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SEMI-PARALLEL MERIDIAN SURFACES IN $\mathbb{E}^4$

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Abstract. In the present article we study a special class of surfaces in the four-dimensional Euclidean space, which are one-parameter systems of meridians of the standard rotational hypersurface. They are called meridian surfaces. We classify semi-parallel meridian surfaces in 4-dimensional Euclidean space $\mathbb{E}^4$.

1. Introduction

Let M be a submanifold of a n-dimensional Euclidean space $\mathbb{E}^n$. Denote by $\overline{R}$ the curvature tensor of the Vander Waerden-Bortoletti connection $\overline{\nabla}$ of M and by h the second fundamental form of M in $\mathbb{E}^n$. The submanifold M is called semi-parallel (or semi-symmetric [15]) if $\overline{R}\cdot h = 0$ [6]. This notion is an extrinsic analogue for semi-symmetric spaces, i.e. Riemannian manifolds for which $R \cdot R = 0$ and a direct generalization of parallel submanifolds, i.e. submanifolds for which $\overline{\nabla}h = 0$. In [6] J. Deprez showed the fact that the submanifold $M \subset \mathbb{E}^n$ is semi-parallel implies that (M, g) is semi-symmetric. For references on semi-symmetric spaces, see [18]; for references on parallel immersions, see [8]. In [6] J. Deprez gave a local classification of semi-parallel hypersurfaces in Euclidean n-space $\mathbb{E}^n$.

Recently, the present authors considered the Wintgen ideal surfaces in Euclidean n-space $\mathbb{E}^n$. They showed that Wintgen ideal surfaces in $\mathbb{E}^n$ satisfying the semi-parallelity condition

$$\overline{R}(X,Y)\cdot h = 0 \tag{1.1}$$

are of flat normal connection [1]. Further, the same authors in [2] proved that the tensor product surfaces in $\mathbb{E}^4$ satisfying the semi-parallelity condition (1.1) are totally umbilical.

In [13] Ganchev and Milousheva constructed special two dimensional surfaces which are one-parameter of meridians of the rotation hypersurfaces in $\mathbb{E}^4$ and called these surfaces *meridian surfaces*. The geometric construction of the meridian surfaces is different from the construction of the standard rotational surfaces with two dimensional axis in $\mathbb{E}^4$ [9]. The same authors classified the meridian surfaces with constant Gauss curvature ($K \neq 0$) and constant mean curvature H [13]. Recently,

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meridian surfaces with 1-type Gauss map were characterized by the present authors and Milousheva in [3]. Further, meridian surfaces were studied in [10] as surfaces in Minkowski 4-space. For more details see also [11], [12] and [17].

In the present study we consider the meridian surfaces in 4-dimensional Euclidean space $\mathbb{E}^4$. We give a classification of this surfaces satisfying the semi-parallelity condition (1.1).

2. Basic Concepts

Let M be a smooth surface in n-dimensional Euclidean space $\mathbb{E}^n$ given with the surface patch $X(u,v) : (u,v) \in D \subset \mathbb{E}^2$. The tangent space to M at an arbitrary point $p = X(u,v)$ of M span $\{X_u, X_v\}$. In the chart (u,v) the coefficients of the first fundamental form of M are given by

$$E = \langle X_u, X_u \rangle, F = \langle X_u, X_v \rangle, G = \langle X_v, X_v \rangle, \tag{2.1}$$

where $\langle , \rangle$ is the Euclidean inner product. We assume that $W^2 = EG - F^2 \neq 0$, i.e. the surface patch $X(u,v)$ is regular. For each $p \in M$, consider the decomposition $T_p\mathbb{E}^n = T_pM \oplus T_p^{\perp}M$ where $T_p^{\perp}M$ is the orthogonal component of the tangent plane T_pM in $\mathbb{E}^n$, that is the normal space of M at p.

Let $\chi(M)$ and $\chi^{\perp}(M)$ be the space of the smooth vector fields tangent and normal to M respectively. Denote by ∇ and $\widetilde{\nabla}$ the Levi-Civita connections on M and $\mathbb{E}^n$, respectively. Given any vector fields X_i and X_j tangent to M consider the second fundamental map $h : \chi(M) \times \chi(M) \to \chi^{\perp}(M)$;

$$h(X_i, X_j) = \widetilde{\nabla}_{X_i} X_j - \nabla_{X_i} X_j; \ 1 \leq i,j \leq 2. \tag{2.2}$$

For any normal vector field N_α, $1 \leq \alpha \leq n-2$, of M, recall the shape operator $A : \chi^{\perp}(M) \times \chi(M) \to \chi(M)$;

$$A_{N_\alpha} X_i = -\widetilde{\nabla}_{N_\alpha} X_i + D_{X_i} N_\alpha; \quad 1 \leq i \leq 2. \tag{2.3}$$

where D denotes the normal connection of M in $\mathbb{E}^n$ [4]. This operator is bilinear, self-adjoint and satisfies the following equation:

$$\langle A_{N_\alpha} X_i, X_j \rangle = \langle h(X_i, X_j), N_\alpha \rangle, \ 1 \leq i,j \leq 2. \tag{2.4}$$

The equation (2.2) is called Gaussian formula, and

$$h(X_i, X_j) = \sum_{\alpha=1}^{n-2} h_{ij}^{\alpha} N_\alpha, \qquad 1 \leq i,j \leq 2 \tag{2.5}$$

where h_{ij}^{α} are the coefficients of the second fundamental form h [4]. If $h = 0$ then M is called totally geodesic. M is totally umbilical if all shape operators are proportional to the identity map. M is an isotropic surface if for each p in M, $\|h(X,X)\|$ is independent of the choice of a unit vector X in T_pM.

If we define a covariant differentiation $\overline{\nabla} h$ of the second fundamental form h on the direct sum of the tangent bundle and normal bundle $TM \oplus T^{\perp}M$ of M by

$$(\overline{\nabla}_{X_i} h)(X_j, X_k) = D_{X_i} h(X_j, X_k) - h(\nabla_{X_i} X_j, X_k) - h(X_j, \nabla_{X_i} X_k), \tag{2.6}$$

for any vector fields X_i,X_j, X_k tangent to M, then we have the Codazzi equation

$$(\overline{\nabla}_{X_i} h)(X_j, X_k) = (\overline{\nabla}_{X_j} h)(X_i, X_k), \tag{2.7}$$

where $\overline{\nabla}$ is called the Vander Waerden-Bortoletti connection of M [4].

We denote by R and $R^{\perp}$ the curvature tensors associated with ∇ and D respectively;

$$R(X_i, X_j)X_k = \nabla_{X_i}\nabla_{X_j}X_k - \nabla_{X_j}\nabla_{X_i}X_k - \nabla_{[X_i,X_j]}X_k, \tag{2.8}$$

$$R^{\perp}(X_i, X_j)N_\alpha = h(X_i, A_{N_\alpha}X_j) - h(X_j, A_{N_\alpha}X_i). \tag{2.9}$$

The equations of Gauss and Ricci are given respectively by

$$\langle R(X_i, X_j)X_k, X_l\rangle = \langle h(X_i, X_l), h(X_j, X_k)\rangle - \langle h(X_i, X_k), h(X_j, X_l)\rangle\,, \tag{2.10}$$

$$\left\langle R^{\perp}(X_i, X_j)N_\alpha, N_\beta\right\rangle = \left\langle [A_{N_\alpha}, A_{N_\beta}]X_i, X_j\right\rangle, \tag{2.11}$$

for the vector fields X_i, X_j, X_k tangent to M and N_α, N_β normal to M [4].

Let us $X_i \wedge X_j$ denote the endomorphism $X_k \longrightarrow \langle X_j, X_k\rangle X_i - \langle X_i, X_k\rangle X_j$. Then the curvature tensor R of M is given by the equation

$$R(X_i, X_j)X_k = \sum_{\alpha=1}^{n-2} (A_{N_\alpha}X_i \wedge A_{N_\alpha}X_j)\, X_k. \tag{2.12}$$

It is easy to show that

$$R(X_i, X_j)X_k = K\,(X_i \wedge X_j)\, X_k, \tag{2.13}$$

where K is the Gaussian curvature of M defined by

$$K = \langle h(X_1, X_1), h(X_2, X_2)\rangle - \|h(X_1, X_2)\|^2 \tag{2.14}$$

(see [14]).

The normal curvature K_N of M is defined by (see [5])

$$K_N = \left\{ \sum_{1=\alpha<\beta}^{n-2} \left\langle R^{\perp}(X_1, X_2)N_\alpha, N_\beta\right\rangle^2 \right\}^{1/2}. \tag{2.15}$$

We observe that the normal connection D of M is flat if and only if $K_N = 0$, and by a result of Cartan, this is equivalent to the diagonalisability of all shape operators A_{N_α} of M, which means that M is a totally umbilical surface in $\mathbb{E}^n$.

3. Semi-parallel Surfaces

Let M be a smooth surface in n-dimensional Euclidean space $\mathbb{E}^n$. Let $\overline{\nabla}$ be the connection of Vander Waerden-Bortoletti of M. The product tensor $\overline{R} \cdot h$ of the curvature tensor $\overline{R}$ with the second fundamental form h is defined by

$$\begin{aligned}(\overline{R}(X_i, X_j)\cdot h)(X_k, X_l) &= \overline{\nabla}_{X_i}(\overline{\nabla}_{X_j}h(X_k, X_l)) - \overline{\nabla}_{X_j}(\overline{\nabla}_{X_i}h(X_k, X_l)) \\ &\quad -\overline{\nabla}_{[X_i,X_j]}h(X_k, X_l),\end{aligned}$$

for all X_i, X_j, X_k, X_l tangent to M.

The surface M is said to be semi-parallel if $\overline{R}\cdot h = 0$, i.e. $\overline{R}(X_i, X_j)\cdot h = 0$ ([15], [6], [7], [16]). It is easy to see that

$$\begin{aligned}(\overline{R}(X_i, X_j)\cdot h)(X_k, X_l) &= R^{\perp}(X_i, X_j)h(X_k, X_l) \\ &\quad \text{-}h(R(X_i, X_j)X_k, X_l)\text{-}h(X_k, R(X_i, X_j)X_l).\end{aligned} \tag{3.1}$$

This notion is an extrinsic analogue for semi-symmetric spaces, i.e. Riemannian manifolds for which $R\cdot R = 0$ and a generalization of parallel surfaces, i.e. $\overline{\nabla}h = 0$ [8].

Substituting (2.5) and (2.4) into (2.9) we get

$$R^{\perp}(X_1,X_2)N_{\alpha}=h_{12}^{\alpha}(h(X_1,X_1)-h(X_2,X_2)+(h_{22}^{\alpha}-h_{11}^{\alpha})h(X_1,X_2). \tag{3.2}$$

Further, by the use of (2.13) we get

$$R(X_1,X_2)X_1=-KX_2, R(X_1,X_2)X_2=KX_1. \tag{3.3}$$

So, substituting (3.2) and (3.3) into (3.1) we obtain

$$\begin{aligned}(\overline{R}(X_1,X_2)\cdot h)(X_1,X_1) &= \left(\sum_{\alpha=1}^{n-2} h_{11}^{\alpha}(h_{22}^{\alpha}-h_{11}^{\alpha})+2K\right)h(X_1,X_2)\\ &\quad+\sum_{\alpha=1}^{n-2} h_{11}^{\alpha}h_{12}^{\alpha}(h(X_1,X_1)-h(X_2,X_2)),\\ (\overline{R}(X_1,X_2)\cdot h)(X_1,X_2) &= \left(\sum_{\alpha=1}^{n-2} h_{12}^{\alpha}(h_{22}^{\alpha}-h_{11}^{\alpha})\right)h(X_1,X_2)\\ &\quad+(\sum_{\alpha=1}^{n-2} h_{12}^{\alpha}h_{12}^{\alpha}\text{-}K)(h(X_1,X_1)\text{-}h(X_2,X_2)),\\ (\overline{R}(X_1,X_2)\cdot h)(X_2,X_2) &= \left(\sum_{\alpha=1}^{n-2} h_{22}^{\alpha}(h_{22}^{\alpha}-h_{11}^{\alpha})-2K\right)h(X_1,X_2)\\ &\quad+\sum_{\alpha=1}^{n-2} h_{22}^{\alpha}h_{12}^{\alpha}(h(X_1,X_1)-h(X_2,X_2)).\end{aligned} \tag{3.4}$$

Semi-parallel surfaces in $\mathbb{E}^n$ are classified by J. Deprez [6]:

Theorem 3.1. [6] *Let M a surface in n-dimensional Euclidean space $\mathbb{E}^n$. Then M is semi-parallel if and only if locally;*

i) M is equivalent to a 2-sphere, or

ii) M has trivial normal connection, or

iii) M is an isotropic surface in $\mathbb{E}^5\subset\mathbb{E}^n$ satisfying $\|H\|^2=3K$.

4. Meridian Surfaces in $\mathbb{E}^4$

In the following sections, we will consider the meridian surfaces in $\mathbb{E}^4$ which were first defined by Ganchev and Milousheva [9]. The meridian surfaces are one-parameter systems of meridians of the standard rotational hypersurface in $\mathbb{E}^4$.

Let $\{e_1,e_2,e_3,e_4\}$ be the standard orthonormal frame in $\mathbb{E}^4$, and $S^2(1)$ be a 2-dimensional sphere in $\mathbb{E}^3=span\{e_1,e_2,e_3\}$, centered at the origin O. We consider a smooth curve $C:r=r(v)$, $v\in J$, $J\subset\mathbb{R}$ on $S^2(1)$, parameterized by the arc-length ($\|(r')^2(v)\|=1$). We denote $t=r'$ and consider the moving frame field $\{t(v),n(v),r(v)\}$ of the curve C on $S^2(1)$. With respect to this orthonormal frame field the following Frenet formulas hold good:

$$\begin{aligned}r'&=t;\\ t'&=\kappa n-r;\\ n'&=-\kappa t,\end{aligned} \tag{4.1}$$

where κ is the spherical curvature of C.

Let $f = f(u),\ g = g(u)$ be smooth functions, defined in an interval $I \subset \mathbb{R}$, such that

$$(f')^2(u) + (g')^2(u) = 1,\ u \in I. \tag{4.2}$$

In [9] Ganchev and Milousheva constructed a surface M^2 in $\mathbb{E}^4$ in the following way:

$$M^2 : X(u,v) = f(u)\, r(v) + g(u)\, e_4, \quad u \in I,\, v \in J. \tag{4.3}$$

The surface M^2 lies on the rotational hypersurface M^3 in $\mathbb{E}^4$ obtained by the rotation of the meridian curve $\alpha : u \to (f(u), g(u))$ around the Oe_4-axis in $\mathbb{E}^4$. Since M^2 consists of meridians of M^3, we call M^2 a *meridian surface* [9]. We denote by κ_α the curvature of meridian curve α, i.e.,

$$\kappa_\alpha = f'(u)g''(u) - f''(u)g(u) = \frac{-f''(u)}{\sqrt{1 - f'^2(u)}}. \tag{4.4}$$

We consider the following orthonormal moving frame fields, X_1, X_2, N_1, N_2 on the meridian surface M^2 such that X_1, X_2 are tangent to M^2 and N_1, N_2 are normal to M^2. The tangent space of M^2 is spanned by the vector fields:

$$\begin{aligned} &X_1 = \tfrac{\partial X}{\partial u}, \quad X_2 = \tfrac{1}{f}\tfrac{\partial X}{\partial v}, \\ &N_1 = n(v), \quad N_2 = -g'(u)\, r(v) + f'(u)\, e_4. \end{aligned} \tag{4.5}$$

By a direct computation we have the components of the second fundamental forms as;

$$\begin{aligned} &h^1_{11} = h^1_{12} = h^1_{21} = 0, \quad h^1_{22} = \tfrac{\kappa}{f}, \\ &h^2_{11} = \kappa_\alpha \quad h^2_{12} = h^2_{21} = 0, \quad h^2_{22} = \tfrac{g'}{f}. \end{aligned} \tag{4.6}$$

Therefore the shape operator matrices of M^2 are of the form

$$A_{N_1} = \begin{bmatrix} 0 & 0 \\ 0 & \frac{\kappa}{f} \end{bmatrix},\ A_{N_2} = \begin{bmatrix} \kappa_\alpha & 0 \\ 0 & \frac{g'}{f} \end{bmatrix} \tag{4.7}$$

and hence we have

$$\begin{aligned} K &= \tfrac{\kappa_\alpha g'}{f}, \\ K_N &= 0, \end{aligned} \tag{4.8}$$

which implies that the meridian surface M^2 is totally umbilical surface in $\mathbb{E}^4$.

In [13] Ganchev and Milousheva constructed three main classes of meridian surfaces:

I. $\kappa = 0$; i.e. the curve C is a great circle on $S^2(1)$. In this case $N_1 = const.$ and M^2 is a planar surface lying in the constant 3-dimensional space spanned by $\{X_1, X_2, N_2\}$. Particularly, if in addition $\kappa_\alpha = 0$, i.e. the meridian curve is a part of a straight line, then M^2 is a developable surface in the 3-dimensional space spanned by $\{X_1, X_2, N_2\}$.

II. $\kappa_\alpha = 0$, i.e. the meridian curve is a part of a straight line. In such a case M^2 is a developable ruled surface. If in addition $\kappa = const.$, i.e. C is a circle on $S^2(1)$, then M^2 is a developable ruled surface in a 3-dimensional space. If $\kappa \neq const.$,i.e. C is not a circle on $S^2(1)$, then M^2 is a developable ruled surface in $\mathbb{E}^4$.

III. $\kappa_\alpha \kappa \neq 0$, i.e. C is not a circle on $S^2(1)$ and α is not a straight line. In this general case the parametric lines of M^2 given by (4.3) are orthogonal and asymptotic.

We prove the following Theorem.

Theorem 4.1. *Let M^2 be a meridian surface in $\mathbb{E}^4$ given with the parametrization (4.3). Then M^2 is semi-parallel if and only if one of the following holds:*

i) M^2 is a developable ruled surface in $\mathbb{E}^3$ or $\mathbb{E}^4$,

ii) the curve C is a circle on $S^2(1)$ with non-zero constant spherical curvature and the meridian curve is determined by

$$f(u) = \pm\sqrt{u^2 - 2au + 2b};\ g(u) = -\sqrt{2b - a^2}\ln\left(u - a - \sqrt{u^2 - 2au + 2b}\right),$$

where $a = const, b = const$. In this case M^2 is a planar surface lying in 3-dimensional space spanned by $\{X_1, X_2, N_2\}$.

Proof. Let M^2 be a meridian surface in $\mathbb{E}^4$ given with the parametrization (4.3). Then by the use of (2.5) with (4.6) we see that

$$\begin{aligned} h(X_1, X_2) &= 0, \\ h(X_1, X_1) - h(X_2, X_2) &= -\frac{\kappa}{f} N_1 + \left(\kappa_\alpha - \frac{g'}{f}\right) N_2. \end{aligned} \tag{4.9}$$

Further, substituting (4.9) and (4.6) into (3.4) and after some computation one can get

$$\begin{aligned} (\overline{R}(X_1, X_2) \cdot h)(X_1, X_1) &= 0, \\ (\overline{R}(X_1, X_2) \cdot h)(X_1, X_2) &= -K\left(-\frac{\kappa}{f} N_1 + \left(\kappa_\alpha - \frac{g'}{f}\right) N_2\right), \\ (\overline{R}(X_1, X_2) \cdot h)(X_2, X_2) &= 0. \end{aligned}$$

Suppose that M^2 is semi-parallel. Then by definition

$$(\overline{R}(X_1, X_2) \cdot h)(X_i, X_j) = 0,\ 1 \leq i, j \leq 2,$$

is satisfied. So, we get

$$K\left(-\frac{\kappa}{f} N_1 + \left(\kappa_\alpha - \frac{g'}{f}\right) N_2\right) = 0.$$

Hence, two possible cases occur: $K = 0$ or $\kappa = 0$ and $\kappa_\alpha - \frac{g'}{f} = 0$. For the first case $\kappa_\alpha = 0$, i.e. the meridian curve is a part of a straight line. In such a case M^2 is a developable ruled surface given in the Case II. For the second case $\kappa = 0$ means that the curve c is a great circle on $S^2(1)$. In this case M^2 lies in the 3-dimensional space spanned by $\{X_1, X_2, N_2\}$. Further, using (4.4) the equation $\kappa_\alpha - \frac{g'}{f} = 0$ can be rewritten in the form

$$f(u)f''(u) - (f'(u))^2 + 1 = 0,$$

which has the solution

$$f(u) = \pm\sqrt{u^2 - 2au + 2b}. \tag{4.10}$$

Consequently, by substituting (4.10) into (4.2) one can get

$$g(u) = -\sqrt{2b - a^2}\ln\left(u - a - \sqrt{u^2 - 2au + 2b}\right).$$

This completes the proof of the theorem. □

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