^{n}\left|\frac{s_i-1}{s_i+1}\right|\right)^2 - \alpha^2|c_1|^2 + \prod_{i=1}^{n}\left|\frac{s_i-1}{s_i+1}\right|\left|\alpha^2\left(2\beta c_2 - c_1^2\right) + \alpha\beta c_1\left(1 + \sum_{i=1}^{n}\frac{4\Re s_i}{|s_i|^2 + 2i\Im s_i - 1}\right)\right|}\right).$$

(10)

The result (10) is sharp for the function given by

$$Z(s) = \beta \frac{1 + \left(\frac{s-\alpha}{s+\alpha}\right)^2 \prod_{i=1}^{n} \frac{\frac{s-\alpha}{s+\alpha} - \frac{s_i-\alpha}{s_i+\alpha}}{1 - \overline{\frac{s_i-\alpha}{s_i+\alpha}}\frac{s-\alpha}{s+\alpha}}}{1 - \left(\frac{s-\alpha}{s+\alpha}\right)^2 \prod_{i=1}^{n} \frac{\frac{s-\alpha}{s+\alpha} - \frac{s_i-\alpha}{s_i+\alpha}}{1 - \overline{\frac{s_i-\alpha}{s_i+\alpha}}\frac{s-\alpha}{s+\alpha}}},$$

(11)

where $s_1, s_2, \ldots, s_n$ are positive real numbers and $n = 1, 3, 5, \ldots$. Here, only odd values have been used for the n parameter since the inequality given in (10) is sharp for only odd values of n.

Proof. Let $f(z)$ be as in above. Also, let $z_1, z_2, \ldots, z_n$ be the zeros of the function $f(z)$ in U that are different from zero. The function

$$B_1(z) = z\prod_{i=1}^{n}\frac{z - z_i}{1 - \overline{z_i}z}$$

(12)

is analytic in U, $|B_1(z)| < 1$ for $|z| < 1$. By the maximum principle, for each $z \in U$, we have

$$|f(z)| \leq |B_1(z)|.$$

(13)

The quotient function

$$g(z) = \frac{f(z)}{B_1(z)}$$

(14)

is analytic in the unit disc U and $|g(z)| < 1$ for $|z| < 1$. In particular, we have

$$g(z)=\frac{Z(\alpha\frac{1+z}{1-z})-\beta}{\left[Z(\alpha\frac{1+z}{1-z})+\beta\right]z\prod_{i=1}^{n}\frac{z-z_i}{1-\overline{z_i}z}}=\frac{c_1\frac{2\alpha z}{(1-z)}+c_2\frac{2^2\alpha^2 z^2}{(1-z)^2}+\ldots}{2\beta+c_1\frac{2\alpha z}{(1-z)}+c_2\frac{2^2\alpha^2 z^2}{(1-z)^2}+\ldots}\frac{1}{z\prod_{i=1}^{n}\frac{z-z_i}{1-\overline{z_i}z}}$$

$$=\frac{c_1\frac{2\alpha}{(1-z)}+c_2\frac{2^2\alpha^2 z}{(1-z)^2}+\ldots}{2\beta+c_1\frac{2\alpha z}{(1-z)}+c_2\frac{2^2\alpha^2 z^2}{(1-z)^2}+\ldots}\frac{1}{\prod_{i=1}^{n}\frac{z-z_i}{1-\overline{z_i}z}}, \tag{15}$$

$$|g(0)|=\frac{\alpha}{\beta}\frac{|c_1|}{\prod_{i=1}^{n}|z_i|} \tag{16}$$

and

$$|g'(0)|=\frac{\left|\alpha^2\left(2\beta c_2-c_1^2\right)+\alpha\beta c_1\left(1+\sum_{i=1}^{n}\left(\frac{1-|z_i|^2}{z_i}\right)\right)\right|}{\beta^2\prod_{i=1}^{n}|z_i|}. \tag{17}$$

In addition, it can be seen that

$$\frac{cf'(c)}{f(c)}=|f'(c)|\geq\left|B_1'(c)\right|=\frac{cB_1'(c)}{B_1(c)}, c\in\partial E. \tag{18}$$

The auxiliary function

$$h(z)=\frac{g(z)-g(0)}{1-\overline{g(0)}g(z)} \tag{19}$$

is an analytic function in the unit disc U, $|h(z)|<1$ for $z\in U$, $|h(0)|=0$ and $|h(c)|=1$ for $c=-1\in\partial U$. From Osserman [12], we obtain

$$\frac{2}{1+|h'(0)|}\leq|h'(-1)|=\frac{1-|g(0)|^2}{\left|1-\overline{g(0)}g(-1)\right|^2}|g'(-1)|\leq\frac{1+|g(0)|}{1-|g(0)|}\left|\frac{f'(-1)}{B_1(-1)}-\frac{f(-1)B_1'(-1)}{B_1^2(-1)}\right|=\frac{1+|g(0)|}{1-|g(0)|}\left(|f'(-1)|-\left|B_1'(-1)\right|\right). \tag{20}$$

Since

$$h'(z)=\frac{1-|g(0)|^2}{\left(1-\overline{g(0)}g(z)\right)^2}g'(z), \tag{21}$$

$$\left|h'(0)\right| = \frac{\left|g'(0)\right|}{1-\left|g(0)\right|^2} = \frac{\dfrac{\left|\alpha^2\left(2\beta c_2 - c_1^2\right)+\alpha\beta c_1\left(1+\sum_{i=1}^{n}\left(\dfrac{1-\left|z_i\right|^2}{z_i}\right)\right)\right|}{\beta^2\prod_{i=1}^{n}\left|z_i\right|}}{1-\left(\dfrac{\alpha}{\beta}\dfrac{\left|c_1\right|}{\prod_{i=1}^{n}\left|z_i\right|}\right)^2}$$

(22)

$$=\prod_{i=1}^{n}\left|z_i\right|\frac{\left|\alpha^2\left(2\beta c_2 - c_1^2\right)+\alpha\beta c_1\left(1+\sum_{i=1}^{n}\left(\dfrac{1-\left|z_i\right|^2}{z_i}\right)\right)\right|}{\beta^2\left(\prod_{i=1}^{n}\left|z_i\right|\right)^2-\alpha^2\left|c_1\right|^2}$$

and

$$\left|B_1'(-1)\right| = 1+\sum_{i=1}^{n}\frac{1-\left|z_i\right|^2}{\left|1+z_i\right|^2},$$

(23)

we obtain

$$\frac{2}{1+\prod_{i=1}^{n}\left|z_i\right|\dfrac{\left|\alpha^2\left(2\beta c_2 - c_1^2\right)+\alpha\beta c_1\left(1+\sum_{i=1}^{n}\left(\dfrac{1-\left|z_i\right|^2}{z_i}\right)\right)\right|}{\beta^2\left(\prod_{i=1}^{n}\left|z_i\right|\right)^2-\alpha^2\left|c_1\right|^2}} \le \frac{1+\dfrac{\alpha}{\beta}\dfrac{\left|c_1\right|}{\prod_{i=1}^{n}\left|z_i\right|}}{1-\dfrac{\alpha}{\beta}\dfrac{\left|c_1\right|}{\prod_{i=1}^{n}\left|z_i\right|}}\left(\frac{\alpha}{\beta}\left|Z'(0)\right|-1-\sum_{i=1}^{n}\frac{1-\left|z_i\right|^2}{\left|1+z_i\right|^2}\right)$$

$$=\frac{\beta\prod_{i=1}^{n}\left|z_i\right|+\alpha\left|c_1\right|}{\beta\prod_{i=1}^{n}\left|z_i\right|-\alpha\left|c_1\right|}\left(\frac{\alpha}{\beta}\left|Z'(0)\right|-1-\sum_{i=1}^{n}\frac{1-\left|z_i\right|^2}{\left|1+z_i\right|^2}\right),$$

(24)

$$\frac{2\left(\beta^2\left(\prod_{i=1}^{n}\left|z_i\right|\right)^2-\alpha^2\left|c_1\right|^2\right)}{\beta^2\left(\prod_{i=1}^{n}\left|z_i\right|\right)^2-\alpha^2\left|c_1\right|^2+\prod_{i=1}^{n}\left|z_i\right|\left|\alpha^2\left(2\beta c_2-c_1^2\right)+\alpha\beta c_1\left(1+\sum_{i=1}^{n}\left(\dfrac{1-\left|z_i\right|^2}{z_i}\right)\right)\right|} \le \frac{\beta\prod_{i=1}^{n}\left|z_i\right|+\alpha\left|c_1\right|}{\beta\prod_{i=1}^{n}\left|z_i\right|-\alpha\left|c_1\right|}\left(\frac{\alpha}{\beta}\left|Z'(0)\right|-1-\sum_{i=1}^{n}\frac{1-\left|z_i\right|^2}{\left|1+z_i\right|^2}\right)$$

(25)

and

$$\frac{2\left(\beta\left(\prod_{i=1}^{n}\left|z_i\right|\right)-\alpha\left|c_1\right|\right)^2}{\beta^2\left(\prod_{i=1}^{n}\left|z_i\right|\right)^2-\alpha^2\left|c_1\right|^2+\prod_{i=1}^{n}\left|z_i\right|\left|\alpha^2\left(2\beta c_2-c_1^2\right)+\alpha\beta c_1\left(1+\sum_{i=1}^{n}\left(\dfrac{1-\left|z_i\right|^2}{z_i}\right)\right)\right|} \le \frac{\alpha}{\beta}\left|Z'(0)\right|-1-\sum_{i=1}^{n}\frac{1-\left|z_i^2\right|}{\left|1+z_i\right|^2}.$$

(26)

Thus, for $z_i = \dfrac{s_i - 1}{si+1}$, we obtain inequality given in (10).

Let's show equality in (10). Let

$$Z\left(\alpha\frac{1+z}{1-z}\right) = \beta\left(1+\frac{2z^2\prod_{i=1}^{n}\frac{z-z_i}{1-\overline{z_i}z}}{1-z^2\prod_{i=1}^{n}\frac{z-z_i}{1-\overline{z_i}z}}\right). \tag{27}$$

Then, we obtain

$$\frac{2\alpha}{(1-z)^2}Z'\left(\alpha\frac{1+z}{1-z}\right) = 2\beta\frac{\left(2z\prod_{i=1}^{n}\frac{z-z_i}{1-\overline{z_i}z}+z^2\sum_{i=1}^{n}\frac{1-|z_i|^2}{\left(1-\overline{z_i}z\right)(z-z_i)}\prod_{i=1}^{n}\frac{z-z_i}{1-\overline{z_i}z}\right)\left(1-z^2\prod_{i=1}^{n}\frac{z-z_i}{1-\overline{z_i}z}\right)}{\left(1-z^2\prod_{i=1}^{n}\frac{z-z_i}{1-\overline{z_i}z}\right)^2}$$
$$+2\beta\frac{\left(2z\prod_{i=1}^{n}\frac{z-z_i}{1-\overline{z_i}z}+z^2\prod_{i=1}^{n}\frac{z-z_i}{1-\overline{z_i}z}\sum_{i=1}^{n}\frac{1-|z_i|^2}{\left(1-\overline{z_i}z\right)(z-z_i)}\right)z^2\prod_{i=1}^{n}\frac{z-z_i}{1-\overline{z_i}z}}{\left(1-z^2\prod_{i=1}^{n}\frac{z-z_i}{1-\overline{z_i}z}\right)^2}. \tag{28}$$

For $z=-1$, we get

$$Z'(0) = 4\frac{\beta}{\alpha}\frac{-\prod_{i=1}^{n}\frac{1+z_i}{1+\overline{z_i}}\left(-2-\sum_{i=1}^{n}\frac{1-|z_i|^2}{\left(1+\overline{z_i}\right)}\frac{1}{1+z_i}\right)}{\left(1+\prod_{i=1}^{n}\frac{1+z_i}{1+\overline{z_i}}\right)^2}. \tag{29}$$

Since $z_1, z_2, ..., z_n$ are positive real numbers, we have

$$|Z'(0)| = \frac{\beta}{\alpha}\left(2+\sum_{i=1}^{n}\frac{1-\frac{s_i-1}{s_i+1}}{1+\frac{s_i-1}{s_i+1}}\right) = \frac{\beta}{\alpha}\left(2+\sum_{i=1}^{n}\frac{1}{s_i}\right). \tag{30}$$

Moreover, since $|c_1| = 0$ and $|c_2| = \dfrac{\beta}{2\alpha^2}\prod_{i=1}^{n}\left|\dfrac{s_i-1}{s_i+1}\right|$, inequality (10) holds.

The general equation for the obtained DPIF, $Z(s)$, in Theorem 3.1 can be rewritten as follows:

$$Z(s) = \frac{b_0 s^{n+2}+b_1 s^n+\ldots+b_{\frac{n+1}{2}}s}{a_0 s^{n+1}+a_1 s^{n-1}+\ldots+a_{\frac{n+1}{2}}} \tag{31}$$

where $b_0 = 1$ for any n value. Other coefficients are obtained using the parameter, n. The general circuit model for this DPIF can be designed as shown in Fig. 1.

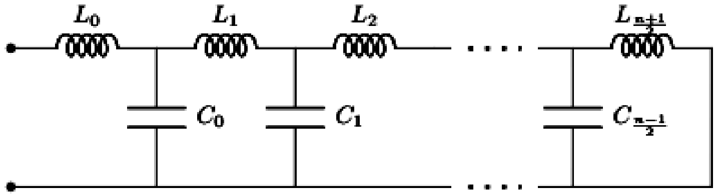

Figure 1. The schematics for the circuit corresponding to the general $Z(s)$ function obtained in Theorem 3.1.

As an example, let us consider the case where $n = 1$. For $n = 1$, the $Z(s)$ function is given as

$$Z_1(s) = \frac{s^3 + \left(\alpha^2 + 2\alpha s_1\right)s}{\left(2\alpha + s_1\right)s^2 + \alpha^2 s_1} \tag{32}$$

and the corresponding circuit is given as in Fig.2.

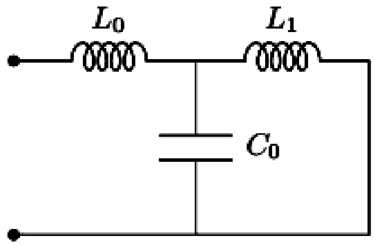

Figure 2. Corresponding circuit for $n = 1$ in Theorem 3.1 where $L_0 = \dfrac{1}{2\alpha + s_1}$ H, $C_0 = \dfrac{1}{2\alpha}\left(\dfrac{2\alpha + s_1}{\alpha + s_1}\right)^2$ F and $L_1 = 2\left(\alpha + s_1\right)^2 / \alpha s_1\left(2\alpha + s_1\right)$ H.

In $Z_1(s)$, α and s_1 are arbitrary positive real numbers. Let us assume that $\alpha = s_1 = 1$. Then,

$$Z_1(s) = \frac{s^3 + 3s}{3s^2 + 1}. \tag{33}$$

For $\alpha = s_1 = 1$; $L_0 = 0.333$ H, $C_0 = 1.125$ F and $L_1 = 2.667$ H. Accordingly, $Z_1(s)$ function has poles at $w = \pm 0.5774$ rad/sec and zeros at $w = \pm 1.7321$ rad/sec. We would like to state that the zeros at negative frequencies and zero at $w = 0$ rad/sec cannot be shown in logarithmic scale. Thus, zeros and poles at the positive frequency locations are shown in Fig. 3.

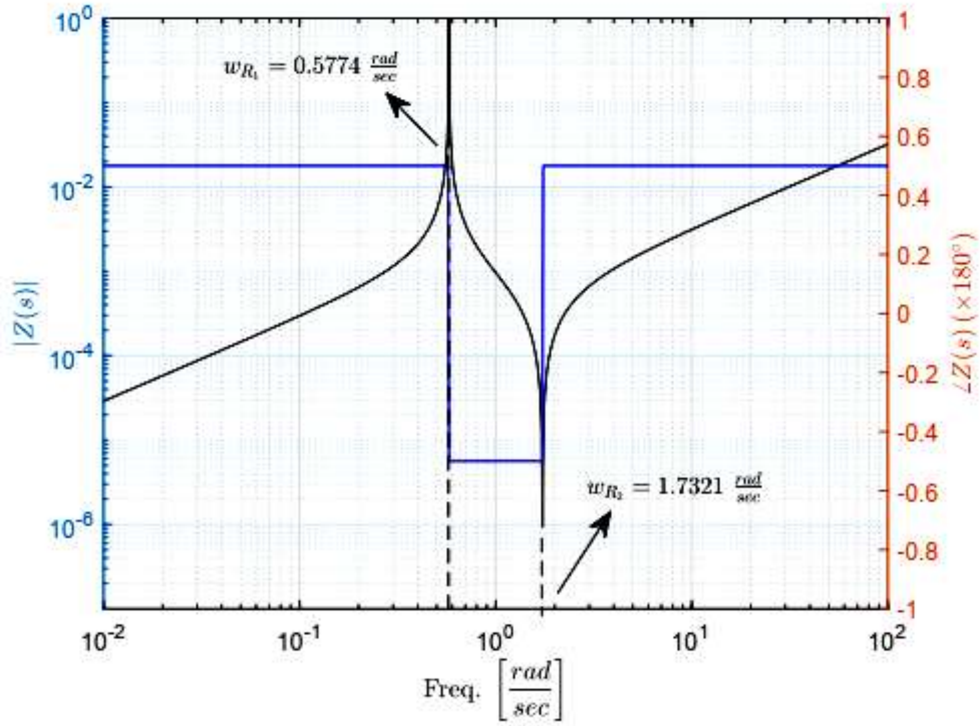

Figure 3. Frequency and phase responses for $n = 1$ in Theorem 3.1 where L_0 = 0.333 H, C_0 = 1.125 F and L_1 = 2.667 H.

For another example, assume that $n = 3$. For $n = 3$, $Z(s)$ function can be determined via analyzing the the general $Z(s)$ function.
The corresponding circuit for $n = 3$ can be designed using the general circuit schematics which was given in Fig. 1. Accordingly, the circuit for $Z_3(s)$ is shown in Fig. 4.

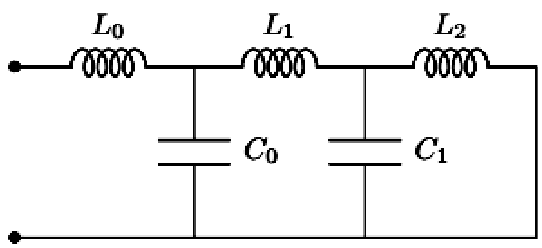

Figure 4. Corresponding circuit for $n = 3$ in Theorem 3.1.

For simplicity, consider that $s_1 = s_2 = s_3 = \alpha = 1$. Then, the $Z_3(s)$ function can be rewritten as follows:

$$Z_3(s) = \frac{s^5 + 10s^3 + 5s}{5s^4 + 10s^2 + 1}, \tag{34}$$

where L_0 = 0.2 H, C_0 = 0.625 F, L_1 = 1.1429 H, C_1 = 1.9141 F and L_2 = 3.6571 H in the corresponding circuit given in Fig. 4. According to coefficients of $Z_3(s)$, there are four poles at ± 0.3249 rad/sec and ± 1.3764 rad/sec. Also, there are five zeros at ± 0.7265 rad/sec, ± 3.0777 rad/sec and 0 rad/sec, respectively. Zeros and poles located at positive frequency bins are shown in Fig. 5.

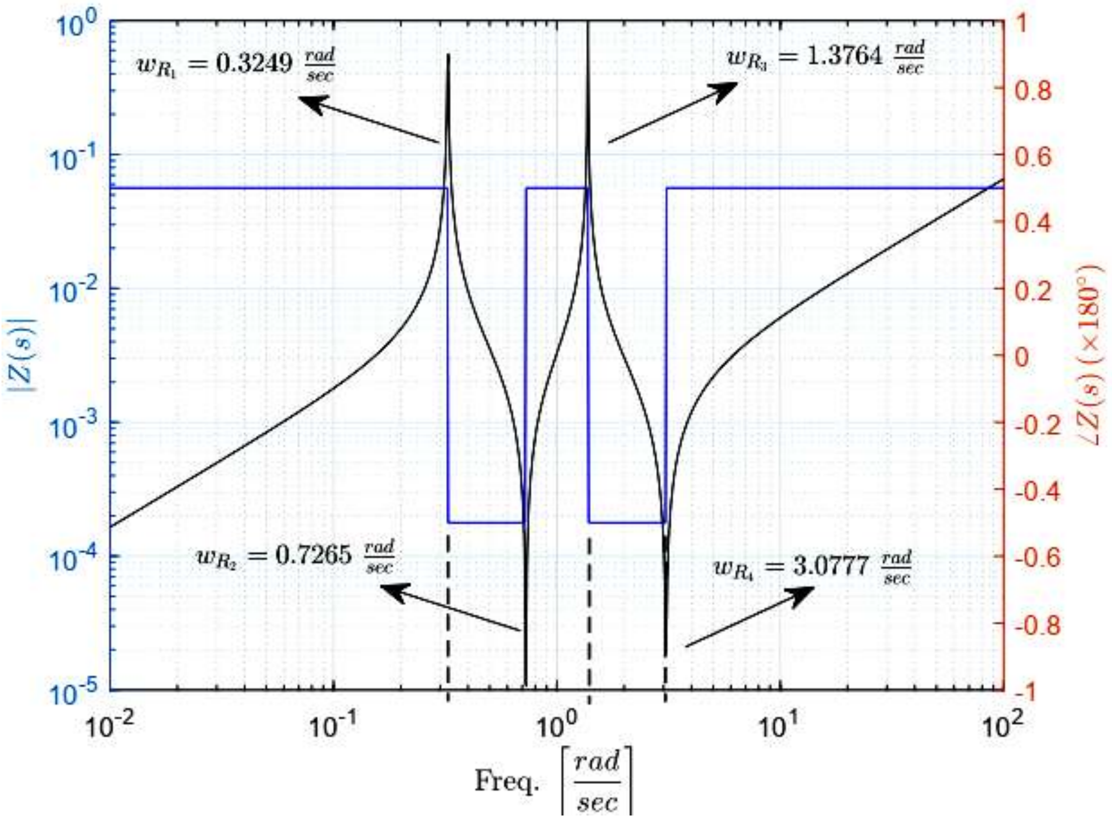

Figure 5. Frequency and phase responses for $n=3$ in Theorem 3.1 where where $L_0 = 0.2$ H, $C_0 = 0.625$ F, $L_1 = 1.1429$ H, $C_1 = 1.9141$ F and $L_2 = 3.6571$ H.

According to Fig. 2 through Fig.5, it is possible to say that the $Z(s)$ function and the corresponding circuit become more complex as n parameter increases. However, increasing the n parameter makes it possible to design multi-notch filters.

4. Conclusions

In this study, boundary analysis for the DPIFs has been presented using the Schwarz lemma. It is assumed that $Z(s)$ is zero at $s=0$ and it is given in the form of $Z(s) = \beta + c_1(s-\alpha) + c_2(s-\alpha)^2 + ...$ where α and β are a positive real numbers. The theorem presented in this study gives a lower bound for the modulus of the derivative of $Z(s)$ evaluated at zero, $|Z'(0)|$. A specific $Z(s)$ function has been obtained by proving the sharpness of the resulting inequality of the presented theorem. After obtaining this specific $Z(s)$ function, the corresponding circuit has been synthesized and it has been graphically analyzed to understand the behaviour its behaviour in frequency domain. According to obtained results, a filter with multiple notches at the frequency domain can be determined by changing the n parameter.

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