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ONE-PARAMETER PLANAR MOTION ON THE GALILEAN PLANE

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Abstract. Müller [2], on the Euclidean plane $\mathbb{E}^2$, introduced the one-parameter planar motions and obtained the relation between absolute, relative, sliding velocities and accelerations. Ergin [3] considered the Lorentzian plane $\mathbb{L}^2$, instead of the Euclidean plane $\mathbb{E}^2$, and introduced the one-parameter planar motions on the Lorentzian plane and also gave the relations between both the velocities and accelerations.

In this paper, one-parameter motions on the Galilean plane $\mathbb{G}^2$ are defined. Also the relations between absolute, relative, sliding velocities and accelerations and pole curves are discussed.

1. INTRODUCTION

We consider $\mathbb{R}^2$ with the bilinear form

$$\langle \mathbf{x}, \mathbf{y} \rangle = x_1 x_2 + \epsilon y_1 y_2 \tag{1.1}$$

where ϵ may be 1,0 or -1 and $\mathbf{x} = (x_1, y_1)$, $\mathbf{y} = (x_2, y_2) \in \mathbb{R}^2$. The distance between two points X and Y is defined by

$$\|\mathbf{x} - \mathbf{y}\| = |\langle \mathbf{x} - \mathbf{y}, \mathbf{x} - \mathbf{y} \rangle|^{\frac{1}{2}} \tag{1.2}$$

where $\mathbf{x}$ and $\mathbf{y}$ are the coordinate vectors of the points X and Y with respect to the coordinate systems in $\mathbb{R}^2$. For $\epsilon = 1$ we have the *Euclidean plane* $\mathbb{E}^2$, for $\epsilon = 0$ we have the *Galilean plane* $\mathbb{G}^2$, and for $\epsilon = -1$ we have the *Minkowskian (or Lorentzian) plane* $\mathbb{L}^2$, (for Lorentzian Plane, see [1]). These are the three Cayley-Klein plane geometries with a parabolic measure of distance. Denote $\mathbb{R}^2$ with the bilinear form (1.1) by $\mathbb{P}_\epsilon$, [4].

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Vectors $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{P}_\epsilon$ are *orthogonal*, written $\mathbf{x} \perp \mathbf{y}$, if $\langle \mathbf{x}, \mathbf{y} \rangle = 0$. Self-orthogonal vectors are called *isotropic*. For $\epsilon = 1$, only the zero vector is isotropic. For $\epsilon = 0$, zero and vertical vectors are isotropic and, for $\epsilon = -1$, zero vectors and vectors parallel to $(\pm 1, 1)$ are isotropic, [4].

The linear transformation $J : \mathbb{P}_\epsilon \longrightarrow \mathbb{P}_\epsilon$ with matrix, also denoted by J,

$$J = \begin{bmatrix} 0 & -\epsilon \\ 1 & 0 \end{bmatrix} \tag{1.3}$$

takes any vector $\mathbf{x}$ to an orthogonal vector $J\mathbf{x}$. It is straight forward to check that, if $\mathbf{x}$ is not isotropic and $\mathbf{y}$ is orthogonal to $\mathbf{x}$, then $\mathbf{y} = kJ\mathbf{x}$ for some real number k. A *circle* is the set of points a given distance from a fixed point, the *center*. The *unit circle* in $\mathbb{P}_\epsilon$ is the set of points with $\|\mathbf{p}\| = 1$. The unit circles on Euclidean, Galilean and Minkowskian planes are shown in Figure 1, [4].

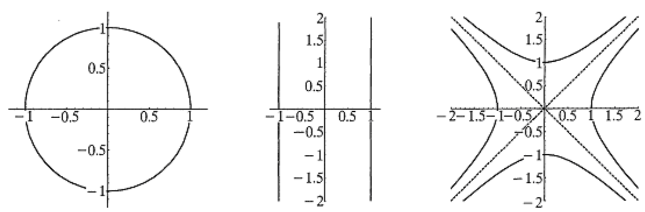

FIGURE 1. The unit circles for $\epsilon = 1, 0, -1$, respectively.

The Galilean unit circle has two branches, the vertical lines $x = \pm 1$, and any point on the $y-$axis is a center. The Minkowskian unit circle has four branches, consisting of a pair of conjugate rectangular hyperbolas with equations $x^2 - y^2 = \pm 1$. Hence, the equation of general unit circle in $\mathbb{P}_\epsilon$ is $x^2 + \epsilon y^2 = \pm 1$. It is not difficult to verify directly from the definition of the matrix exponential as $e^A = \sum \frac{A^n}{n!}$ that

$$J = \begin{bmatrix} \cos_\epsilon \phi & -\epsilon \sin_\epsilon \phi \\ \sin_\epsilon \phi & \cos_\epsilon \phi \end{bmatrix} \tag{1.4}$$

where

$$\cos_\epsilon \phi = \sum_{n=0}^{\infty} \frac{(-\epsilon)^n \phi^{2n}}{(2n)!}, \qquad \sin_\epsilon \phi = \sum_{n=0}^{\infty} \frac{(-\epsilon)^n \phi^{2n+1}}{(2n+1)!}. \tag{1.5}$$

For $\epsilon = 1$ these are the usual cosine and sine functions, for $\epsilon = -1$ they are hyperbolic cosine and sine and for $\epsilon = 0$ they are just

$$\begin{array}{l} \cos_0 \phi = 1 \\ \sin_0 \phi = \phi \end{array}, \quad \text{for all } \phi. \tag{1.6}$$

In all case, we obtain

$$\cos_\epsilon^2 \phi + \epsilon \sin_\epsilon^2 \phi = 1 \tag{1.7}$$

and

$$\partial_\phi \cos_\epsilon \phi = -\epsilon \sin_\epsilon \phi, \quad \partial_\phi \sin_\epsilon \phi = \cos_\epsilon \phi. \tag{1.8}$$

Equating corresponding entries of matrix equation

$$e^{J(\phi+\psi)} = e^{J\phi} e^{J\psi} \tag{1.9}$$

gives the sum formulae

$$\begin{aligned} \cos_\epsilon(\phi+\psi) &= \cos_\epsilon \phi \cos_\epsilon \psi - \epsilon \sin_\epsilon \phi \sin_\epsilon \psi \\ \sin_\epsilon(\phi+\psi) &= \sin_\epsilon \phi \cos_\epsilon \psi + \cos_\epsilon \phi \sin_\epsilon \psi \end{aligned}, \tag{1.10}$$

[4].

1.1. **Galilean Metric and Galilean Transformation.** The Galilean norm of $\mathbf{x} = (x, y) \in \mathbb{G}^2$ is defined by $\|\mathbf{x}\|_g = \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle_g} = |x|$. Furthermore, if $\|\mathbf{x}\|_g = 1$, $\mathbf{x}$ is called a *unit vector*, where $\langle , \rangle_g$ is called the *Galilean inner product* for $\epsilon = 0$ in the equation (1.1).

On the Galilean Plane, the distance $d(X, Y)$ between two points $X = (x_1, y_1)$ and $Y = (x_2, y_2)$ is defined by the formula

$$d(X, Y) = \|\mathbf{YX}\|_g = \|\mathbf{x} - \mathbf{y}\|_g = \sqrt{\langle \mathbf{x} - \mathbf{y}, \mathbf{x} - \mathbf{y} \rangle_g} = |x_1 - x_2| \tag{1.11}$$

and it equals the signed length of the projection $\mathbf{PQ}$ of the segment $\mathbf{XY}$ on the $x-$axis (Fig. 2), [5].

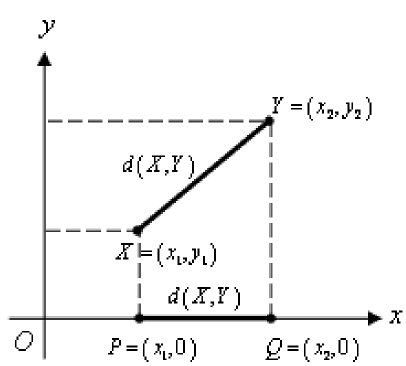

FIGURE 2. The distance on Galilean plane.

If the distance $d(X, Y)$ between the points X and Y is zero, i.e., $x_1 = x_2$, then X and Y belong to the same special line (parallel to the $y-$axis; Fig. 3). For such points it makes sense to define the *special distance*

$$\delta\left(X,Y\right)=\left|y_1-y_2\right|, \tag{1.12}$$

[5].

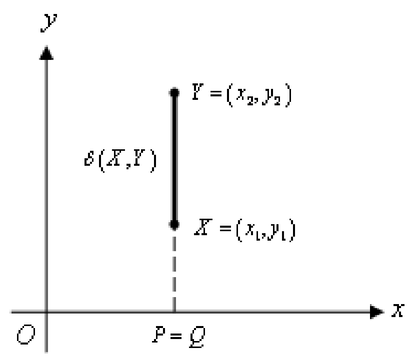

FIGURE 3. The special distance on Galilean plane.

Taking φ as the rotation angle between $\mathbf{x}=(x,y)$ and $\mathbf{x}'=(x',y')$ (Fig. 4), we can write

$$\begin{aligned} x &= r\cos g\theta \\ y &= r\sin g\theta \end{aligned} \qquad \text{and} \qquad \begin{aligned} x' &= r\cos g\left(\theta+\varphi\right) \\ y' &= r\sin g\left(\theta+\varphi\right), \end{aligned}$$

where $\cos g$ and $\sin g$ are $\cos_0$ and $\sin_0$ (for $\epsilon=0$, in the equations (1.4-1.10)), respectively.

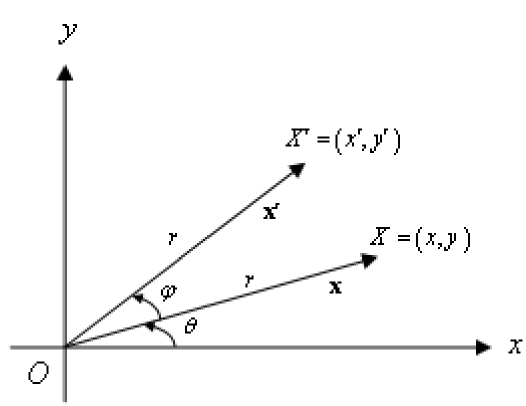

FIGURE 4. The rotation on Galilean plane.

Then, using the equation (1.10), for $\epsilon = 0$, we obtain

$$\begin{aligned} x' &= x \cos g\varphi + y0 \\ y' &= x \sin g\varphi + y \cos g\varphi. \end{aligned}$$

From the equation (1.6) (since for all φ, $\cos g\varphi = 1$ and $\sin g\varphi = \varphi$), we get

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \varphi & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

or

$$\begin{aligned} x' &= x \\ y' &= \varphi x + y. \end{aligned} \tag{1.13}$$

Then, from composed of the transformation (or the shear)

$$\begin{aligned} x_1 &= x \\ y_1 &= \varphi x + y \end{aligned}$$

and the transformation (or translation)

$$\begin{aligned} x' &= x_1 + a \\ y' &= y_1 + b\ , \end{aligned}$$

we arrive at the formulae

$$\begin{aligned} x' &= x + a \\ y' &= \varphi x + y + b\ . \end{aligned} \tag{1.14}$$

The equation (1.14) is called *Galilean transformation* and we remark that the transformation (1.14) map
a) lines onto lines,
b) parallel lines onto parallel lines,
c) collinear segments onto collinear segments,
d) a figure onto a figure of the same area.
This Galilean transformation belong to the kinematics on $\mathbb{G}^2$. Under the Galilean Transformation examining the motion of points of $\mathbb{G}^2$ and establishing the invariants are the kinematic geometry of $\mathbb{G}^2$. These are called, in other words, *the Galilean geometry*, [5].

2. KINEMATICS ON THE GALILEAN PLANE

In kinematics, the one-parameter planar motions on the Euclidean plane were given by Müller [2]. Then, the one-parameter planar motions on the Lorentzian plane were given by Ergin [3].

In this section, the one-parameter planar motions on the Galilean plane $\mathbb{G}^2$ are defined. Then, the relations between both velocities and accelerations of a point under the one-parameter planar Galilean motions are obtained.

I

Let $\mathbb{G}$ and $\mathbb{G}'$ be moving and fixed Galilean planes and $\{O;\mathbf{g}_1,\mathbf{g}_2\}$ and $\{O';\mathbf{g}_1',\mathbf{g}_2'\}$ be their coordinate systems, respectively. By taking

$$\mathbf{OO}' = \mathbf{u} = u_1\mathbf{g}_1 + u_2\mathbf{g}_2 \ , \quad \text{for} \quad u_1, u_2 \in \mathbb{R} \tag{2.1}$$

the motion defined by the transformation

$$\mathbf{x}' = \mathbf{x} - \mathbf{u} \tag{2.2}$$

is called a *one-parameter planar Galilean motion* and denoted by $B = \mathbb{G}/\mathbb{G}'$, where $\mathbf{x}, \mathbf{x}'$ are the coordinate vectors with respect to the moving and fixed rectangular coordinate systems of a point $X = (x_1, x_2) \in \mathbb{G}$, respectively (Fig. 5). Also the rotation angle φ and the vectors $\mathbf{x}, \mathbf{x}'$ and $\mathbf{u}$ are continuously differentiable functions of a time parameter t. Furthermore, at the initial time $t = 0$ the coordinate systems coincide. Taking $\varphi = \varphi(t)$ as the rotation angle between $\mathbf{g}_1$ and $\mathbf{g}_1'$ (Fig. 5), we can write

$$\begin{aligned} \mathbf{g}_1 &= \mathbf{g}_1' + \varphi\mathbf{g}_2' \\ \mathbf{g}_2 &= \qquad \mathbf{g}_2' \ . \end{aligned} \tag{2.3}$$

In this study we assume that

$$\dot{\varphi}(t) = \frac{d\varphi}{dt} \neq 0 \ , \tag{2.4}$$

where " . " denotes the derivation with respect to " t " and $\dot{\varphi}(t)$ is called the *angular velocity* of the motion $B = \mathbb{G}/\mathbb{G}'$.

Differentiating the equations (2.1) and (2.3) with respect to t, the *derivative formulae* of the motion $B = \mathbb{G}/\mathbb{G}'$ are obtained as follows

$$\begin{aligned} \dot{\mathbf{g}}_1 &= \dot{\varphi}\mathbf{g}_2 \\ \dot{\mathbf{g}}_2 &= \mathbf{0} \end{aligned} \tag{2.5}$$

and

$$\dot{\mathbf{u}} = \dot{u}_1\mathbf{g}_1 + (\dot{u}_2 + u_1\dot{\varphi})\,\mathbf{g}_2 \ . \tag{2.6}$$

Now, we will define velocities of a point $X \in \mathbb{G}$ using the derivative formulae of the motion $B = \mathbb{G}/\mathbb{G}'$:
The velocity of the point X with respect to $\mathbb{G}$ is known as the *relative velocity* $\mathbf{V}_r$

and it is defined by $\frac{d\mathbf{x}}{dt} = \dot{x}$. Also, for the relative velocity $\mathbf{V}_r$, we can write

$$\mathbf{V}_r = \dot{x}_1\mathbf{g}_1 + \dot{x}_2\mathbf{g}_2 \ . \tag{2.7}$$

Moreover, if we differentiate the equation (2.2) with respect to t, the *absolute velocity* of the point $X \in \mathbb{G}$ is found as

$$\mathbf{V}_a = -\dot{u}_1\mathbf{g}_1 + \left(-\dot{u}_2 - u_1\dot{\varphi} + x_1\dot{\varphi}\right)\mathbf{g}_2 + \mathbf{V}_r \ . \tag{2.8}$$

From the equation (2.8), we get the *sliding velocity*

$$\mathbf{V}_f = -\dot{u}_1\mathbf{g}_1 + \left(-\dot{u}_2 - u_1\dot{\varphi} + x_1\dot{\varphi}\right)\mathbf{g}_2 \ . \tag{2.9}$$

So we can give the following theorem using the equation (2.7), (2.8) and (2.9).

Theorem 2.1. *Let X be a moving point on the plane $\mathbb{G}$ and $\mathbf{V}_r, \mathbf{V}_a$ and $\mathbf{V}_f$ be the relative, absolute and sliding velocities of X, respectively, under the one-parameter planar motion $B = \mathbb{G}/\mathbb{G}'$. Then*

$$\mathbf{V}_a = \mathbf{V}_f + \mathbf{V}_r \ . \tag{2.10}$$

The proof is obvious by using the definitions of velocities above. □

For a general planar motions, there is a point that does not move, which means that its coordinates are the same in both reference coordinate systems $\{O; \mathbf{g}_1, \mathbf{g}_2\}$ and $\{O'; \mathbf{g}'_1, \mathbf{g}'_2\}$. This point is called the *pole point* or the *instantaneous rotation pole center*, (Fig. 5). In this case, we obtain

$$\mathbf{V}_f = \mathbf{0}$$

or

$$\begin{cases} -\dot{u}_1 = 0 \\ -\dot{u}_2 - u_1\dot{\varphi} + x_1\dot{\varphi} = 0. \end{cases}$$

Then for the pole point $P = (p_1, p_2) \in \mathbb{G}$ of the motion $B = \mathbb{G}/\mathbb{G}'$, we have

$$P ... \begin{cases} p_1 = \frac{\dot{u_2}(t)}{\dot{\varphi}(t)} \\ p_2 = p_2\left(\lambda\left(t\right)\right) \end{cases} , \quad \text{for } \lambda \in \mathbb{R}. \tag{2.11}$$

Result 2.1. *Invariant points on both planes at any instant t of $B = \mathbb{G}/\mathbb{G}'$ lie on line parallel to $y-$axis on the plane $\mathbb{G}$. That is, there is only pole line on the plane $\mathbb{G}$ at any instant t. For all $t \in I$, this pole lines are parallel to $y-$axis and each other and they constitute bundles of parallel lines.*

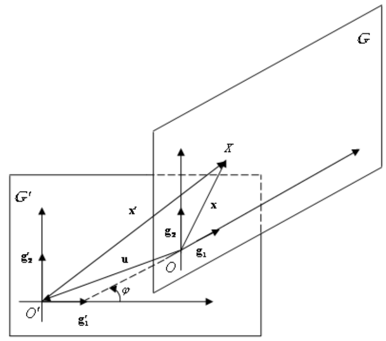

FIGURE 5. The motion $B = \mathbb{G}/\mathbb{G}'$.

Using equations (2.9) and (2.11), for the sliding velocity, we can rewrite

$$\mathbf{V}_f = \{0\mathbf{g}_1 + (x_1 - p_1)\,\mathbf{g}_2\}\,\dot{\varphi}\ . \tag{2.12}$$

Now, we can give the following results by the equation (2.12):

Corollary 2.1. *During the one-parameter plane motion $B = \mathbb{G}/\mathbb{G}'$, the pole ray $\mathbf{PX} = (x_1 - p_1)\,\mathbf{g}_1 + (x_2 - p_2)\,\mathbf{g}_2$ and the sliding velocity $\mathbf{V}_f = \{0\mathbf{g}_1 + (x_1 - p_1)\,\mathbf{g}_2\}\,\dot{\varphi}$ are perpendicular vectors in the sense of Galilean geometry. That is, $\langle \mathbf{V}_f, \mathbf{PX}\rangle_g = 0$. Then under the motion $B = \mathbb{G}/\mathbb{G}'$, the focus of the points $X \in \mathbb{G}$ is an orbit curve that it's normal pass through the rotation pole P.*

Corollary 2.2. *Under the motion $B = \mathbb{G}/\mathbb{G}'$, the Galilean norm of the sliding velocity $\mathbf{V}_f$ is*

$$\|\mathbf{V}_f\|_\delta = \|\mathbf{PX}\|_g\,|\dot{\varphi}|\,.$$

That is, during the motion $B = \mathbb{G}/\mathbb{G}'$, all of the orbits of the points $X \in \mathbb{G}$ are such curves whose normal lines pass thoroughly the pole P. At any instant t, the motion $B = \mathbb{G}/\mathbb{G}'$ is a Galilean instantaneous rotation with the angular velocity $\dot{\varphi}$ about the pole point P .

II

In this section, we will define relative, absolute, sliding and Coriolis acceleration vectors, during the one-parameter planar motion $B = \mathbb{G}/\mathbb{G}'$.

Let X be a moving point of $\mathbb{G}$. Then the acceleration of the point X with respect to $\mathbb{G}$ is known as the *relative acceleration* and it is defined by $\frac{d^2\mathbf{x}}{dt^2} = \ddot{\mathbf{x}} = \dot{\mathbf{V}}_r$. Also, for the relative acceleration $\mathbf{b}_r$, we can write

$$\mathbf{b}_r = \ddot{x}_1\mathbf{g}_1 + \ddot{x}_2\mathbf{g}_2\ . \tag{2.13}$$

The acceleration of the point X with respect to $\mathbb{G}'$ is known as the *absolute acceleration* and it is defined by

$$\mathbf{b}_a = \dot{\mathbf{V}}_a = \ddot{x}_1\mathbf{g}_1 + \{(x_1 - p_1)\ddot{\varphi} - \dot{p}_1\dot{\varphi} + 2\dot{x}_1\dot{\varphi} + \ddot{x}_2\}\,\mathbf{g}_2 \tag{2.14}$$

In the equation (2.14), the expression

$$\mathbf{b}_f = 0\mathbf{g}_1 + \{(x_1 - p_1)\ddot{\varphi} - \dot{p}_1\dot{\varphi}\}\,\mathbf{g}_2 \tag{2.15}$$

is called the *sliding acceleration* and

$$\mathbf{b}_c = 0\mathbf{g}_1 + (2\dot{x}_1\dot{\varphi})\,\mathbf{g}_2 \tag{2.16}$$

is called the *Coriolis acceleration* of the one-parameter planar motion $B = \mathbb{G}/\mathbb{G}'$. So, we can give the following theorem and corollary using the equations (2.7), (2.14), (2.15) and (2.16):

Theorem 2.2. *Let X be a moving point on the plane $\mathbb{G}$. Then,*

$$\mathbf{b}_a = \mathbf{b}_f + \mathbf{b}_c + \mathbf{b}_r\ , \tag{2.17}$$

during the one-parameter planar motion $B = \mathbb{G}/\mathbb{G}'$. □

Corollary 2.3. *During the motion $B = \mathbb{G}/\mathbb{G}'$, the Coriolis acceleration vector $\mathbf{b}_c$ and the relative velocity vector $\mathbf{V}_r$ are perpendicular to each other in the sense of Galilean geometry, i.e. $\langle \mathbf{V}_r, \mathbf{b}_c \rangle_g = 0$.*

Under the one-parameter planar motion $B = \mathbb{G}/\mathbb{G}'$, the acceleration pole is characterized by vanishing the sliding acceleration. Therefore, if we take $\mathbf{b}_f = \mathbf{0}$, the *acceleration pole point* $Q = (q_1, q_2) \in \mathbb{G}$ of the motion $B = \mathbb{G}/\mathbb{G}'$, we get

$$Q \ldots \begin{cases} q_1 = p_1(t) + \dot{p}_1(t)\,\frac{\dot{\varphi}(t)}{\ddot{\varphi}(t)} \\ q_2 = q_2(\mu(t)) \end{cases} , \quad \text{for } \mu \in \mathbb{R}\ . \tag{2.18}$$

Result 2.2. *Invariant points on both planes at any instant t* of *$B = \mathbb{G}/\mathbb{G}'$ lie on line parallel to $y-axis$ on the plane $\mathbb{G}$. That is, there is only acceleration pole line on the plane $\mathbb{G}$ at any instant t.For all $t \in I$, this acceleration pole lines are parallel to $y-axis$ and each other and they constitute bundles of parallel lines.*

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