CONSTRUCTIVE MATHEMATICAL ANALYSIS
2 (2019), No. 3, pp. 124-129
http://dergipark.gov.tr/cma
ISSN 2651-2939

A Quantitative Variant of Voronovskaja's Theorem for King-Type Operators

ZOLTÁN FINTA*

ABSTRACT. In this note we establish a quantitative Voronovskaja theorem for modified Bernstein polynomials using the first order Ditzian-Totik modulus of smoothness.

Keywords: Bernstein operators, Voronovskaja theorem, King operators, First order Ditzian-Totik modulus of smoothness

2010 Mathematics Subject Classification: 41A25, 41A36.

1. INTRODUCTION

The Bernstein polynomials are defined by

$$(B_nf)(x) \equiv B_n(f;x) = \sum_{k=0}^{n} p_{n,k}(x) f\left(\frac{k}{n}\right), \tag{1.1}$$

where $p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}$, $x \in [0,1]$, $f \in C[0,1]$ and $n \geq 1$. Among the properties of Bernstein polynomials we mention the following asymptotic formula, called Voronovskaja's theorem: *if f is bounded on $[0,1]$, differentiable in some neighborhood of $x \in [0,1]$, and has second derivative $f''(x)$, then*

$$\lim_{n\to\infty} n\left((B_nf)(x) - f(x)\right) = \frac{1}{2}x(1-x)f''(x). \tag{1.2}$$

Further properties:

$$(B_ne_0)(x) = 1, \quad (B_ne_1)(x) = x \quad \text{and} \quad (B_ne_2)(x) = x^2 + \frac{1}{n}x(1-x),$$

where $e_i(x) = x^i$, $x \in [0,1]$ and $i \in \{0,1,2,\ldots\}$. In [8] King constructed a Bernstein-type operator, which preserves the functions e_0 and e_2. By modification of $f\left(\frac{k}{n}\right)$ in (1.1), Aldaz et al. [1] defined Bernstein-King-type operators possessing e_0 and e_j as fixed points, where $j \in \{2,3,\ldots\}$ is arbitrary. These operators are given by

$$(U_{n,j}f)(x) \equiv U_{n,j}(f;x) = \sum_{k=0}^{n} p_{n,k}(x) f(a_{n,k}) \tag{1.3}$$

(see [1, Proposition 11]), where $x \in [0,1]$, $f \in C[0,1]$ and

$$a_{n,k} = \sqrt[j]{\frac{k(k-1)\ldots(k-j+1)}{n(n-1)\ldots(n-j+1)}}, \quad n \geq j \geq 2. \tag{1.4}$$

The operators $U_{n,j}$ are linear and positive, $U_{n,j}e_0 = e_0$ and $U_{n,j}e_j = e_j$, respectively.

Received: 13 April 2019; Accepted: 11 July 2019; Published Online: 13 July 2019
*Corresponding author: Z. Finta; fzoltan@math.ubbcluj.ro
DOI: 10.33205/cma.553427

The goal of the paper is to obtain a quantitative Voronovskaja-type theorem for $U_{n,j}$ with the aid of the first order Ditzian-Totik modulus of smoothness defined by

$$\omega_{\varphi}^{1}(f;\delta) = \sup_{0<h\leq\delta} \sup_{x\pm\frac{1}{2}h\varphi(x)} \left| f\left(x+\frac{1}{2}h\varphi(x)\right) - f\left(x-\frac{1}{2}h\varphi(x)\right) \right|, \tag{1.5}$$

where $\varphi(x) = \sqrt{x(1-x)}$, $x \in [0,1]$. It is known [2, Theorem 2.1.1] that (1.5) is equivalent with the $K-$functional

$$K_{1,\varphi}(f;\delta) = \inf_{g\in W(\varphi)} \{\|f-g\| + \delta\|\varphi g'\|\}, \quad \delta > 0,$$

where $W(\varphi) = \{g \,|\, g \in AC_{loc}[0,1], \|\varphi g'\| < \infty\}$ and $g \in AC_{loc}[0,1]$ means that g is absolutely continuous in every closed interval $[a,b] \subseteq [0,1]$, i.e. there exists $C_1 > 0$ such that

$$C_1^{-1}\omega_{\varphi}^{1}(f;\delta) \leq K_{1,\varphi}(f;\delta) \leq C_1\omega_{\varphi}^{1}(f;\delta). \tag{1.6}$$

It is worth mentioning that Floater obtained a generalization of (1.2) in [4], dealing with the asymptotic behavior of differentiated Bernstein polynomials. Different quantitative versions of Floater's theorem were established in [5], [6], [7] and [3].

2. Main result

In the sequel we need some lemmas.

Lemma 2.1. *The inequalities* $0 \leq 1 - x^n - (1-x)^n \leq nx(1-x)$ *hold true for all* $x \in [0,1]$ *and* $n = 1, 2, \ldots$

Proof. For $x \in [0,1]$, we have

$$x^n + (1-x)^n \leq x + (1-x) = 1. \tag{2.1}$$

For the second inequality, we have

$$\begin{aligned} 1 - x^n - (1-x)^n &= (1-x)(1+x+\ldots+x^{n-1}) - (1-x)^n \\ &= (1-x)[1+x+\ldots+x^{n-1} - (1-x)^{n-1}] \leq nx(1-x) \end{aligned}$$

iff $1+x+\ldots+x^{n-1} \leq nx + (1-x)^{n-1}$. We prove the former inequality by induction on n. If $n = 1$, then $1 \leq x+1$; we suppose that $1+x+\ldots+x^{n-1} \leq nx + (1-x)^{n-1}$. Then, by (2.1),

$$\begin{aligned} & 1+x+\ldots+x^{n-1}+x^n \\ \leq\ & nx + (1-x)^{n-1} + x^n = (n+1)x + (1-x)^{n-1} - x + x^n \\ =\ & (n+1)x + (1-x)^n + (1-x)^{n-1} - (1-x)^n - x + x^n \\ =\ & (n+1)x + (1-x)^n + x(1-x)^{n-1} - x + x^n \\ =\ & (n+1)x + (1-x)^n - x(1-x^{n-1} - (1-x)^{n-1}) \leq (n+1)x + (1-x)^n, \end{aligned}$$

which was to be proved. □

Lemma 2.2. *For the operator* $U_{n,j}$ *defined by (1.3)-(1.4) and* $x \in [0,1]$, *we have*

a) $0 \leq U_{n,j}(xe_0 - e_1; x) \leq \frac{1}{n}(j-1)$;
b) $U_{n,j}((e_1 - xe_0)^2; x) \leq \frac{2}{n}((j-1)^2+1)\varphi^2(x)$;
c) $U_{n,j}((e_1 - xe_0)^4; x) \leq \frac{8}{n^2}((j-1)^2+1)$.

Proof. Because $U_{n,j}$ is linear and preserves the functions e_0 and e_j, we obtain

$$
\begin{aligned}
U_{n,j}(xe_0 - e_1; x) &= x - U_{n,j}(e_1; x) = \sum_{k=0}^{n} p_{n,k}(x)\frac{k}{n} - \sum_{k=0}^{n} p_{n,k}(x)a_{n,k} \\
&= \sum_{k=0}^{n} p_{n,k}(x)\left(\frac{k}{n} - a_{n,k}\right). \qquad (2.2)
\end{aligned}
$$

For $k \in \{j, j+1, \dots, n\}$, we have $\frac{k-j+1}{n-j+1} \leq \dots \leq \frac{k-1}{n-1} \leq \frac{k}{n}$. Hence

$$
\begin{aligned}
0 \leq \frac{k}{n} - a_{n,k} &\leq \frac{k}{n} - \frac{k-j+1}{n-j+1} = (j-1)\frac{n-k}{n(n-j+1)} \\
&\leq (j-1)\frac{n-j}{n(n-j+1)} \leq \frac{j-1}{n}. \qquad (2.3)
\end{aligned}
$$

Therefore, in view of (2.2) and (2.3), we get

$$
\begin{aligned}
0 \leq U_{n,j}(xe_0 - e_1; x) &= \sum_{k=1}^{j-1} p_{n,k}(x)\frac{k}{n} + \sum_{k=j}^{n-1} p_{n,k}(x)\left(\frac{k}{n} - a_{n,k}\right) \\
&\leq \sum_{k=1}^{j-1} p_{n,k}(x)\frac{j-1}{n} + \sum_{k=j}^{n-1} p_{n,k}(x)\frac{j-1}{n} \leq \frac{j-1}{n}\sum_{k=0}^{n} p_{n,k}(x) = \frac{j-1}{n}.
\end{aligned}
$$

$b)$ Taking into account the inequality $(a+b)^2 \leq 2(a^2+b^2)$, (2.3) and Lemma 2.1, we find that

$$
\begin{aligned}
U_{n,j}((e_1 - xe_0)^2; x) &= \sum_{k=0}^{n} p_{n,k}(x)(a_{n,k} - x)^2 \\
&\leq 2\sum_{k=0}^{n} p_{n,k}(x)\left(a_{n,k} - \frac{k}{n}\right)^2 + 2\sum_{k=0}^{n} p_{n,k}(x)\left(\frac{k}{n} - x\right)^2 \\
&= 2\sum_{k=1}^{j-1} p_{n,k}(x)\left(\frac{k}{n}\right)^2 + 2\sum_{k=j}^{n-1} p_{n,k}(x)\left(a_{n,k} - \frac{k}{n}\right)^2 + \frac{2}{n}x(1-x) \\
&\leq 2\left(\frac{j-1}{n}\right)^2\sum_{k=1}^{j-1} p_{n,k}(x) + 2\left(\frac{j-1}{n}\right)^2\sum_{k=j}^{n-1} p_{n,k}(x) + \frac{2}{n}x(1-x) \\
&= 2\left(\frac{j-1}{n}\right)^2(1-(1-x)^n - x^n) + \frac{2}{n}x(1-x) \leq 2\left(\frac{j-1}{n}\right)^2 nx(1-x) + \frac{2}{n}x(1-x) \\
&= \frac{2}{n}((j-1)^2+1)\varphi^2(x).
\end{aligned}
$$

c) In view of $(a+b)^4 \le 8(a^4+b^4)$, (2.3) and $\sum_{k=0}^{n}(k-nx)^4 p_{n,k}(x) = 3n^2\varphi^4(x)+n(\varphi^2(x)-6\varphi^4(x))$, we obtain

$$\begin{aligned}
U_{n,j}((e_1-xe_0)^4;x) &= \sum_{k=0}^{n} p_{n,k}(x)(a_{n,k}-x)^4 \\
&\le 8\sum_{k=0}^{n} p_{n,k}(x)\left(a_{n,k}-\frac{k}{n}\right)^4 + 8\sum_{k=0}^{n} p_{n,k}(x)\left(\frac{k}{n}-x\right)^4 \\
&= 8\sum_{k=1}^{j-1} p_{n,k}(x)\left(\frac{k}{n}\right)^4 + 8\sum_{k=j}^{n-1} p_{n,k}(x)\left(a_{n,k}-\frac{k}{n}\right)^4 + \frac{8}{n^4}\sum_{k=0}^{n} p_{n,k}(x)\,(k-nx)^4 \\
&\le 8\left(\frac{j-1}{n}\right)^4 \sum_{k=1}^{j-1} p_{n,k}(x) + 8\left(\frac{j-1}{n}\right)^4 \sum_{k=j}^{n-1} p_{n,k}(x) \\
&\quad + \frac{8}{n^4}(3n^2\varphi^4(x)+n(\varphi^2(x)-6\varphi^4(x))) \\
&\le 8\left(\frac{j-1}{n}\right)^4 + \frac{8}{n^4}4n^2\varphi^2(x) \le 8\left(\frac{j-1}{n}\right)^4 + \frac{8}{n^2} \le \frac{8}{n^2}((j-1)^4+1).
\end{aligned}$$

This completes the proof of the lemma. □

The main result is the following theorem.

Theorem 2.1. *Let $U_{n,j}$ be given by (1.3)-(1.4). Then there exists $C_2 > 0$ depending only on j such that*

$$\begin{aligned}
\Big| n(U_{n,j}(f;x)-f(x)) &+ f'(x)nU_{n,j}(xe_0-e_1;x) - \frac{1}{2}f''(x)nU_{n,j}((e_1-xe_0)^2;x) \Big| \\
&\le C_2\omega_\varphi^1\left(f'';\frac{1}{\sqrt{n}}\right)
\end{aligned} \tag{2.4}$$

for all $x \in [0,1]$, $f \in C^2[0,1]$ and $n \ge j \ge 2$. Furthermore

$$0 \le \liminf_{n\to\infty} nU_{n,j}(xe_0-e_1;x) \le \limsup_{n\to\infty} nU_{n,j}(xe_0-e_1;x) \le j-1 \tag{2.5}$$

and

$$\begin{aligned}
0 &\le \liminf_{n\to\infty} nU_{n,j}((e_1-xe_0)^2;x) \\
&\le \limsup_{n\to\infty} nU_{n,j}((e_1-xe_0)^2;x) \le \frac{1}{2}((j-1)^2+1).
\end{aligned} \tag{2.6}$$

Proof. Because $U_{n,j}(f;0) = f(0)$ and $U_{n,j}(f;1) = f(1)$, the estimate (2.4) is satisfied for $x \in \{0,1\}$. Now let $x \in (0,1)$ and $t \in [0,1]$. By Taylor's formula, we have

$$f(t) = f(x) + f'(x)(t-x) + \frac{1}{2}f''(x)(t-x)^2 + \int_x^t (f''(u)-f''(x))(t-u)\,du.$$

Hence

$$\left| U_{n,j}(f;x) - f(x) + f'(x)U_{n,j}(xe_0 - e_1;x) - \frac{1}{2}f''(x)U_{n,j}((e_1 - xe_0)^2;x) \right|$$
$$= \left| U_{n,j}\left(\int_x^t (f''(u) - f''(x))(t-u)\,du; x\right) \right|$$
$$\leq U_{n,j}\left(\left| \int_x^t |f''(u) - f''(x)||t-u|\,du \right|; x\right). \tag{2.7}$$

On the other hand

$$\left| \int_x^u \frac{dv}{\varphi(v)} \right| \leq \varphi^{-1}(x)|u-x|^{1/2} \left| \int_x^u \frac{dv}{|u-v|^{1/2}} \right| \leq 2\varphi^{-1}(x)|u-x|, \quad x \in (0,1), u \in [0,1]$$

(cf. [2, Lemma 9.6.1]). Hence, for all $g \in W(\varphi)$, we have

$$\begin{aligned}
&\left| \int_x^t |f''(u) - f''(x)||t-u|\,du \right| \\
&\leq \left| \int_x^t |f''(u) - g(u)||t-u|\,du \right| + \left| \int_x^t |g(u) - g(x)||t-u|\,du \right| \\
&\quad + \left| \int_x^t |g(x) - f''(x)||t-u|\,du \right| \\
&\leq \frac{1}{2}(t-x)^2\|f''-g\| + \left| \int_x^t \left| \int_x^u g'(v)\,dv \right| |t-u|\,du \right| + \frac{1}{2}(t-x)^2\|f''-g\| \\
&\leq (t-x)^2\|f''-g\| + \|\varphi g'\| \left| \int_x^t \left| \int_x^u \frac{dv}{\varphi(v)} \right| |t-u|\,du \right| \\
&\leq (t-x)^2\|f''-g\| + 2\varphi^{-1}(x)\|\varphi g'\| \left| \int_x^t |u-x||t-u|\,du \right| \\
&\leq (t-x)^2\|f''-g\| + 2\varphi^{-1}(x)|t-x|^3\|\varphi g'\|. \qquad (2.8)
\end{aligned}$$

Combining (2.7), (2.8), Hölder's inequality and Lemma 2.2, we get

$$\begin{aligned}
&\left| U_{n,j}(f;x) - f(x) + f'(x)U_{n,j}(xe_0 - e_1;x) - \frac{1}{2}f''(x)U_{n,j}((e_1 - xe_0)^2;x) \right| \\
&\leq \|f''-g\|U_{n,j}((e_1 - xe_0)^2;x) + 2\varphi^{-1}(x)\|\varphi g'\|U_{n,j}(|e_1 - xe_0|^3;x) \\
&\leq \|f''-g\|U_{n,j}((e_1 - xe_0)^2;x) \\
&\quad + 2\varphi^{-1}(x)\|\varphi g'\| \left(U_{n,j}((e_1 - xe_0)^2;x)\right)^{1/2} \left(U_{n,j}((e_1 - xe_0)^4;x)\right)^{1/2} \\
&\leq \frac{2}{n}((j-1)^2+1)\varphi^2(x)\|f''-g\| \\
&\quad + 2\varphi^{-1}(x)\|\varphi g'\| \sqrt{\frac{2}{n}((j-1)^2+1)}\varphi(x)\frac{2\sqrt{2}}{n}\sqrt{(j-1)^4+1} \\
&\leq \frac{8}{n}\sqrt{(j-1)^2+1}\sqrt{(j-1)^4+1}\left(\|f''-g\| + \frac{1}{\sqrt{n}}\|\varphi g'\|\right).
\end{aligned}$$

Taking the infimum on the right hand side over all $g \in W(\varphi)$, we find

$$\left| n(U_{n,j}(f;x) - f(x)) + f'(x) n U_{n,j}(xe_0 - e_1;x) - \frac{1}{2} f''(x) n U_{n,j}((e_1 - xe_0)^2;x) \right|$$
$$\leq 8\sqrt{(j-1)^2+1}\sqrt{(j-1)^4+1} K_{1,\varphi}\left(f; \frac{1}{\sqrt{n}}\right).$$

Hence, by (1.6), we obtain the estimation (2.4).
Finally, the estimations (2.5) follow from Lemma 2.2, a). Again, due to Lemma 2.2, b), we obtain

$$nU_{n,j}((e_1 - xe_0)^2;x) \leq 2((j-1)^2+1)\varphi^2(x) \leq \frac{1}{2}((j-1)^2+1).$$

Hence we find the estimations (2.6), which completes the proof of the theorem. □

Corollary 2.1. *There exists $C_3 > 0$ such that*

$$\left| n(U_{n,2}(f;x) - f(x)) + (f'(x) - xf''(x)) n U_{n,2}(xe_0 - e_1;x) \right| \leq C_3 \omega_\varphi^1\left(f''; \frac{1}{\sqrt{n}}\right)$$

for all $x \in [0,1]$, $f \in C^2[0,1]$ and $n \geq 2$. Furthermore

$$0 \leq \liminf_{n\to\infty} nU_{n,2}(xe_0 - e_1;x) \leq \limsup_{n\to\infty} nU_{n,2}(xe_0 - e_1;x) \leq 1.$$

Proof. It follows immediately from Theorem 2.1, taking into account that $U_{n,2}(e_0;x) = 1$, $U_{n,2}(e_2;x) = x^2$ and

$$\begin{aligned} U_{n,2}((e_1 - xe_0)^2;x) &= U_{n,2}(e_2;x) - 2xU_{n,2}(e_1;x) + x^2 U_{n,2}(e_0;x) \\ &= 2x(x - U_{n,2}(e_1;x)) = 2xU_{n,2}(xe_0 - e_1;x). \end{aligned}$$

□

References

[1] J. M. Aldaz, O. Kounchev and H. Render: *Shape preserving properties of generalized Bernstein operators on extended Chebyshev spaces*. Numer. Math. **114** (2009), 1–25.
[2] Z. Ditzian and V. Totik: *Moduli of Smoothness*. Springer, New York, 1987.
[3] Z. Finta: *On generalized Voronovskaja theorem for Bernstein polynomials*. Carpathian J. Math. **28** (2012), 231–238.
[4] M. S. Floater: *On the convergence of derivatives of Bernstein approximation*. J. Approx. Theory. **134** (2005), 130–135.
[5] H. Gonska and I. Raşa: *Asymptotic behavior of differentiated Bernstein polynomials*. Math. Vesnik. **61** (2009), 53–60.
[6] H. Gonska and G. Tachev: *A quantitative variant of Voronovskaja's theorem*. Result. Math. **53** (2009), 287–294.
[7] H. Gonska, M. Heilmann and I. Raşa: *Asymptotic behavior of differentiated Bernstein polynomials revisited*. General Math. **18** (2010), 45–53.
[8] J. P. King: *Positive linear operators which preserve x^2*. Acta Math. Hungar. **99** (2003), 203–208.

Babeş-Bolyai University
Department of Mathematics
Str M. Kogălniceanu No 1, 400084, Cluj-Napoca, Romania
Email address: fzoltan@math.ubbcluj.ro