Turkish Journal of Science&Technology
14(2), 69-78, 2019

Comparison of Maximum Likelihood and Bayes Estimators Under Symmetric and Asymmetric Loss Functions by means of Tierney Kadane's Approximation for Weibull Distribution

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(Geliş/Received: 24/12/2018; Kabul/Accepted: 22/05/2019)

Abstract: In this study, it is considered the problem of comparing the performances of the Maximum Likelihood (ML) and Bayes estimators under symmetric and asymmetric loss function for the unknown parameters of Weibull distribution. ML estimators are computed by using the Newton Raphson method. Bayesian estimations under Squared, Linex and General Entropy loss functions by using Jeffrey's extension prior are introduced with Tierney Kadane approximation for Weibull distribution. For different sample sizes, estimators are compared to obtain the best estimator in terms of mean squared errors using a Monte Carlo simulation study.

Key words: Tierney-Kadane's approximation, Bayes estimation, Weibull distribution, Maximum likelihood estimation, Loss functions.

Weibull Dağılımı için Tierney Kadane'in Yaklaşımı ile Simetrik ve Asimetrik Kayıp Fonksiyonları Altında En Çok Olabilirlik ve Bayes Tahmin Edicilerinin Karşılaştırılması

Öz: Bu çalışmada, Weibull dağılımının bilinmeyen parametreleri için simetrik ve asimetrik kayıp fonksiyonları altında Bayes tahmin edicileri ve en çok olabilirlik tahmin edicilerinin karşılaştırılması problemi düşünülmüştür. En çok olabililirlik tahmin edicileri Newton Raphson methodu ile hesaplanmıştır.Weibull dağılımı için Tierney Kadane'in yaklaşımı ile Karesel, Linex ve Genel entropy kayıp fonksiyonları altında Bayes tahmin edicileri Jeffrey'in genişletilmiş önseli kullanılarak elde edilmiştir.Farklı örnek boyutları için Monte Carlo Simulasyon çalışması ile en iyi tahmin ediciyi elde etmek için tahmin ediciler hata kareler ortalamaları bakımından karşılaştırılmıştır.

Anahtar Kelimeler: Tierney Kadane'in yaklaşımı, Bayes tahmini, Weibull dağılımı, En çok olabilirlik tahmin edicisi, Kayıp fonksiyonları.

1.Introduction

Weibull distribution has become a popular tool for modeling life data and improving growth in the field of reliability. The Weibull distribution is generally used in reliability. The Weibull distribution can be used to model a variety of life behaviors. A Weibull distribution is the values of the shape parameter β, and the scale parameter α, affects the characteristics life of the distribution, the failure rate, the reliability function [1]. There are many studies about parameters for Weibull Distribution. Some of articles Nadarajah et.al. [2], Zhang et.al. [3], Guure et.al [4], Rasheed et.al [5], Pandey and Rao [6], Rasheed and F.Naji [7], Arshad and Abdalghani [8], Meena et.al [9] and Arshad and Misra [10]. The main advantage of Weibull analysis provides accurate failure analysis and failure forecasts with extremely small samples. ML estimation has been the most generally used method for estimating the parameters of the Weibull distribution. Bayes estimator for exponential distribution with an extension of Jeffreys' prior information was considered [11]. [1]The cumulative distribution function (CDF), probability density function (pdf), reliability function and hazard function of an X random variable having $W(\alpha,\beta)$ are as follows.

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$$F(x) = 1 - e^{-\left(\frac{x}{\alpha}\right)^{\beta}} \qquad \alpha > 0\,,\ \beta > 0\,,\ x > 0 \tag{1}$$

$$f(x) = \beta \alpha^{-\beta} x^{\beta-1} e^{-\left(\frac{x}{\alpha}\right)^{\beta}} \qquad \alpha > 0\,,\ \beta > 0\,,\ x > 0 \tag{2}$$

$$R(x) = e^{-\left(\frac{x}{\alpha}\right)^{\beta}} \tag{3}$$

$$h(x) = \beta \alpha^{-\beta} x^{\beta-1} \tag{4}$$

,where

- *for β<1 h(t)* is decreasing,
- *for β>1 h(t)* is increasing,
- *for β=1 h(t)* is constant.

As in almost all branches of science, one of the main objectives of statistics is to have information about the working population. In other words, knowing the unknown (parameters) of the population. That is, to make estimates about the parameters of the population. If so, it may be desirable to have the best estimator. There are many methods used in the literature.In this article, it is considered ML and Bayes estimators.

The aim of the study is to compare the ML and Bayes estimators under three loss functions by means of Tierney Kadane approximation using Jeffrey' s extension prior. The plan of the manuscript is as follows. In section 2, ML estimates of the parameters of Weibull distribution are reviewed. In section 3, Bayes estimators under Squared, Linex and General Entropy loss functions by using Tierney-Kadane approximation are obtained. In section 4, the simulation study is given. Finally, in section 5, the conclusion part is presented.

2.Preliminary

F(x): Distribution function
f(x): Probability density function
L(θ): The likelihood function depending on θ
ℓ(θ): The log-likelihood function depending on θ
π(θ): The prior distribution function depending on θ
π(θ/x): The posterior distribution function depending on θ
L(.,.):Loss function
$\hat{\alpha}$: Maximum likelihood estimation of α parameter

3.Material and Method

The ML Method the basic principle of the likelihood method is the selection of the sampling value corresponding to the values with the highest probability of obtaining the sampling values (or probability densities) as an estimate for the unknown parameter by looking at the sample values. The ML method is a method used to find predictors. The Bayesian approach is fundamentally different from other methods. In this approach, it is assumed in addition that θ is itself a random variable (though unobservable) with a known distribution. This prior distribution (specified according to the problem) is modified in light of the data to determine a posterior distribution (the conditional distribution of θ given the data), which summarizes what can be said about θ on the basis of the assumptions made and the data [12]. In addition to the primary distribution, a posterior distribution is used that reflects the sample information. The Bayesian estimation is considered to be the expected value of the posterior distribution under the lost function of interest.

4.Proposed Method

4.1.Tierney Kadane's approximation

Tierney and Kadane [13] is one of the methods to find the approximate value of the mathematical explanations as the ratio of two integrals given in Equations (19), (22) and (25). Although the Lindley approach plays an important role in the prediction of Bayes, this approach can only be used to obtain derivatives 1 and 2. For this reason, Tierney and Kadane proposed a new approach in 1986 (Tierney-Kadane approach) which makes it possible to take the 3rd derivative of the log-likelihood function. In this approach, there is a faster convergence than Lindley approximation [14]. This approximation can be written as follows for a case with two parameters. $u(\alpha,\beta)$ is any function of α and β , $\ell\left(\alpha,\beta\middle|\underset{-}{x}\right)$ is defined in Eq., (10) , $\rho(\alpha,\beta)$ is logarithm joint prior distribution and defined as follows. For details see Tierney and Kadene [13].

$$\rho(\alpha,\beta)=\ln\left(\pi(\alpha,\beta)\right)=-c\ln(\alpha)-c\ln(\beta) \tag{5}$$

$$l(\alpha,\beta)=\frac{1}{n}\{\rho(\alpha,\beta)+\ell(\alpha,\beta)\} \tag{6}$$

$$l^*(\alpha,\beta)=\frac{1}{n}\log u(\alpha,\beta)+l(\alpha,\beta) \quad . \tag{7}$$

Then Tierney Kadane's Bayes estimator of $u(\alpha,\beta)$ is defined as follows.

$$\begin{aligned}\hat{u}_{BS}(\alpha,\beta)&=E\left(u(\alpha,\beta)\middle|\underset{-}{x}\right)=\frac{\int e^{nl^*(\alpha,\beta)}d(\alpha,\beta)}{\int e^{nl(\alpha,\beta)}d(\alpha,\beta)} \quad .\\&=\left(\frac{\det\Sigma^*}{\det\Sigma}\right)^{1/2}\exp\left[n\left(l^*\left(\hat{\alpha}_{l^*},\hat{\beta}_{l^*}\right)-l\left(\hat{\alpha}_l,\hat{\beta}_l\right)\right)\right]\end{aligned} \tag{8}$$

$\left(\hat{\alpha}_{l^*},\hat{\beta}_{l^*}\right)$ and $\left(\hat{\alpha}_l,\hat{\beta}_l\right)$ maximize $l^*(\alpha,\beta)$ and $l(\alpha,\beta)$, respectively. Σ^* and Σ are minus the inverse Hessians of $l^*(\alpha,\beta)$ and $l(\alpha,\beta)$ at $\left(\hat{\alpha}_{l^*},\hat{\beta}_{l^*}\right)$ and $\left(\hat{\alpha}_l,\hat{\beta}_l\right)$, respectively. Σ is defined as follows;

$$\Sigma=\begin{bmatrix}-\partial^2 l/\partial\alpha^2 & -\partial^2 l/\partial\alpha\partial\beta\\ -\partial^2 l/\partial\alpha\partial\beta & -\partial^2 l/\partial\beta^2\end{bmatrix}^{-1} \tag{9}$$

4.2.Maximum likelihood estimation

Let $X_1,X_2,...,X_n$ be independent random variables having W distribution with α , β parameters. Then the log-likelihood function is given by

$$\begin{aligned}\ell\left(\alpha,\beta|\underline{x}\right)&=\ln\left(L\left(\alpha,\beta|\underline{x}\right)\right)\\&=n\ln(\beta)-n\beta\ln(\alpha)-\left(\sum_{i=1}^{n}\left(\frac{x_i}{\alpha}\right)^{\beta}\right)+(\beta-1)\left(\sum_{i=1}^{n}\ln x_i\right)\end{aligned} \tag{10}$$

Differentiating the log-likelihood function $\ell\left(\alpha,\beta\middle|\underset{-}{x}\right)$ partially with respect to α , β parameters and then equating to zero, following non-linear equations is obtained and these equations can be solved with the Newton-Raphson method [15].

$$\frac{\partial\ell\left(\alpha,\beta|\underline{x}\right)}{\partial\alpha}=-\frac{n\beta}{\alpha}-\left(\sum_{i=1}^{n}\left(\frac{\left(\frac{x_i}{\alpha}\right)^{\beta}\beta}{\alpha}\right)\right)=0 \tag{11}$$

$$\frac{\partial\ell\left(\alpha,\beta|\underline{x}\right)}{\partial\beta}=\frac{n}{\beta}-nl\,\mathrm{n}(\alpha)-\left(\sum_{i=1}^{n}\left(\frac{x_i}{\alpha}\right)^{\beta}\ln\left(\frac{x_i}{\alpha}\right)\right)+\sum_{i=1}^{n}\ln(x_i)=0 \tag{12}$$

4.3.Bayesian estimation symmetric and asymmetric loss functions

Let $X = (X_1, X_2, ..., X_n)$ is a random sample taken from Weibull(α,β) distribution. For Bayesian estimation of the parameters, it is needed for prior distributions for these parameters. In this study, as a prior distribution, Jeffrey's extension priors are used and these are as follows [16].

$$\pi_1(\alpha) \propto \left(\frac{1}{\alpha}\right)^c \tag{13}$$

$$\pi_2(\beta) \propto \left(\frac{1}{\beta}\right)^c \tag{14}$$

Prior and posterior distributions of $\alpha,\ \beta$ parameters are

$$\pi(\alpha,\beta) \propto \left(\frac{1}{\alpha}\right)^c \left(\frac{1}{\beta}\right)^c \propto \left(\frac{1}{\alpha\beta}\right)^c \tag{15}$$

$$\begin{aligned}\pi(\alpha,\beta|\underline{x}) &= \frac{f\left((\alpha,\beta);\underline{x}\right)}{f\left(\underline{x}\right)} \\ &= \frac{\beta^n \alpha^{-n\beta} e^{-\left(\left(\sum_{i=1}^{n} x_i\right)/\alpha\right)^{\beta}} \prod_{i=1}^{n} x_i^{\beta-1} \left(\frac{1}{\alpha\beta}\right)^c}{\int_0^\infty \int_0^\infty \beta^n \alpha^{-n\beta} e^{-\left(\left(\sum_{i=1}^{n} x_i\right)/\alpha\right)^{\beta}} \prod_{i=1}^{n} x_i^{\beta-1} \left(\frac{1}{\alpha\beta}\right)^c d\alpha d\beta}\end{aligned} \tag{16}$$

respectively.

The Squared error loss function is a symmetric function and introduced by [17] and [18]. Let any function of α and β is $u(\alpha,\beta) = u$. The squared loss function is as follows:

$$L_1(\hat{u}_{BS} - u) = (\hat{u}_{BS} - u)^2 \tag{17}$$

The value which is minimize the expected value of squared loss function is

$$\hat{u}_{BS}(\alpha,\beta) = E\left[u(\alpha,\beta)|\underline{x}\right] \tag{18}$$

In this case, Bayes estimator of $u(\alpha,\beta)$ under squared error loss function which is a symmetric loss function is obtained as follows.

$$\begin{aligned}\hat{u}_{BS}(\alpha,\beta) &= E\left[u(\alpha,\beta)|\underline{x}\right] \\ &= \int_0^\infty \int_0^\infty u(\alpha,\beta)\,\pi\left(\alpha,\beta|\underline{x}\right) d\alpha d\beta \\ &= \frac{\int_0^\infty \int_0^\infty u\left(\alpha,\beta/\underline{x}\right) e^{\left[\ell\left(\alpha,\beta/\underline{x}\right)+\rho(\alpha,\beta)\right]} d\alpha d\beta}{\int_0^\infty \int_0^\infty e^{\left[\ell\left(\alpha,\beta/\underline{x}\right)+\rho(\alpha,\beta)\right]} d\alpha d\beta}\end{aligned} \tag{19}$$

$\ell\left(\alpha,\beta|\underline{x}\right)$ is log-likelihood function, $\rho\left(\alpha,\beta|\underline{x}\right)$ is the logarithm of a joint prior distribution. The Linex loss function is an asymmetric function and introduced by Varian [19]. Zellner [20] is studied about Bayes estimation under linex loss function. Let any function of α and β is $u(\alpha,\beta)$ and "a" arbitrary constant. The Linex loss function is defined as follows.

$$L_2(\Delta)\ \alpha\ \exp(a\Delta)-a\Delta-1; \quad a \neq 0, \tag{20}$$

$\Delta = \hat{u}(\alpha,\beta) - u(\alpha,\beta)$. Then, posterior mean of Linex loss function is given as;

$$E_\theta\left[L_2\left(\hat{u}-u\right)\right] \propto \exp\left(a\hat{u}\right)E_\theta[\exp(-au)] - a\left(\hat{u}-E_\theta(u)\right) - 1 \tag{21}$$

$\hat{u} = \hat{u}(\alpha,\beta)$ and $u = u(\alpha,\beta)$, $\hat{u}_{BL}$ which minimize this posterior mean is Bayes estimator of u and is obtained as follows,

$$\hat{u}_{BL}(\alpha,\beta) = -\frac{1}{a}\ln E\left[\exp\left(-au(\alpha,\beta)\right)|\underline{x}\right]$$

$$= -\frac{1}{a}\ln\left(\frac{\int_0^\infty\int_0^\infty \exp\left(-au(\alpha,\beta)\right)e^{\left[\ell\left(\alpha,\beta|\underline{x}\right)+\rho(\alpha,\beta)\right]}d\alpha d\beta}{\int_0^\infty\int_0^\infty e^{\left[\ell\left(\alpha,\beta|\underline{x}\right)+\rho(\alpha,\beta)\right]}d\alpha d\beta}\right) \tag{22}$$

General Entropy loss function is an asymmetric function and suggested by Calabria and Pulcinia [21]. Dey and Liao [22] are studied with Bayes estimation under the General Entropy loss function. Let any function of α , β is $u(\alpha,\beta)$ and "k" arbitrary constant. General Entropy loss function is defined as follows.

$$L_3(\hat{u},u) \propto \left(\frac{\hat{u}}{u}\right)^k - k\,ln\left(\frac{\hat{u}}{u}\right) - 1 \tag{23}$$

Then, posterior mean of General Entropy loss function is given as;

$$E_\theta\left[L_3\left(\hat{u},u\right)\right] \propto E\left(\frac{\hat{u}}{u}\right)^k - kE\left[\ln\left(\hat{u}\right) - \ln(u)\right] - 1 \tag{24}$$

$\hat{u} = \hat{u}(\alpha,\beta)$ and $u = u(\alpha,\beta)$. Then, $\hat{u}_{BGE}$ which minimize this posterior mean is Bayes estimator of u and is obtained as follows.

$$\hat{u}_{BGE}(\alpha,\beta) = \left\{E\left\{\left[u(\alpha,\beta)\right]^{-k}|\underline{x}\right\}\right\}^{-\frac{1}{k}}$$

$$= \left\{\frac{\int_0^\infty\int_0^\infty \left[u(\alpha,\beta)\right]^{-k} e^{\left[\ell\left(\alpha,\beta|\underline{x}\right)+\rho(\alpha,\beta)\right]}d\alpha d\beta}{\int_0^\infty\int_0^\infty e^{\left[\ell\left(\alpha,\beta|\underline{x}\right)+\rho(\alpha,\beta)\right]}d\alpha d\beta}\right\}^{-\frac{1}{k}} \tag{25}$$

It is very difficult to solve the equations (19), (22) and (25) in closed-form. Because of this reason, the Bayes Estimators of $u(\alpha,\beta)$ can be obtained using Tierney-Kadane's approximation.

4.4. Mathematical equations

The partial derivatives related to $l^*(\alpha,\beta)$, $l(\alpha,\beta)$, Σ^* and Σ are defined as follows;

$$l(\alpha,\beta)=\frac{1}{n}\left[n\ln(\beta)-n\beta\ln(\alpha)-\left(\sum_{i=1}^{n}\left(\frac{x_i}{\alpha}\right)^{\beta}\right)+(\beta-1)\left(\sum_{i=1}^{n}\ln x_i\right)-c\ln(\alpha)-c\ln(\beta)\right] \tag{26}$$

$$\frac{\partial^2 l}{\partial\alpha^2}=\frac{\frac{n\beta}{\alpha^2}-\left(\sum_{i=1}^{n}\left(\frac{\left(\frac{x_i}{\alpha}\right)^{\beta}\beta^2}{\alpha^2}+\frac{\left(\frac{x_i}{\alpha}\right)^{\beta}\beta}{\alpha^2}\right)\right)+\frac{c}{\alpha^2}}{n} \tag{27}$$

$$\frac{\partial^2 l}{\partial\alpha\beta}=\frac{-\frac{n}{\alpha}-\left(\sum_{i=1}^{n}\left(\frac{\left(\frac{x_i}{\alpha}\right)^{\beta}\ln\left(\frac{x_i}{\alpha}\right)\beta}{\alpha}+\frac{\left(\frac{x_i}{\alpha}\right)^{\beta}}{\alpha}\right)\right)}{n} \tag{28}$$

$$\frac{\partial^2 l}{\partial\beta^2}=\frac{-\frac{n}{\beta^2}-\left(\sum_{i=1}^{n}\left(\frac{x_i}{\alpha}\right)^{\beta}\ln\left(\frac{x_i}{\alpha}\right)^{2}\right)+\frac{c}{\beta^2}}{n} \tag{29}$$

Bayes estimators for α, β parameters using Eq. (9) are found as follows.

i. If $u(\alpha,\beta)=\alpha$

$$\Sigma_1^*=\begin{bmatrix}-\partial^2 l_1^*/\partial\alpha^2 & -\partial^2 l_1^*/\partial\alpha\partial\beta\\ -\partial^2 l_1^*/\partial\alpha\partial\beta & -\partial^2 l_1^*/\partial\beta^2\end{bmatrix}^{-1} \tag{30}$$

$$\hat{\alpha}_B=\left(\frac{\det\Sigma_1^*}{\det\Sigma}\right)^{1/2}\exp\left[n\left(l_1^*\left(\hat{\alpha}_{l_1^*},\hat{\beta}_{l_1^*}\right)-l\left(\hat{\alpha}_l,\hat{\beta}_l\right)\right)\right] \tag{31}$$

$$l_1^*(\alpha,\beta)=\frac{1}{n}\log\alpha+l(\alpha,\beta) \tag{32}$$

The partial derivatives related to l_1^* are given as,

$$\frac{\partial^2 l_1^*}{\partial\alpha^2}=-\frac{1}{n\alpha^2}+\frac{\frac{n\beta}{\alpha^2}-\left(\sum_{i=1}^{n}\left(\frac{\left(\frac{x_i}{\alpha}\right)^{\beta}\beta^2}{\alpha^2}+\frac{\left(\frac{x_i}{\alpha}\right)^{\beta}\beta}{\alpha^2}\right)\right)+\frac{c}{\alpha^2}}{n} \tag{33}$$

$$\frac{\partial^2 l_1^*}{\partial\alpha\beta}=\frac{-\frac{n}{\alpha}-\left(\sum_{i=1}^{n}\left(-\frac{(\frac{x_i}{\alpha})^{\beta}\ln(\frac{x_i}{\alpha})\beta}{\alpha}-\frac{(\frac{x_i}{\alpha})^{\beta}}{\alpha}\right)\right)}{n} \tag{34}$$

$$\frac{\partial^2 l_1^*}{\partial \beta^2} = -\frac{1}{n\beta^2} + \frac{-\frac{n}{\beta^2} - \left(\sum_{i=1}^{n} \left(\frac{x_i}{\alpha} \right)^{\beta} \ln\left(\frac{x_i}{\alpha} \right)^2 \right) + \frac{c}{\beta^2}}{n} \tag{35}$$

ii. If $u(\alpha, \beta) = \beta$

$$\Sigma_2^* = \begin{bmatrix} -\partial^2 l_2^* / \partial \alpha^2 & -\partial^2 l_2^* / \partial \alpha \partial \beta \\ -\partial^2 l_2^* / \partial \alpha \partial \beta & -\partial^2 l_2^* / \partial \beta^2 \end{bmatrix}^{-1} \tag{36}$$

$$\hat{\beta}_B = \left(\frac{\det \Sigma_2^*}{\det \Sigma} \right)^{1/2} \exp\left[n\left(l_2^*\left(\hat{\alpha}_{l_2^*}, \hat{\beta}_{l_2^*} \right) - l\left(\hat{\alpha}_l, \hat{\beta}_l \right) \right) \right], l_2^*(\alpha, \beta) = \frac{1}{n} \ln \beta + l(\alpha, \beta) \tag{37}$$

The partial derivatives related to l_2^* are given as,

$$\frac{\partial^2 l_2^*}{\partial \beta^2} = -\frac{1}{n\beta^2} + \frac{-\frac{n}{\beta^2} - \left(\sum_{i=1}^{n} \left(\frac{x_i}{\alpha} \right)^{\beta} \ln\left(\frac{x_i}{\alpha} \right)^2 \right) + \frac{c}{\beta^2}}{n}, \quad \frac{\partial^2 l_2^*}{\partial \alpha \partial \beta} = \frac{-\frac{n}{\alpha} - \left(\sum_{i=1}^{n} \left(-\frac{(\frac{x_i}{\alpha})^{\beta} \ln(\frac{x_i}{\alpha})\beta}{\alpha} - \frac{(\frac{x_i}{\alpha})^{\beta}}{\alpha} \right) \right)}{n} \tag{38}$$

5.Appendix

In this section, ML and approximate Bayes Estimators by Tierney-Kadane' approximation are obtained under Squared error loss function, Linex loss function and General Entropy loss function for unknown parameters of W distribution and results are compared in terms of mean squared error by using Monte Carlo simulation method. Mean squared error (MSE) is defined as follows;Let θ is the true parameter value and $\hat{\theta}_{(i)}$ $\left(i = 1, 2, ..., 10000\right)$ is the estimation value in i^{th} replication. Then the MSE for Tierney-Kadane approximations can be written as,

$$MSE = \frac{1}{10000} \sum_{i=1}^{10000} \left(\hat{\theta}_{(i)} - \theta \right)^2 \tag{39}$$

Simulation steps are as follows:Step1: It is generated data from W distribution with $\alpha = 1, 1.5,\ \beta = 1.3, 1.7$ for the sample size n=20, 30, 50, 100.Step 2: ML estimates for parameters are computed by a solution of non-linear Eqs. (11-12) by using the Newton-Raphson method.Step 3: Tierney-Kadane Bayes estimates are computed for parameters under Squared error, Linex $a = \pm 0.8, \pm 1.5$ and General entropy $k = \pm 0.8, \pm 1.5$ loss functions using Jeffrey's extension prior (c=0.2).Step 4: Means squared errors are computed over 10000 replications by using Eq. (39).

6.Conclusion

In this study, approximate Bayes estimators under Squared error, Linex and General entropy loss functions obtained by using the Tierney-Kadane's method and ML's for W distribution with parameters are compared. The ML's of the unknown parameters are computed by using the Newton Raphson method. The approximate Bayes estimators are compared with the ML's in terms of MSE by using the Monte Carlo simulation method. As seen from Table 1 and Table 2, the performances of Bayes estimates for parameters and general entropy loss function are generally better than others in terms of MSEs. In addition, MSEs of ML and approximate Bayes estimates obtained under different loss functions are decreased when n is increased. Furthermore, MSEs of estimators are close to each other for large n values.İt is seen that the minimum MSE is reached even if the parameter values change when looking at Figures 1 and 2.In general, the ML estimators and estimators are obtained under the quadratic loss function are almost the same as MSE, and in some cases the linex loss function has the same MSE with general entropy loss function, while general entropy loss function often has a smaller MSE.

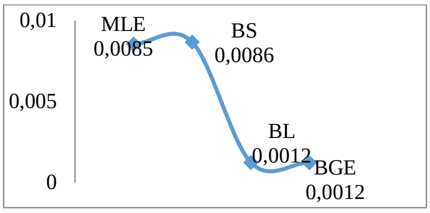

Figure 1. MSEs values for α=1.5 and β=1.7

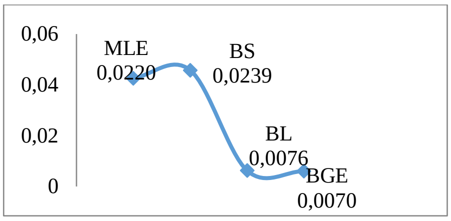

Figure 2. MSEs values for α=1 and β=1.3

Table 1. Mean estimates and MSEs for parameters of Weibull distribution $(a, k \pm 0.8)$

n	α	c	β	α ,	ML	BS	BL	BGE	ML	BS	BL	BGE
					$a = k = 0.8$				$a = k = -0.8$			
30	1	0.2	1.3	α_{MSE}	**0.022004**	**0.023935**	**0.007610**	**0.007089**	0.021749	0.023826	0.007772	0.006984
				α_{ME}	1.000476	1.033536	1.029985	1.020257	1.002649	1.035844	1.043469	1.032701
				β_{MSE}	0.046422	0.045170	**0.008289**	**0.006964**	0.046640	0.045415	0.008809	0.007107
				β_{ME}	1.364397	1.355320	1.347874	1.338887	1.363272	1.354210	1.370014	1.350367
	1.5	0.2	1.7	α_{MSE}	0.028945	0.030532	0.004226	0.004118	0.028336	0.030021	0.004380	0.004034
				α_{ME}	1.499102	1.531427	1.520957	1.516731	1.500584	1.533068	1.544775	1.530689
				β_{MSE}	0.080435	0.078229	0.008121	0.007032	0.077744	0.075671	0.008789	0.006926
				β_{ME}	1.788931	1.778696	1.754710	1.748446	1.783719	1.773503	1.800806	1.769439
50	1	0.2	1.3	α_{MSE}	0.012859	0.013555	0.004334	0.004115	**0.013030**	**0.013757**	**0.004447**	**0.004146**
				α_{ME}	1.000961	1.020859	1.019286	1.013165	1.001479	1.021337	1.025957	1.019487
				β_{MSE}	0.024779	0.024334	**0.004512**	**0.004090**	0.025153	0.024688	**0.004752**	**0.004157**
				β_{ME}	1.337451	1.331815	1.327961	1.322621	1.339448	1.333842	1.342863	1.331561
	1.5	0.2	1.7	α_{MSE}	0.017565	0.018109	0.002539	0.002490	0.017126	0.017705	0.002557	0.002444
				α_{ME}	1.498704	1.518280	1.512414	1.509669	1.499594	1.519100	1.526108	1.517694
				β_{MSE}	0.042597	0.041880	**0.004467**	**0.004129**	0.041346	0.040578	**0.004640**	**0.004015**
				β_{ME}	1.746877	1.740578	1.727203	1.723297	1.750987	1.744673	1.760201	1.742276
100	1	0.2	1.3	α_{MSE}	**0.006377**	**0.006567**	0.002105	0.002038	**0.006410**	**0.006578**	**0.002117**	**0.002047**
				α_{ME}	1.001266	1.011215	1.010638	1.007478	1.000161	1.010111	1.012450	1.009195
				β_{MSE}	0.011376	0.011256	0.002109	0.002000	0.011323	0.011212	0.002142	0.002006
				β_{ME}	1.319785	1.316903	1.315156	1.312507	1.318482	1.315603	1.319924	1.314472
	1.5	0.2	1.7	α_{MSE}	0.008430	0.008578	**0.001210**	**0.001198**	**0.008551**	**0.008667**	**0.001242**	**0.001220**
				α_{ME}	1.499987	1.509782	1.507014	1.505567	1.498607	1.508401	1.511918	1.507702
				β_{MSE}	0.019095	0.018908	**0.002055**	**0.001963**	**0.019217**	**0.019037**	**0.002139**	**0.001993**
				β_{ME}	1.724503	1.721269	1.714866	1.712890	1.723307	1.720076	1.727521	1.718894

Table 2. Mean estimates and MSEs for parameters of Weibull distribution $(a, k \pm 1.5)$

n	α	c	β	α β	ML	BS	BL	BGE	ML	BS	BL	BGE
					$a = k = 1.5$				$a = k = -1.5$			
20	1	0.2	1.3	α_{MSE}	0.031708	0.035951	0.038618	0.035970	0.032481	0.037079	0.044535	0.036985
				α_{ME}	0.999643	1.049293	1.027367	1.013366	1.002285	1.052084	1.080924	1.062769
				β_{MSE}	0.082520	0.079692	0.051532	0.039886	0.077694	0.074972	0.053068	0.037289
				β_{ME}	1.398143	1.385038	1.348326	1.339240	1.395629	1.382525	1.433962	1.395356
	1.5	0.2	1.7	α_{MSE}	0.042242	0.045765	0.021244	0.021252	0.043484	0.047233	0.025538	0.021656
				α_{ME}	1.499223	1.547709	1.514314	1.512944	1.500617	1.549138	1.585814	1.557600
				β_{MSE}	0.138847	0.133927	0.044144	0.039669	0.143650	0.138642	0.060878	0.039865
				β_{ME}	1.832416	1.817573	1.742855	1.747999	1.835069	1.820249	1.906590	1.835203
30	1	0.2	1.3	α_{MSE}	0.022542	0.024529	0.027120	0.025402	0.021511	0.023501	0.027633	0.024488
				α_{ME}	1.001508	1.034497	1.021067	1.011375	1.001393	1.034568	1.053076	1.041488
				β_{MSE}	0.046899	0.045606	0.028794	0.025092	0.045804	0.044595	0.031058	0.024463
				β_{ME}	1.366987	1.357906	1.335148	1.328881	1.362667	1.353607	1.385489	1.361996
	1.5	0.2	1.7	α_{MSE}	0.028042	0.029417	0.014015	0.014109	0.028327	0.029834	0.015692	0.014197
				α_{ME}	1.496258	1.528779	1.507386	1.506154	1.498274	1.530752	1.554361	1.536279
				β_{MSE}	0.077124	0.075157	0.026333	0.024567	**0.079072**	**0.077009**	0.032630	0.024385
				β_{ME}	1.779187	1.768993	1.722547	1.725018	1.782831	1.772606	1.824999	1.782292
50	1	0.2	1.3	α_{MSE}	**0.012965**	**0.013626**	0.015186	0.014596	0.013297	0.014032	0.016086	0.014812
				α_{ME}	1.000247	1.020112	1.012661	1.006556	1.001272	1.021225	1.032093	1.025302
				β_{MSE}	0.025682	0.025224	0.016344	0.014963	0.024191	0.023777	**0.016282**	**0.014135**
				β_{ME}	1.338425	1.332813	1.320102	1.316247	1.334954	1.329338	1.347366	1.334284
	1.5	0.2	1.7	α_{MSE}	0.017214	0.017837	0.008670	0.008626	0.016956	0.017379	0.008946	0.008523
				α_{ME}	1.501041	1.520582	1.508192	1.507317	1.495978	1.515490	1.529296	1.518751
				β_{MSE}	0.043107	0.042340	0.015477	0.014722	0.042769	0.042030	0.017263	0.014501
				β_{ME}	1.750684	1.744389	1.717540	1.718807	1.748588	1.742281	1.771790	1.747976
100	1	0.2	1.3	α_{MSE}	0.006467	0.006627	0.007423	0.007281	0.006479	0.006639	0.007546	0.007273
				α_{ME}	0.999718	1.009684	1.006190	1.003023	0.999741	1.009707	1.015036	1.011711
				β_{MSE}	0.011125	0.011022	**0.007212**	**0.006964**	0.010893	0.010792	**0.007310**	**0.006836**
				β_{ME}	1.317081	1.314195	1.308198	1.306236	1.316845	1.313959	1.322612	1.316400
	1.5	0.2	1.7	α_{MSE}	0.008538	0.008688	0.004291	0.004268	0.008427	0.008561	0.004339	0.004206
				α_{ME}	1.500520	1.510306	1.504310	1.503789	1.499575	1.509369	1.516190	1.510981
				β_{MSE}	**0.019582**	**0.019384**	0.007306	0.007112	0.019062	0.018878	**0.007567**	**0.006923**
				β_{ME}	1.725885	1.722650	1.709719	1.710240	1.723895	1.720668	1.734779	1.723475

ML:Maximum likelihood estimation
BS:Bayes estimation under squared error loss function
BGE:Bayes estimation under general entropy loss function
BL:Bayes estimation under linex loss function
α_{MSE} :MSEs for α parameter
β_{MSE} : MSEs for α parameter
α_{ME} : Mean estimate for α parameter
β_{ME} : Mean estimate for β parameter

Acknowledge

This study is presented as a master thesis by G.G, Selcuk University, Konya, TURKEY. 2016.

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