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ON RANDERS CHANGE OF m-TH ROOT FINSLER METRICS

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Abstract. In this paper, we consider Randers change of m-th root Finsler metrics. We find necessary and sufficient condition under which a Randers change of an m-th root metric be locally dually flat. Then we prove that the Rander change of an m-th root Finsler metric is locally projectively flat if and only if it is locally Minkowskian.

1. Introduction

A change of Finsler metric $F \to \bar{F}$ is called a Randers change of F, if

$$\bar{F}(x,y) = F(x,y) + \beta(x,y), \tag{1.1}$$

where $\beta(x,y) = b_i(x)y^i$ is a 1-form on a smooth manifold M. It is easy to see that, if $\sup_{F(x,y)=1} |b_i(x)y^i| < 1$, then $\bar{F}$ is again a Finsler metric. Hashiguchi-Ichijyō showed that if β is closed, then $\bar{F}$ is pointwise projective to F. The notion of a Randers change has been proposed by Matsumoto, named by Hashiguchi-Ichijyō and studied in detail by Shibata [7][9][12]. If F reduces to a Riemannian metric then $\bar{F}$ reduces to a Randers metric. Due to this reason the transformation (1.1) has been called the Randers change of Finsler metric. For other Finslerian transformations see [12][17].

The Randers change is projective if and only if $b_i(x)$ is locally a gradient vector field. According to Hashiguchi-Ichjyo, a Randers change is projective, if and only if $b_{i|j} = b_{j|i}$, that is $b_i(x)$ is locally a gradient vector field and symbols "|" mean the covariant derivatives in F with respect to Berwald connection [7]. It is remarkable that, if F is absolutely homogeneous then the necessary and sufficient condition for $\bar{F}$ to have reversible geodesics is that β is closed and it is a first integral of the geodesic flow of $\bar{F}$ [6]. Consider the Randers metric $F = \alpha + \beta$, where $\alpha = \sqrt{a_{ij}(x)y^iy^j}$ and $||\beta|| := |a_{ij}b^ib^j| < 1$. If β is a closed 1-form, then F has reversible geodesics and if it is parallel with respect to α (i.e., $b_{i|j} = 0$) then F has strictly reversible geodesics.

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In [2], Amari-Nagaoka introduced the notion of dually flat Riemannian metrics when they study the information geometry on Riemannian manifolds. Information geometry has emerged from investigating the geometrical structure of a family of probability distributions and has been applied successfully to various areas including statistical inference, control system theory and multi-terminal information theory [1]. In Finsler geometry, Shen extends the notion of locally dually flatness for Finsler metrics [11]. A Finsler metric F on a manifold M is said to be locally dually flat if at any point there is a coordinate system (x^i) in which the spray coefficients are in the following form $G^i = -\frac{1}{2}g^{ij}H_{y^j}$ where $H = H(x, y)$ is a C^∞ homogeneous scalar function on TM_0. Such a coordinate system is called an adapted coordinate system [14]. Indeed, a Finsler metric F on an open subset $U \subset \mathbb{R}^n$ is called dually flat if it satisfies

$$\frac{\partial^2 F^2}{\partial x^k \partial y^l} y^k = 2\frac{\partial F^2}{\partial x^l}.$$

Let (M, F) be a Finsler manifold of dimension n, TM its tangent bundle and (x^i, y^i) the coordinates in a local chart on TM. Let F be the following function on M, by $F = \sqrt[m]{A}$, where A is given by $A := a_{i_1 \dots i_m}(x) y^{i_1} y^{i_2} \dots y^{i_m}$ with $a_{i_1 \dots i_m}$ symmetric in all its indices (for example see [3][4][5][10][13][14][15][16]). Then F is called an m-th root Finsler metric. Suppose that A_{ij} define a positive definite tensor and A^{ij} denotes its inverse. For an m-th root metric F, put

$$A_i = \frac{\partial A}{\partial y^i}, \quad A_{ij} = \frac{\partial^2 A}{\partial y^j \partial y^j}, \quad A_{x^i} = \frac{\partial A}{\partial x^i}, \quad A_0 = A_{x^i} y^i.$$

In this paper, we consider Randers change of an m-th root Finsler metric and find necessary and sufficient condition under which a Randers change of an m-th root metric be locally dually flat. More precisely, we prove the following.

Theorem 1.1. *Let $F = \sqrt[m]{A}$ be an m-th root Finsler metric on an open subset $U \subset \mathbb{R}^n$, where A is irreducible. Suppose that $\bar{F} = F + \beta$ be Randers change of F where $\beta = b_i(x) y^i$. Then $\bar{F}$ is locally dually flat if and only if there exists a 1-form $\theta = \theta_l(x) y^l$ on U such that the following hold*

$$\beta_{0l}\beta + \beta_l\beta_0 = 2\beta\beta_{x^l}, \tag{1.2}$$

$$A_{x^l} = \frac{1}{3m}[mA\theta_l + 2\theta A_l], \tag{1.3}$$

$$(\frac{1}{m} - 1)A_l A^{-1} A_0 \beta + (A_0\beta)_l - 3A_{x^l}\beta + A_l\beta_0 = A(2\beta_{x^l} - \beta_{0l}), \tag{1.4}$$

where $\beta_{0l} = \beta_{x^k y^l} y^k$, $\beta_{x^l} = (b_i)_{x^l} y^i$, $\beta_0 = \beta_{x^l} y^i$ and $\beta_{0l} = (b_l)_0$.

A Finsler metric is said to be locally projectively flat if at any point there is a local coordinate system in which the geodesics are straight lines as point sets. It is known that a Finsler metric $F(x, y)$ on an open domain $U \subset \mathbb{R}^n$ is locally projectively flat if and only if

$$G^i = P y^i,$$

where $P(x, \lambda y) = \lambda P(x, y)$, $\lambda > 0$ [8]. Projectively flat Finsler metrics on a convex domain in $\mathbb{R}^n$ are regular solutions to Hilbert's Fourth Problem: determine the metrics on an open subset in $\mathbb{R}^n$, whose geodesics are straight lines.

Theorem 1.2. *Let $F = \sqrt[m]{A}$ be an m-th root Finsler metric on an open subset $U \subset \mathbb{R}^n$, where A is irreducible. Suppose that $\bar{F} = F + \beta$ be Randers change of F where $\beta = b_i(x)y^i$. Then $\bar{F}$ is locally projectively flat if and only if it is locally Minkowskian.*

2. Proof of the Theorem 1.1

In this section, we will prove a generalized version of Theorem 1.1. Indeed we find necessary and sufficient condition under which a Randers change of an generalized m-th root metric be locally dually flat. Let F be a scalar function on TM defined by following

$$F = \sqrt{A^{2/m} + B},$$

where A and B are given by

$$A := a_{i_1 \dots i_m}(x) y^{i_1} \dots y^{i_m}, \quad B := b_{ij}(x) y^i y^j. \tag{2.1}$$

Then F is called generalized m-th root Finsler metric. Suppose that the matrix (A_{ij}) defines a positive definite tensor and (A^{ij}) denotes its inverse. Then the following hold

$$g_{ij} = \frac{A^{\frac{2}{m}-2}}{m^2}[mAA_{ij} + (2-m)A_iA_j] + b_{ij},$$
$$A_i = \frac{\partial A}{\partial y^i}, \quad A_{ij} = \frac{\partial^2 A}{\partial y^j \partial y^j}, \quad B_i = \frac{\partial B}{\partial y^i}, \quad B_{ij} = \frac{\partial^2 B}{\partial y^j \partial y^j},$$
$$A_{x^i} = \frac{\partial A}{\partial x^i}, \quad A_0 = A_{x^i} y^i, \quad B_{x^i} = \frac{\partial B}{\partial x^i}, \quad B_0 = B_{x^i} y^i.$$

Now, we are going to prove the following.

Theorem 2.1. *Let $F = \sqrt{A^{2/m} + B}$ be an generalized m-th root Finsler metric on an open subset $U \subset \mathbb{R}^n$, where A is irreducible. Suppose that $\bar{F} = F + \beta$ be Randers change of F where $\beta = b_i(x)y^i$. Then $\bar{F}$ is locally dually flat if and only if there exists a 1-form $\theta = \theta_l(x)y^l$ on U such that the following holds*

$$\beta_{0l}\beta + \beta_l\beta_0 + B_{0l} = 2\big[\beta\beta_{x^l} + B_{x^l}\big], \tag{2.2}$$

$$A_{x^l} = \frac{1}{3m}[mA\theta_l + 2\theta A_l], \tag{2.3}$$

$$\Upsilon_l \Upsilon_0 \beta = 2\Upsilon\Big[(\Upsilon_{0l}\beta + \Upsilon_0\beta_l + \Upsilon_l\beta_0 - 2\Upsilon_{x^l}\beta) + 2\Upsilon(\beta_{0l} - 2\beta_{x^l})\Big], \tag{2.4}$$

where $\beta_{0l} = \beta_{x^k y^l} y^k$, $\beta_{x^l} = (b_i)_{x^l} y^i$, $\beta_0 = (b_i)_0 y^i$, $\beta_{0l} = (b_l)_0$, $\Upsilon := A^{\frac{2}{m}} + B$ and

$$\Upsilon_p := \frac{2}{m} A^{\frac{2}{m}-1} A_p + B_p,$$
$$\Upsilon_{0p} := \frac{2}{m} A^{\frac{2}{m}-2} \big[(\frac{2}{m} - 1) A_p A_0 + A A_{0p}\big] + B_{0p}.$$

To prove Theorem 2.1, we need the following.

Lemma 2.1. *Suppose that the equation $\Phi A^{\frac{2}{m}-2} + \Psi A^{\frac{1}{m}-1} + \Theta = 0$ holds, where Φ, Ψ, Θ are polynomials in y and $m > 2$. Then $\Phi = \Psi = \Theta = 0$.*

Proof of Theorem 2.1: Let $\bar{F}$ be a locally dually flat metric. We have

$$\begin{aligned}\bar{F}^2 &= A^{\frac{2}{m}} + B + 2\beta(A^{\frac{2}{m}} + B)^{1/2} + \beta^2,\\ (\bar{F}^2)_{x^k} &= \frac{2}{m}A^{\frac{2}{m}-1}A_{x^k} + B_{x^k} + (A^{\frac{2}{m}} + B)^{-1/2}(\frac{2}{m}A^{\frac{2}{m}-1}A_{x^k} + B_{x^k})\beta\\ &+ \quad 2(A^{\frac{2}{m}} + B)^{1/2}\beta_{x^k} + 2\beta_{x^k}\beta.\end{aligned}$$

Then

$$\begin{aligned}[\bar{F}^2]_{x^k y^l}y^k = \quad & \frac{2}{m}A^{\frac{2}{m}-2}\big[(\frac{2}{m}-1)A_lA_0 + AA_{0l}\big] + 2(\beta_{0l}\beta + \beta_l\beta_0) + B_{0l}\\ & -\frac{1}{2}(A^{\frac{2}{m}} + B)^{-3/2}\Upsilon_l\Upsilon_0\beta + (A^{\frac{2}{m}} + B)^{-1/2}\Upsilon_0\beta_l\\ & + (A^{\frac{2}{m}} + B)^{-1/2}\Upsilon_{0l}\beta + (A^{\frac{2}{m}} + B)^{-1/2}\Upsilon_l\beta_0 + 2(A^{\frac{2}{m}} + B)^{1/2}\beta_{0l}.\end{aligned}$$

Thus, we get

$$\begin{aligned}&\frac{1}{m}A^{\frac{2}{m}-2}\Bigg[(\frac{2}{m}-1)A_lA_0 + AA_{0l} - 2AA_{x^k}\Bigg]\\ &+(A^{\frac{2}{m}} + B)^{-3/2}\Bigg[\frac{-1}{2}\Upsilon_l\Upsilon_0\beta + (A^{\frac{2}{m}} + B)(\Upsilon_{0l}\beta + \Upsilon_0\beta_l + \Upsilon_l\beta_0 - 2\Upsilon_{x^l}\beta)\\ &+2(A^{\frac{2}{m}} + B)^2(\beta_{0l} - 2\beta_{x^l})\Bigg] + 2(\beta_{0l}\beta + \beta_l\beta_0 - 2\beta_{x^l}\beta) + B_{0l} - 2B_{x^l} = 0.\end{aligned}$$

By Lemma 2.1, we have

$$(\frac{2}{m}-1)A_lA_0 + AA_{0l} = 2AA_{x^k}, \tag{2.5}$$

$$\frac{-1}{2}\Upsilon_l\Upsilon_0\beta + C[\Upsilon_{0l}\beta + \Upsilon_0\beta_l + \Upsilon_l\beta_0 - 2\Upsilon_{x^l}\beta] = 2C^2(2\beta_{x^l} - \beta_{0l}), \tag{2.6}$$

$$2(\beta_{0l}\beta + \beta_l\beta_0 - 2\beta_{x^l}\beta) = 2B_{x^l} - B_{0l}, \tag{2.7}$$

One can rewrite (2.5) as follows

$$A(2A_{x^l} - A_{0l}) = (\frac{2}{m}-1)A_lA_0. \tag{2.8}$$

Irreducibility of A and

$$deg(A_l) = m-1$$

imply that there exists a 1-form $\theta = \theta_l y^l$ on U such that

$$A_0 = \theta A. \tag{2.9}$$

Plugging (2.9) into (2.8), we get

$$A_{0l} = A\theta_l + \theta A_l - A_{x^l}. \tag{2.10}$$

Substituting (2.9) and (2.10) into (2.8) yields (2.3). The converse is a direct computation. This completes the proof. □

3. Proof of the Theorem 1.2

In this section, we will prove a generalized version of Theorem 1.2. Indeed we study the Randers change of an generalized m-th root metric

$$F=\sqrt{A^{\frac{2}{m}}+B},$$

where A and B are given by

$$A:=a_{i_1\cdots i_m}(x)y^{i_1}...y^{i_m},\quad B:=b_{ij}(x)y^iy^j$$

and A is irreducible. More precisely, we prove the following.

Theorem 3.1. *Let $F=\sqrt{A^{\frac{2}{m}}+B}$ be an generalized m-th root Finsler metric on an open subset $U\subset\mathbb{R}^n$, where A is irreducible. Suppose that $\bar{F}=F+\beta$ be Randers change of F where $\beta=b_i(x)y^i$. Then $\bar{F}$ is locally projectively flat if and only if it is locally Minkowskian.*

To prove Theorem 3.1, we need the following.

Lemma 3.1. *Let (M,F) be a Finsler manifold. Suppose that $\bar{F}=F+\beta$ be a Randers change of F. Then $\bar{F}$ is a projectively flat Finsler metric if and only if the following holds*

$$F_{0l}-F_{x^l}=(b_i)_{x^l}y^i-(b_l)_0. \tag{3.1}$$

In local coordinates (x^i,y^i), the vector filed

$$\mathbf{G}=y^i\frac{\partial}{\partial x^i}-2G^i\frac{\partial}{\partial y^i}$$

is a global vector field on TM_0, where $G^i=G^i(x,y)$ are local functions on TM_0 given by following

$$G^i:=\frac{1}{4}g^{il}\left\{\frac{\partial^2F^2}{\partial x^k\partial y^l}y^k-\frac{\partial F^2}{\partial x^l}\right\},\quad y\in T_xM.$$

A Finsler metric F is called a Berwald metric if

$$G^i=\frac{1}{2}\Gamma^i_{jk}(x)y^jy^k$$

is quadratic in $y\in T_xM$ for any $x\in M$. The projection of an integral curve of $\mathbf{G}$ is called a geodesic in M. In local coordinates, a curve $c(t)$ is a geodesic if and only if its coordinates $(c^i(t))$ satisfy $\ddot{c}^i+2G^i(\dot{c})=0$ [18].

Now, by using Lemma 3.1, we are going to prove the following.

Proposition 3.1. *Let $F=\sqrt{A^{\frac{2}{m}}+B}$ be an generalized m-th root Finsler metric on an open subset $U\subset\mathbb{R}^n$, where A is irreducible, $m>4$ and $B\neq 0$. Suppose that $\bar{F}=F+\beta$ be Randers change of F where $\beta=b_i(x)y^i$. In that case, if $\bar{F}$ is projectively flat metric then F reduces to a Berwald metric.*

Proof. By Lemma 3.1, we get

$$F_{x^l} = \frac{2A^{2/m}A_{x^l} + mAB_{x^l}}{2mA\sqrt{A^{\frac{2}{m}} + B}}$$

and

$$\begin{aligned} F_{x^k y^l} y^k = & -\frac{1}{4}(A^{\frac{2}{m}} + B)^{-1/2}\Big[(\frac{2A^{2/m}A_0}{mA} + B_0)(\frac{2A^{2/m}A_l}{mA} + B_l)(A^{\frac{2}{m}} + B)^{-1}\Big] \\ & +\frac{1}{2}(A^{\frac{2}{m}} + B)^{-1/2}\Big[(\frac{4A^{2/m}A_0A_l}{m^2A^2} + \frac{2A^{2/m}A_{0l}}{mA} - \frac{2A^{2/m}A_0A_l}{mA^2} + B_{0l})\Big]. \end{aligned}$$

By (3.1), we obtain the following

$$\Phi A^{\frac{2}{m}} + \Psi A^{\frac{4}{m}} + \Theta = 0,$$

where

$$\begin{aligned} \Phi &= -\frac{1}{2}mA\Big[A_0B_l + B_oA_l + 2B(A_{x^l} - A_{0l}) + mA(B_{x^l} - B_{0l})\Big] - (m-2)A_0A_lB, \\ \Psi &= mA(A_{0l} - A_{x^l}) - (m-1)A_0A_l, \\ \Theta &= \frac{1}{4}m^2A^2\Big[- 2B_{0l}B + B_0B_l + 2B_{x^l}B\Big] + (b_l)_0 - (b_i)_{x^l}y^i. \end{aligned}$$

By Lemma 2.1, we have

$$\Phi = 0, \tag{3.2}$$

$$\Psi = 0, \tag{3.3}$$

$$\Theta = 0. \tag{3.4}$$

By (3.3), we result that

$$mA(A_{0l} - A_{x^l}) = (m-1)A_0A_l. \tag{3.5}$$

Then irreducibility of A and $deg(A_l) = m-1 < deg(A)$ implies that A_0 is divisible by A. This means that, there is a 1-form $\theta = \theta_l y^l$ on U such that,

$$A_0 = 2mA\theta. \tag{3.6}$$

Substituting (3.6) into (3.5), yields

$$A_{0l} = A_{x^l} + 2(m-1)\theta A_l. \tag{3.7}$$

Plugging (3.6) and (3.7) into (3.2), we get

$$mA(2\theta B_l - B_{0l} + B_{x^l}) = A_l(4B\theta - B_0). \tag{3.8}$$

Clearly, the right side of (3.8) is divisible by A. Since A is irreducible, deg(A_l) and deg($2\theta B - \frac{1}{2}B$) are both less than deg(A), then we have have

$$B_0 = 4B\theta. \tag{3.9}$$

By (3.6) and (3.9), we get the spray coefficients $G^i = Py^i$ with $P = \theta$. Then F is a Berwald metric. □

Proof of Theorem 3.1: By Proposition 3.1, if F is projectively flat then it reduces to a Berwald metric. Now, if $m > 4$ then by Numata's Theorem every Berwald metric of non-zero scalar flag curvature $\mathbf{K}$ must be Riemaniann. This is contradicts with our assumption. Then $\mathbf{K} = 0$, and in this case F reduces to a locally Minkowskian metric. □

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