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ON THE TANGENT BUNDLE WITH DEFORMED SASAKI METRIC

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Abstract. In the present paper, we consider a deformation (in the horizontal bundle) of Sasaki metric on the tangent bundle TM over an $n-$dimensional Riemannian manifold (M, g). We firstly study some properties of deformed Sasaki metric which is pure with respect to some paracomplex structures on TM. Finally conditions for deformed Sasaki metric to be recurrent or pseudo symmetric are given.

1. Introduction

Let (M, g) be an $n-$dimensional Riemannian manifold and TM be its tangent bundle. Some pseudo-Riemannian and Riemannian metrics on the tangent bundles TM of a Riemannian manifold (M, g) were obtained by considering the natural lifts of the metric from the base manifold to TM (see, [1, 2, 3, 12, 14, 16, 17, 18, 19, 20, 21, 24, 30]). Among them, we quote the metric of Sasaki and the complete lift of the metric g. In [24], S. Sasaki constructed a Riemannian metric ${}^{S}g$ on the tangent bundle TM of a Riemannian manifold (M, g), which depends closely on the base metric g. Today this metric is a standard notion in differential geometry called the Sasaki metric. The Sasaki metric ${}^{S}g$ has been extensively studied by several authors and in many different contexts. In [31](see also [32, 33], B. V. Zayatuev introduced a deformation (in the horizontal bundle) of Sasaki metric on the tangent bundle TM given by

$$\begin{aligned} {}^{S}g_f\left({}^{H}X,{}^{H}Y\right) &= {}^{V}(fg\,(X,Y)), \\ {}^{S}g_f\left({}^{H}X,{}^{V}Y\right) &= {}^{S}g_f\left({}^{V}X,{}^{H}Y\right) = 0, \\ {}^{S}g_f\left({}^{V}X,{}^{V}Y\right) &= {}^{V}(g\,(X,Y)), \end{aligned} \tag{1.1}$$

where $f > 0$, $f \in C^{\infty}(M)$. For $f = 1$, it follows that ${}^{S}g_f = {}^{S}g$, i.e. the metric ${}^{S}g_f$ is a generalization of the Sasaki metric ${}^{S}g$. In [27], J. Wang and Y. Wang called this metric the rescaled Sasaki metric and studied geodesics and some curvature properties for the rescaled Sasaki metric. For the rescaled Sasaki type metric on

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the cotangent bundle, see [13]. In the present paper, we call this metric deformed Sasaki metric.

Our purpose is to give some results concerning the deformed Sasaki metric on TM. The present paper is organized as follows: In section 2, we review some introductory materials concerning with the tangent bundle TM over an n-dimensional Riemannian manifold M. In section 3, we investigate the paraholomorphy property of the deformed Sasaki metric by using some compatible paracomplex structures on TM and give some remarks concerning the twin Norden metric of ${}^{S}g_{f}$. Also we construct an almost product connection and give conditions for the almost product connection to be symmetric. Section 4 deals with curvature properties of the deformed Sasaki metric.

Throughout this paper, all manifolds, tensor fields and connections are always assumed to be differentiable of class C^{∞}. Also, we denote by $\Im_{q}^{p}(M)$ the set of all tensor fields of type (p,q) on M, and by $\Im_{q}^{p}(TM)$ the corresponding set on the tangent bundle TM. The Einstein summation convention is used, the range of the indices i,j,s being always $\{1,2,...,n\}$

2. Preliminaries

The tangent bundle of a smooth $n-$dimensional Riemannian manifold may be endowed with a structure of $2n-$dimensional smooth manifold, induced by the structure on the base manifold. If (M,g) is a smooth Riemannian manifold of dimension n, we denote its tangent bundle by $\pi : TM \to M$. A system of local coordinates $\left(U,x^{i}\right),\ i=1,...,n$ in M induces on TM a system of local coordinates $\left(\pi^{-1}\left(U\right),\ x^{i},\ x^{\bar{i}}=y^{i}\right),\ \bar{i}=n+i=n+1,...,2n$, where $x^{\bar{i}}=y^{i}$ is the components of vectors u in each tangent space $T_{x}M,\ x\in U$ with respect to the natural frame $\left\{\frac{\partial}{\partial x^{i}}\right\}$.

Let $X=X^{i}\frac{\partial}{\partial x^{i}}$ be the local expressions in U of a vector field X on M. Then the vertical lift ${}^{V}X$ and the horizontal lift ${}^{H}X$ of X are respectively given, with respect to the induced coordinates, by

$$ {}^{V}X = X^{i}\partial_{\bar{i}}, \tag{2.1} $$

and

$$ {}^{H}X = X^{i}\partial_{i} + y^{s}\Gamma_{sj}^{h}X^{j}\partial_{\overline{h}}, \tag{2.2} $$

where $\partial_{i}=\frac{\partial}{\partial x^{i}}$, $\partial_{\bar{i}}=\frac{\partial}{\partial x^{\bar{i}}}$ and Γ_{ij}^{h} are the coefficients of the Levi-Civita connection ∇ of g.

The Lie bracket operation of vertical and horizontal vector fields on TM is given by the formulas [10]

$$ \left\{ \begin{array}{l} \left[{}^{H}X,{}^{H}Y\right] = {}^{H}\left[X,Y\right] - {}^{V}\left(R(X,Y)u\right) \\ \left[{}^{H}X,{}^{V}Y\right] = {}^{V}\left(\nabla_{X}Y\right) \\ \left[{}^{V}X,{}^{V}Y\right] = 0 \end{array} \right. \tag{2.3} $$

for any $X,Y\ \Im_{0}^{1}(M)$, where R is the Riemannian curvature of g defined by $R\left(X,Y\right)=\left[\nabla_{X},\nabla_{Y}\right]-\nabla_{[X,Y]}$.

3. SOME RESULTS CONCERNING THE DEFORMED SASAKI METRIC

An almost paracomplex manifold is an almost product manifold (M_{2k}, φ), $\varphi^2 = id$, $\varphi \neq \pm id$ such that the two eigenbundles T^+M_{2k} and T^-M_{2k} associated to the two eigenvalues +1 and -1 of φ, respectively, have the same rank. An almost paracomplex Norden manifold (M_{2k}, φ, g) is a real $2k$-dimensional differentiable manifold M_{2k} with an almost paracomplex structure φ and a Riemannian metric g such that:

$$g(\varphi X, Y) = g(X, \varphi Y)$$

for all $X, Y \in \Im_0^1(M)$. A para-Kähler-Norden (paraholomorphic Norden) manifold can be defined as a triple (M_{2k}, φ, g) which consists of a smooth manifold M_{2k} endowed with an almost paracomplex structure φ and a Norden metric g such that $\nabla\varphi = 0$, where ∇ is the Levi-Civita connection of g. It is well known that the condition $\nabla\varphi = 0$ is equivalent to paraholomorphy of the Norden metric g [22], i.e. $\Phi_\varphi g = 0$, where Φ_φ is the Tachibana operator [25, 29]: $(\Phi_\varphi g)(X, Y, Z) = (\varphi X)(g(Y, Z)) - X(g(\varphi Y, Z)) + g((L_Y\varphi)X, Z) + g(Y, (L_Z\ \varphi)X)$. Also note that $G(Y, Z) = g(\varphi Y, Z)$ is the twin Norden metric.

Now, let us consider an almost paracomplex structure J on TM defined by

$$\begin{cases} J(^HX) = -^H X \\ J(^VX) = {}^VX \end{cases} \tag{3.1}$$

for all $X, Y \in \Im_0^1(M)$ [8]. We put

$$A\left(\widetilde{X}, \widetilde{Y}\right) = {}^Sg_f\left(J\widetilde{X}, \widetilde{Y}\right) - {}^Sg_f\left(\widetilde{X}, J\widetilde{Y}\right)$$

for any $\widetilde{X}, \widetilde{Y} \in \Im_0^1(TM)$. For all vector fields $\widetilde{X}$ and $\widetilde{Y}$ which are of the form ${}^VX$, ${}^VY$ or ${}^HX$, ${}^HY$, from (1.1) and (3.1), we have $A\left(\widetilde{X}, \widetilde{Y}\right) = 0$, i.e. ${}^Sg_f$ is pure with respect to J. Hence we have the following theorem:

Theorem 3.1. *Let (M, g) be a Riemannian manifold and TM be its tangent bundle equipped with the deformed Sasaki metric ${}^Sg_f$ and the paracomplex structure J defined by (3.1). The triple $(TM, J, {}^S g_f)$ is an almost paracomplex Norden manifold.*

We now are interested in the paraholomorphy property of the deformed Sasaki metric ${}^Sg_f$ with respect to the almost paracomplex structure J. Having determined both the deformed Sasaki metric ${}^Sg_f$ and the almost paracomplex structure J and by using the fact that ${}^VX^V(g(Y, Z)) = 0$ and ${}^HX^V(fg(Y, Z)) = {}^V(X(fg(Y, Z)))$ we calculate

$$\begin{aligned}(\Phi_J{}^Sg_f)(\tilde{X}, \tilde{Y}, \tilde{Z}) &= (J\tilde{X})({}^Sg_f(\tilde{Y}, \tilde{Z})) - \tilde{X}({}^Sg_f(J\tilde{Y}, \tilde{Z})) \\ &+ {}^Sg_f((L_{\tilde{Y}}J)\tilde{X}, \tilde{Z}) + {}^Sg_f(\tilde{Y}, (L_{\tilde{Z}}J)\tilde{X})\end{aligned}$$

for all $\tilde{X}, \tilde{Y}, \tilde{Z} \in \Im_0^1(TM)$. Then we get

$$\begin{aligned}
(\Phi_J {}^S g_f)({}^V X, {}^V Y, {}^H Z) &= 0, \\
(\Phi_J {}^S g_f)({}^V X, {}^V Y, {}^V Z) &= 0, \\
(\Phi_J {}^S g_f)({}^V X, {}^H Y, {}^V Z) &= 0, \\
(\Phi_J {}^S g_f)({}^V X, {}^H Y, {}^H Z) &= 0, \\
(\Phi_J {}^S g_f)({}^H X, {}^V Y, {}^H Z) &= -2 {}^S g_f({}^V Y, {}^V (R(X,Z)u)) \\
(\Phi_J {}^S g_f)({}^H X, {}^V Y, {}^V Z) &= 0, \\
(\Phi_J {}^S g_f)({}^H X, {}^H Y, {}^H Z) &= 0, \\
(\Phi_J {}^S g_f)({}^H X, {}^H Y, {}^V Z) &= -2 {}^S g_f({}^V (R(X,Y)u), {}^V Z).
\end{aligned}$$

Therefore, we have the following result.

Theorem 3.2. *Let (M, g) be a Riemannian manifold and let TM be its tangent bundle equipped with the deformed Sasaki metric ${}^S g_f$ and the paracomplex structure J defined by (3.1). The triple $\left(TM, J, {}^S g_f\right)$ is a para-Kähler-Norden (or paraholomorphic Norden) manifold if and only if M is flat.*

Remark 3.1. Let (M, g) be a Riemannian manifold and let TM be its tangent bundle equipped with the deformed Sasaki metric ${}^S g_f$. The diagonal lift ${}^D I$ of the identity tensor field $I \in \Im_1^1(M)$ defined by the following properties

$$\begin{aligned}
{}^D I {}^H X &= {}^H X \\
{}^D I {}^V X &= -{}^V X
\end{aligned} \tag{3.2}$$

is an almost paracomplex structure on TM. In fact ${}^D I$ satisfies $({}^D I)^2 = I_{TM}$. Also, the deformed Sasaki metric ${}^S g_f$ is pure with respect to ${}^D I$, e.i. the triple $\left(TM, {}^D I, {}^S g_f\right)$ is an almost paracomplex Norden manifold. Finally, by using $\Phi-$operator, we can say that the deformed Sasaki metric ${}^S g_f$ is paraholomorphic if and only if M is flat.

Remark 3.2. Another almost paracomplex structure on TM is defined by

$$\begin{cases} J_S({}^H X) = {}^V X \\ J_S({}^V X) = {}^H X \end{cases} \tag{3.3}$$

for all $X, Y \in \Im_0^1(M)$. The deformed Sasaki metric ${}^S g_f$ is pure with respect to J_S if and only if $f = 1$. Conversely, in the case of $f \neq 1$, the deformed Sasaki metric ${}^S g_f$ is never pure with respect to J_S. If $f = 1$, ${}^S g_f$ is the Sasaki metric. In [23], the author and collaborator investigate the paraholomorphy of the Sasaki metric with respect to J_S. Also, for the paraholomorphy properties of the Cheeger-Gromoll type metrics on TM, see [12].

Consider the almost paracomplex Norden manifold $\left(TM, {}^D I, {}^S g_f\right)$. The twin metric tensor of ${}^S g_f$ is the metric defined by

$$\widetilde{G} = {}^S g_f({}^D I \widetilde{X}, \widetilde{Y})$$

for all $\widetilde{X}, \widetilde{Y} \in \Im_0^1(TM)$. In view of (1.1) and (3.1), we see that

$$
\begin{aligned}
\widetilde{G}({}^H X, {}^H Y) &= {}^S g_f({}^D I\, {}^H X, {}^H Y) ={}^V (f g(X,Y)),\\
\widetilde{G}({}^H X, {}^V Y) &= {}^S g_f({}^D I\, {}^H X, {}^V Y) = 0,\\
\widetilde{G}({}^V X, {}^H Y) &= {}^S g_f({}^D I\, {}^V X, {}^H Y) = 0,\\
\widetilde{G}({}^V X, {}^V Y) &= {}^S g_f({}^D I\, {}^V X, {}^V Y) = -{}^V (g(X,Y).
\end{aligned}
\tag{3.4}
$$

Remark 3.3. Let (M,g) be a Riemannian manifold and let TM be its tangent bundle equipped with the deformed Sasaki metric ${}^S g_f$. If $f = 1$, then the triple $(TM, J_S, \widetilde{G})$ is an almost para-Hermitian manifold, where $\widetilde{G}$ is a twin metric defined by (3.4) and J_S an almost paracomplex structure defined by (3.3).

Another almost paracomplex Norden manifold $\left(TM, J_S, {}^S g\right)$ $(f = 1)$, the twin metric tensor of ${}^S g$ is defined by

$$
\begin{aligned}
\widetilde{K}({}^H X, {}^H Y) &= {}^S g(J_S {}^H X, {}^H Y) = 0,\\
\widetilde{K}({}^H X, {}^V Y) &= {}^S g(J_S {}^H X, {}^V Y) ={}^V (g(X,Y)),\\
\widetilde{K}({}^V X, {}^H Y) &= {}^S g(J_S {}^V X, {}^H Y) ={}^V (g(X,Y)),\\
\widetilde{K}({}^V X, {}^V Y) &= {}^S g(J_S {}^V X, {}^V Y) = 0.
\end{aligned}
\tag{3.5}
$$

Remark 3.4. Let (M,g) be a Riemannian manifold and let TM be its tangent bundle equipped with the Sasaki metric ${}^S g$. The triple $(TM, J, \widetilde{K})$ is an almost para-Hermitian manifold, where $\widetilde{K}$ is a twin metric defined by (3.5) and J is an almost paracomplex structure defined by (3.1).

For the Levi-Civita connection of the deformed Sasaki metric we give the next proposition.

Proposition 3.1. [27] *Let (M,g) be a Riemannian manifold and equip its tangent bundle TM with the deformed Sasaki metric ${}^S g_f$. Then the corresponding Levi-Civita connection $\widetilde{\nabla}^f$ satisfies the following:*

$$
\left\{
\begin{array}{l}
i)\ \widetilde{\nabla}^f_{{}^H X}\ {}^H Y = {}^H(\nabla_X Y) +{}^H (A_f(X,Y)) - \dfrac{1}{2}\ {}^V(R(X,Y)u)\\
ii)\ \widetilde{\nabla}^f_{{}^H X}\ {}^V Y = {}^V(\nabla_X Y) + \dfrac{1}{2f}\ {}^H(R(u,Y)X),\\
iii)\ \widetilde{\nabla}^f_{{}^V X}\ {}^H Y = \dfrac{1}{2f}\ {}^H(R(u,X)Y),\\
iv)\ \widetilde{\nabla}^f_{{}^V X}\ {}^V Y = 0
\end{array}
\right.
\tag{3.6}
$$

for any $X, Y \in \Im_0^1(M)$, where $A_f(X,Y) = \frac{1}{2f}\ (X(f)Y + Y(f)X - g(X,Y) \circ (df)^*)$ is a tensor field of type $(1,2)$ and $A_f(X,Y) = A_f(Y,X)$.

Let now consider the almost product structure J defined by (3.1) and the Levi-Civita connection $\widetilde{\nabla}^f$ given by Proposition 3.1. We define a tensor field of type $(1,2)$ on TM by

$$
\widetilde{S}(\widetilde{X}, \widetilde{Y}) = \frac{1}{2}\{(\widetilde{\nabla}^f_{J\widetilde{Y}} J)\widetilde{X} + J((\widetilde{\nabla}^f_{\widetilde{Y}} J)\widetilde{X}) - J((\widetilde{\nabla}^f_{\widetilde{X}} J)\widetilde{Y})\}
$$

for all $\widetilde{X}, \widetilde{Y} \in \Im_0^1(TM)$. Then the linear connection

$$
\overline{\nabla}_{\widetilde{X}} \widetilde{Y} = \widetilde{\nabla}^f_{\widetilde{X}} \widetilde{Y} - \widetilde{S}(\widetilde{X}, \widetilde{Y})
\tag{3.7}
$$

is an almost product connection on TM (for almost product connection, see [15]).

Theorem 3.3. *Let (M,g) be a Riemannian manifold and let TM be its tangent bundle equipped with the deformed Sasaki metric ${}^{S}g_f$ and the almost product structure J defined by (3.1). Then the almost product connection $\overline{\nabla}$ constructed by the Levi-Civita connection $\widetilde{\nabla}^f$ of the deformed Sasaki metric ${}^{S}g_f$ and the almost product structure J is as follows:*

$$(3.8)\qquad \begin{cases} i)\ \overline{\nabla}_{{}^{H}X}{}^{H}Y = {}^{H}(\nabla_X Y) + {}^{H}(A_f(X,Y)) \\ ii)\ \overline{\nabla}_{{}^{H}X}{}^{V}Y = {}^{V}(\nabla_X Y), \\ iii)\ \overline{\nabla}_{{}^{V}X}{}^{H}Y = \frac{3}{2f}\,{}^{H}(R(u,X)Y), \\ iv)\ \overline{\nabla}_{{}^{V}X}{}^{V}Y = 0. \end{cases}$$

Denoting by $\overline{T}$, the torsion tensor of $\overline{\nabla}$, we have from (3.1), (3.7) and (3.8)

$$\begin{aligned} \overline{T}({}^{V}X,{}^{V}Y) &= 0, \\ \overline{T}({}^{V}X,{}^{H}Y) &= \frac{3}{2f}\,{}^{H}(R(u,X)Y), \\ \overline{T}({}^{H}X,{}^{H}Y) &= {}^{V}(R(X,Y)u). \end{aligned}$$

Hence we have the theorem below

Theorem 3.4. *Let (M,g) be a Riemannian manifold and let TM be its tangent bundle. The almost product connection $\overline{\nabla}$ constructed by the Levi-Civita connection $\widetilde{\nabla}^f$ of the deformed Sasaki metric ${}^{S}g_f$ and and the almost product structure J is symmetric if and only if M is flat.*

As is well-known, If there ezists a symmetric almost product connection on M then the almost product structure J is integrable [15]. The converse is also true [11]. Thus we get the following conclusions.

Corollary 3.1. *Let (M,g) be a Riemannian manifold and TM be its tangent bundle equipped with the deformed Sasaki metric ${}^{S}g_f$ and the paracomplex structure J defined by (3.1). The triple $(TM, J, {}^{S}g_f)$ is a paracomplex Norden manifold if and only if M is flat.*

Corollary 3.2. *Let (M,g) be a Riemannian manifold and let TM be its tangent bundle equipped with the Sasaki metric ${}^{S}g$. The triple $(TM, J, \widetilde{K})$ is a para-Hermitian manifold if and only if M is flat, where $\widetilde{K}$ is a twin metric defined by (3.5) and J is an almost paracomplex structure defined by (3.1).*

Similarly, we consider the almost product structure ${}^{D}I$ or J_S and the Levi-Civita connection $\widetilde{\nabla}^f$ of the deformed Sasaki metric ${}^{S}g_f$, other almost product connections can be constructed.

4. Curvature properties of the deformed Sasaki metric

The local symmetry is one of the fundamental notions in Riemannian geometry. The study of Riemannian symmetric manifolds began with the work of E. Cartan [5]. A Riemannian manifold (M,g) is said to be locally symmetric due to E. Cartan if its curvature tensor R satisfies the relation $\nabla R = 0$, where ∇ denotes the operator of covariant differentiation with respect to the metric tensor g. The class of Riemannian symmetric manifolds is a very natural generalization of the class of manifolds of constant curvature. During the last five decades the notion of locally symmetric manifolds has been weakened by many authors in several ways to a different extent such as recurrent manifold by A. G. Walker [28], pseudo symmetric manifold in the sense of R. Deszcz [9], pseudosymmetric manifold in the sense of M. C. Chaki [6], generalized pseudo symmetric manifold by M. C. Chaki [7], and weakly symmetric manifolds by L. Tamassy and T. Q. Binh [26]. Note that the notion of pseudo symmetric manifold studied in particular by R. Deszcz is different from that of M. C. Chaki. The aim of the section is to study conditions for the curvature $\widetilde{R}^f$ of ${}^S g_f$ to be recurrent or pseudo symmetric. In this section, we employ the method proposed by T. Q. Binh and L. Tamassy [4].

First we shall give the following proposition related to the curvature $\widetilde{R}^f$ of ${}^S g_f$ for later use.

Proposition 4.1. [27] *Let (M, g) be a Riemannian manifold and $\widetilde{R}^f$ be the Riemann curvature tensor of the tangent bundle $(TM, {}^S g_f)$ equipped with the deformed Sasaki metric. Then the following formulae hold*

$$\widetilde{R}^f({}^V X, {}^V Y)^V Z = 0, \tag{4.1}$$

$$\widetilde{R}^f({}^H X, {}^V Y)^V Z = {}^H[-\frac{1}{2f} R(Y,Z)X - \frac{1}{4f^2} R(u,Y)(R(u,Z)X)], \tag{4.2}$$

$$\begin{aligned} \widetilde{R}^f({}^V X, {}^V Y)^H Z &= {}^H[\frac{1}{f} R(X,Y)Z - \frac{1}{4f^2} R(u,Y)(R(u,X)Z) \\ &\quad + \frac{1}{4f^2} R(u,X)(R(u,Y)Z)], \end{aligned} \tag{4.3}$$

$$\begin{aligned} \widetilde{R}^f({}^H X, {}^V Y)^H Z &= {}^H[(\nabla_X \frac{1}{2f}) R(u,Y)Z + \frac{1}{2f} (\nabla_X R)(u,Y)Z \\ &\quad + A_f(X, \frac{1}{2f} R(u,Y)Z) - \frac{1}{2f} R(u,Y)(A_f(X,Z))] \\ &\quad + {}^V[\frac{1}{2} R(X,Z)Y + \frac{1}{4f} R(R(u,Y)Z,X)u], \end{aligned} \tag{4.4}$$

$$\begin{aligned} \widetilde{R}^f({}^H X, {}^H Y)^V Z &= {}^H[(\nabla_X \frac{1}{2f}) R(u,Z)Y + \frac{1}{2f} (\nabla_X R)(u,Z)Y \\ &\quad - (\nabla_Y \frac{1}{2f}) R(u,Z)X - \frac{1}{2f} (\nabla_Y R)(u,Z)X) \\ &\quad + \frac{1}{2f} A_f(X, R(u,Z)Y) - \frac{1}{2f} A_f(Y, R(u,Z)X)] \\ &\quad + {}^V[R(X,Y)Z + \frac{1}{4f} R(R(u,Z)Y,X)u \\ &\quad - \frac{1}{4f} R(R(u,Z)X,Y)u] \end{aligned} \tag{4.5}$$

$$(4.6)\quad \widetilde{R}^f({}^HX,{}^HY){}^HZ = {}^H[R(X,Y)Z+\frac{1}{2f}R(u,R(X,Y)u)Z$$
$$+\frac{1}{4f}R(u,R(Z,Y)u)X+\frac{1}{4f}R(u,R(X,Z)u)Y$$
$$+(\nabla_X A_f)(Y,Z)-(\nabla_Y A_f)(X,Z)$$
$$+A_f(X,A_f(Y,Z))-A_f(Y,A_f(X,Z))]$$
$$+{}^V[\frac{1}{2f}(\nabla_Z R)(X,Y)u+\frac{1}{2}R(Y,A_f(X,Z))u$$
$$-\frac{1}{2}R(X,A_f(Y,Z))u].$$

4.1. Firstly we shall study conditions for the curvature $\widetilde{R}^f$ of ${}^Sg_f$ to be recurrent. If $\widetilde{R}^f$ is recurrent then there exists a 1-form α on TM such that

$$(\widetilde{\nabla}^f_{\widetilde{W}}\widetilde{R}^f)(\widetilde{X},\widetilde{Y})\widetilde{Z}=\alpha(\widetilde{W})\widetilde{R}^f(\widetilde{X},\widetilde{Y})\widetilde{Z} \tag{4.7}$$

for all $\widetilde{X},\widetilde{Y},\widetilde{W},\widetilde{Z}\in\Im^1_0(TM)$. If we write ${}^HW,{}^HX,{}^VY,{}^VZ$ insted of $\widetilde{W},\widetilde{X},\widetilde{Y},\widetilde{Z}$ respectively, we get

$$\alpha({}^HW)(\widetilde{R}^f({}^HX,{}^VY){}^VZ)\mid_u=[(\widetilde{\nabla}^f_{{}^HW}\widetilde{R}^f)({}^HX,{}^VY){}^VZ]\mid_u .$$

Using $i)$ and $iii)$ of (3.6) and (4.2), we get

$$\alpha({}^HW)(\widetilde{R}^f({}^HX,{}^VY){}^VZ)\mid_u=-\widetilde{\nabla}^f_{{}^HW}{}^H(\frac{1}{2f}R(Y,Z)X)\mid_u$$
$$-\widetilde{\nabla}^f_{{}^HW}{}^H(\frac{1}{4f^2}R(u,Y)(R(u,Z)X))\mid_u-\widetilde{R}^f({}^H(\nabla_W X),{}^VY){}^VZ\mid_u$$
$$-\widetilde{R}^f({}^H(A_f(W,X)),{}^VY){}^VZ\mid_u+\frac{1}{2}\widetilde{R}^f({}^V(R(W,X)u),{}^VY){}^VZ\mid_u$$
$$-\widetilde{R}^f({}^HX,{}^V(\nabla_W Y)){}^VZ\mid_u-\frac{1}{2f}\widetilde{R}^f({}^HX,{}^H(R(u,Y)W)){}^VZ\mid_u$$
$$-\widetilde{R}^f({}^HX,{}^VY){}^V(\nabla_W Z)\mid_u-\frac{1}{2f}\widetilde{R}^f({}^HX,{}^VY){}^H(R(u,Z)W)\mid_u .$$

As for the vertical part of the both sides of the equation above note that

$$0 = \frac{1}{4f}R(W,R(Y,Z)X)u+\frac{1}{8f^2}R(W,R(u,Y)(R(u,Z)X))u$$
$$-\frac{1}{2f}R(X,R(u,Y)W)Z-\frac{1}{8f^2}R(R(u,Z)(R(u,Y)W),X)u$$
$$+\frac{1}{8f^2}R(R(u,Z)X,R(u,Y)W)u-\frac{1}{4f}R(X,R(u,Z)W)Y$$
$$-\frac{1}{8f^2}R(R(u,Y)(R(u,Z)W),X)u.$$

Substituting respectively $Y=u$ and $Z=u$ in the last equation, we obtain the following relations:

$$\frac{1}{4f}R(W,R(u,Z)X)u-\frac{1}{4f}R(X,R(u,Z)W)u=0 \tag{4.8}$$

and

$$\frac{1}{4f}R(W,R(Y,u)X)u-\frac{1}{2f}R(X,R(u,Y)W)u=0.$$

Let us change Y with Z in the second equation above, then we have

$$\frac{1}{4f}R(W,R(Z,u)X)u-\frac{1}{2f}R(X,R(u,Z)W)u=0. \tag{4.9}$$

The sum of the equations (4.8) and (4.9) is $R(X,R(u,Z)W)u=0$. With $W=X$ and taking the $g-$product with Z allow $g(R(u,Z)X,R(u,Z)X)=0$ which means that $R(u,Z)X=0$, i.e. (M,g) is flat.

Let (M,g) be flat. In the case, the Riemann curvature tensor $\widetilde{R}^f$of the tangent bundle $(TM,^S g_f)$ holds

$$\begin{aligned}\widetilde{R}^f(^HX,^HY)^HZ &= {}^H[(\nabla_X A_f)(Y,Z)-(\nabla_Y A_f)(X,Z)\\ &\quad +A_f(X,A_f(Y,Z))-A_f(Y,A_f(X,Z))]\end{aligned}$$

and all others zero. If

$$(\nabla_X A_f)(Y,Z)-(\nabla_Y A_f)(X,Z))+A_f(X,A_f(Y,Z))-A_f(Y,A_f(X,Z))=0,$$

then $(TM,^S g_f)$ is flat. Hence we have the result below.

Theorem 4.1. *Let (M,g) be a Riemannian manifold and TM be its tangent bundle equipped with the deformed Sasaki metric ${}^Sg_f$. The tangent bundle $(TM,^S g_f)$ is recurrent if (M,g) is flat and $(\nabla_X A_f)(Y,Z)-(\nabla_Y A_f)(X,Z))+A_f(X,A_f(Y,Z))-A_f(Y,A_f(X,Z))=0$, where $A_f(X,Y)=\frac{1}{2f}\,(X(f)Y+Y(f)X-g(X,Y)\circ(df)^*)$ is a $(1,2)-$tensor field. Thus $(TM,^S g_f)$ is flat.*

In view of Theorem 4.1, we have the following conculasion.

Corollary 4.1. *Let (M,g) be a Riemannian manifold and TM be its tangent bundle equipped with the deformed Sasaki metric ${}^Sg_f$. Suppose that $f=C(constant)$. Then the tangent bundle $(TM,^S g_f)$ is recurrent if snd only if (M,g) is flat. Thus $(TM,^S g_f)$ is flat.*

4.2. Another curvature property is pseudo symmetry. Now we investigate conditions for the curvature $\widetilde{R}^f$ of ${}^Sg_f$ to be pseudo symmetric in the sense of M. C. Chaki. The tangent bundle $(TM,^S g_f)$ is called pseudo symmetric, if there exists a 1-form α and a vector field $\overline{A}$ on TM such that

$$\begin{aligned}&(\widetilde{\nabla}^f_{\widetilde{W}}\widetilde{R}^f)(\widetilde{X},\widetilde{Y})\widetilde{Z}\\ &= 2\alpha(\widetilde{W})\widetilde{R}^f(\widetilde{X},\widetilde{Y})\widetilde{Z}+\alpha(\widetilde{X})\widetilde{R}^f(\widetilde{W},\widetilde{Y})\widetilde{Z}+\alpha(\widetilde{Y})\widetilde{R}^f(\widetilde{X},\widetilde{W})\widetilde{Z}\\ &\quad+\alpha(\widetilde{Z})\widetilde{R}^f(\widetilde{X},\widetilde{Y})\widetilde{W}+^S g_f(\widetilde{R}^f(\widetilde{X},\widetilde{Y})\widetilde{Z},\widetilde{W})\overline{A},\end{aligned} \tag{4.10}$$

where $\overline{A}$ is the ${}^Sg_f$-dual vector field of the $1-$form α, e.i. ${}^Sg_f(\widetilde{X},\overline{A})=\alpha(\widetilde{X})$. We consider the condition (4.9) for ${}^HW,^H X,^V Y,^V Z$, we have

$$\begin{aligned}&2\alpha(^HW)\widetilde{R}^f(^HX,^VY)^VZ+\alpha(^HX)\widetilde{R}^f(^HW,^VY)^VZ+\alpha(^VY)\widetilde{R}^f(^HX,^HW)^VZ\\ &+\alpha(^VZ)\widetilde{R}^f(^HX,^VY)^HW+^S g_f(\widetilde{R}^f(^HX,^VY)^VZ,^HW)\overline{A}\\ =\ & -\widetilde{\nabla}^f_{^HW}{}^H(\frac{1}{2f}R(Y,Z)X)\mid_u-\widetilde{\nabla}^f_{^HW}{}^H(\frac{1}{4f^2}R(u,Y)(R(u,Z)X))\mid_u\end{aligned}$$

$$
\begin{aligned}
&-\widetilde{R}^f({}^H(\nabla_W X),{}^V Y)^V Z \mid_u - \widetilde{R}^f({}^H(A_f(W,X)),{}^V Y)^V Z \mid_u \\
&+\frac{1}{2}\widetilde{R}^f({}^V(R(W,X)u),{}^V Y)^V Z \mid_u - \widetilde{R}^f({}^H X,{}^V(\nabla_W Y))^V Z \mid_u \\
&-\frac{1}{2f}\widetilde{R}^f({}^H X,{}^H(R(u,Y)W))^V Z \mid_u - \widetilde{R}^f({}^H X,{}^V Y)^V(\nabla_W Z) \mid_u \\
&-\frac{1}{2f}\widetilde{R}^f({}^H X,{}^V Y)^H(R(u,Z)W) \mid_u .
\end{aligned}
$$

For the vertical part of the both sides of the last equation, by the formulas of the curvature tensor $\widetilde{R}^f$ and the connection $\widetilde{\nabla}^f$ we get

$$
\begin{aligned}
&\alpha({}^V Y)[R(X,W)Z + \frac{1}{4f}R(R(u,Z)W,X)u - \frac{1}{4f}R(R(u,Z)X,W)u] \\
&+\alpha({}^V Z)[\frac{1}{2}R(X,W)Y + \frac{1}{4f}R(R(u,Y)W,X)u] \\
&-fg(\frac{1}{2f}R(Y,Z)X + \frac{1}{4f^2}R(u,Y)(R(u,Z)X,W)\overline{A} \\
= &\ \frac{1}{4f}R(W,R(Y,Z)X)u + \frac{1}{8f^2}R(W,R(u,Y)(R(u,Z)X))u \\
&-\frac{1}{2f}R(X,R(u,Y)W)Z - \frac{1}{8f^2}R(R(u,Z)(R(u,Y)W),X)u \\
&+\frac{1}{8f^2}R(R(u,Z)X,R(u,Y)W)u - \frac{1}{4f}R(X,R(u,Z)W)Y \\
&-\frac{1}{8f^2}R(R(u,Y)(R(u,Z)W),X)u.
\end{aligned}
$$

By putting $Y = u$, respectively $Z = u$, we get

$$
\begin{aligned}
&\alpha({}^V u)[R(X,W)Z + \frac{1}{4f}R(R(u,Z)W,X)u - \frac{1}{4f}R(R(u,Z)X,W)u] \\
&\alpha({}^V Z)\frac{1}{2}R(X,W)u - \frac{1}{2}g(R(u,Z)X,W)\overline{A} \\
= &\ \frac{1}{4f}R(W,R(u,Z)X)u - \frac{1}{4f}R(X,R(u,Z)W)u
\end{aligned} \tag{4.11}
$$

and

$$
\begin{aligned}
&\alpha({}^V Y)R(X,W)u + \alpha({}^V u)[\frac{1}{2}R(X,W)Y + \frac{1}{4f}R(R(u,Y)W,X)u] \\
&-\frac{1}{2}g(R(Y,u)X,W)\overline{A} \\
= &\ \frac{1}{4f}R(W,R(Y,u)X)u - \frac{1}{2f}R(X,R(u,Y)W)u.
\end{aligned}
$$

In last equation we replace Y with Z :

$$
\begin{aligned}
&\alpha({}^V Z)R(X,W)u + \alpha({}^V u)[\frac{1}{2}R(X,W)Z + \frac{1}{4f}R(R(u,Z)W,X)u] \\
&-\frac{1}{2}g(R(Z,u)X,W)\overline{A} \\
= &\ \frac{1}{4f}R(W,R(Z,u)X)u - \frac{1}{2f}R(X,R(u,Z)W)u.
\end{aligned} \tag{4.12}
$$

and by adding (4.11) and (4.12) we have

$$\begin{aligned}(4.13)\qquad & \alpha({}^V Z)[\frac{1}{2}R(X,W)u + R(X,W)u] + \alpha({}^V u)[R(X,W)Z \\ & +\frac{1}{2}R(X,W)Z + \frac{1}{2f}R(R(u,Z)W,X)u - \frac{1}{4f}R(R(u,Z)X,W)u] \\ = & -\frac{3}{4f}R(X,R(u,Z)W)u.\end{aligned}$$

Putting $Z = u$, we obtain $3\alpha({}^V u)R(X,W)u = 0$ and if $\alpha({}^V u) \neq 0$ we have $R(X,W)u = 0$. Suppose now that $\alpha({}^V u) = 0$, then (4.13) transforms into the following relation:

$$\begin{aligned} & \alpha({}^V Z)[\frac{1}{2}R(X,W)u + R(X,W)u] \\ = & -\frac{3}{4f}R(X,R(u,Z)W)u.\end{aligned}$$

By setting $W = X$ we get $R(X,R(u,Z)X)u = 0$. We take the g-product with Z, it follows that $g(R(u,Z)X, R(u,Z)X) = 0$ which means that $R(u,Z)X = 0$. Again the g-product with an arbitrary Y and using property of the curvature tensor gives $g(R(X,Y)u, Z) = 0$ for any vector field Z. Thus $R(X,Y)u = 0$ for every X, Y and u. Hence (M,g) is flat.

Let (M,g) be flat. Then, the Riemann curvature tensor $\widetilde{R}^f$ of the tangent bundle $(TM,{}^S g_f)$ transform into the following:

$$\begin{aligned}\widetilde{R}^f({}^H X,{}^H Y)^H Z = & {}^H[(\nabla_X A_f)(Y,Z) - (\nabla_Y A_f)(X,Z) \\ & +A_f(X,A_f(Y,Z)) - A_f(Y,A_f(X,Z))]\end{aligned}$$

and all others zero. If

$$(\nabla_X A_f)(Y,Z) - (\nabla_Y A_f)(X,Z)) + A_f(X,A_f(Y,Z)) - A_f(Y,A_f(X,Z)) = 0,$$

then $(TM,{}^S g_f)$ is flat. Consequently we have the following theorem.

Theorem 4.2. *Let (M,g) be a Riemannian manifold and TM be its tangent bundle equipped with the deformed Sasaki metric ${}^S g_f$. The tangent bundle $(TM,{}^S g_f)$ is pseudo symmetric if (M,g) is flat and $(\nabla_X A_f)(Y,Z) - (\nabla_Y A_f)(X,Z)) + A_f(X,A_f(Y,Z)) - A_f(Y,A_f(X,Z)) = 0$, where $A_f(X,Y) = \frac{1}{2f}\,(X(f)Y + Y(f)X - g(X,Y)\circ(df)^*)$ is a $(1,2)-$tensor field. Thus $(TM,{}^S g_f)$ is flat.*

Theorem 4.2 immediately gives the following conculasion.

Corollary 4.2. *Let (M,g) be a Riemannian manifold and TM be its tangent bundle equipped with the deformed Sasaki metric ${}^S g_f$. Suppose that $f = C(constant)$. Then the tangent bundle $(TM,{}^S g_f)$ is pseudo symmetric if and only if (M,g) is flat. Thus $(TM,{}^S g_f)$ is flat.*

For $f = 1$, we get the T. Q. Binh and L. Tamassy result. Their result is as follows:

Corollary 4.3. *If $(TM,{}^S g)$ is a recurrent or pseudo symmetric Riemannian manifold, then (M,g) must be flat and thus $(TM,{}^S g)$ must be flat too. The converse is trivially true.*

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