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RIEMANNIAN HYPERSURFACES WITH CONSTANT SCALAR CURVATURE IN A HESSIAN MANIFOLD OF CONSTANT CURVATURE

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Abstract. We investigated hypersurfaces with constant scalar curvature in a Hessian manifold of constant curvature and obtained two theorems on hypersurfaces of Hessian manifolds with non-negative constant curvature.

1. Introduction

Let M^{n+1} be a flat affine manifold with flat affine connection D. Among Riemannian metrics on M^{n+1} there exists an important class of Riemannian metrics compatible with the flat affine connection D. A Riemannian metric g on M^{n+1} is said to be Hessian metric if g is locally expressed by $g = D^2u$ where u is a local smooth function.We call such a pair $(D\,,\,g)$ a Hessian structure on M^{n+1} and a triple $\left(M^{n+1}\,,\,D\,,\,g\right)$ a Hessian manifold, [1], [2], [3], [4]. Geometry of Hessian manifold is deeply related to Kählerian geometry and affine differential geometry [2]. It is also a fruitful area for differential geometry. Hessian manifolds with constant sectional curvature has interesting applications with different aspects [5], [6], [7]. It is well known that a compact convex hypersurface with constant mean curvature in a Euclidean space is a sphere. On the other hand Simons [8] has recently done an important suggestive contribution to the study of minimal submanifolds in a Riemannian manifold, in which he has given a formula for the Laplacian of the square of the norm of the second fundamental form of the submanifold. Under the stimulus of the Simons' study Do Cormo, Chern and Kobayashi [9] and Nomizu and Smyth [10], using the similar formula to that of Simons, have obtained some theorems on a compact minimal submanifold or a complete hypersurface with constant mean curvature in a Riemannian manifold of constant curvature. Nakagawa and Yokote [11] generalize by applying a formula of Simons' type to a compact hypersurface with constant scalar curvature in a Riemannian manifold of constant curvature. Then Omachi [12] has obtained some results in the case of hypersurfaces in a space of non-negative constant curvature making use of harmonic curvature.

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In this paper our main purpose is to investigate hypersurfaces with constant scalar curvature in a Hessian manifold of constant curvature by a close analogy with above studies. Also we generalize Nakagawa and Yokote's result to the special type of a Hessian manifold.

2. **Preliminaries**

Let $M^{\,n+1}$ be a Hessian manifold with Hessian structure $(D\,,\,g\,)$. We express various geometric concepts for the Hessian structure $(D\,,\,g)$ in terms of affine coordinate system $\left\{ x^1\,,\,\ldots\,,\,x^{n+1} \right\}$ with respect to D , i.e $D\,dx^A = 0$. Here A, B, C,... run from 1 to $n+1$. i) The Hessian metric ;

$$g_{AB} = \frac{\partial^2 u}{\partial x^A \partial x^B}$$

ii) Let γ be a tensor field of type $(1,2)$ defined by

$$\gamma\,(X,Y) = \triangledown_X Y - D_X Y$$

where ∇ is the Riemannian connection for g . Then we have

$$\gamma^A_{BC} = \Gamma^A_{BC} = \frac{1}{2} g^{AD} \frac{\partial g_{DB}}{\partial x^C}$$

$$\gamma_{ABC} = \frac{1}{2}\frac{\partial g_{AB}}{\partial x^C} = \frac{1}{2}\frac{\partial^3 u}{\partial x^A \partial x^B \partial x^C}$$

$$\gamma_{ABC} = \gamma_{BAC} = \gamma_{CBA}$$

where Γ^A_{BC} are the Christoffel ' s symbols of ∇ . iii)Define a tensor field S of type $(1,3)$ by

$$S = D_\gamma$$

and call it the Hessian curvature tensor for $(\,D\,,\,g\,)$. Then we have

$$S^A_{BCE} = \frac{\partial \gamma^A_{BE}}{\partial x^C}$$

$$S_{ABCE} = \frac{1}{2}\frac{\partial^4 u}{\partial x^A \partial x^B \partial x^C \partial x^E} - \frac{1}{2} g^{DF} \frac{\partial^3 u}{\partial x^A \partial x^C \partial x^D} - \frac{\partial^3 u}{\partial x^B \partial x^E \partial x^F}$$

$$S_{ABCE} = S_{AECB} = S_{CBAE} = S_{BAEC} = S_{CEAB}.$$

iv) The Riemannian curvature tensor for ∇ ;

$$R^A_{BCE} = \gamma^A_{DC}\gamma^D_{BE} - \gamma^A_{DE}\gamma^D_{BC}\,,$$

$$R_{ABCE} = \frac{1}{2}\left(S_{BACE} - S_{ABCE}\right) \tag{2.1}$$

[4].

Definition 2.1. For a non-zero contravariant symmetric tensor ξ_x of degree 2 at x we set

$$h\left(\xi_x\right) = \frac{\langle \varsigma\left(\xi_x\right), \xi_x \rangle}{\langle \xi_x\,, \xi_x \rangle}$$

and call it the Hessian sectional curvature in the direction ξ_x. Here ς is a endomorphism with respect to the inner product $\langle , \rangle$ induced by the Hessian metric g [4].

Theorem 2.1. *Let $\left(M^{n+1}, D, g\right)$ be a Hessian manifold of dimension ≥ 2. If the Hessian sectional curvature $h\left(\xi_x\right)$ depends only x then $\left(M^{n+1}, D, g\right)$ is of constant Hessian sectional curvature .$\left(M^{n+1}, D, g\right)$ is of constant Hessian sectional curvature c if and only if*

$$S_{ABCE} = \frac{c}{2}\left(g_{AB}g_{CE} + g_{AE}g_{CB}\right). \tag{2.2}$$

[4].

Corollary 2.1. *If a Hessian manifold $\left(M^{n+1}, D, g\right)$ is a space of constant Hessian sectional curvature c, then the Riemannian manifold $\left(M^{n+1}, g\right)$ is a space of constant sectional curvature $-\frac{c}{4}$* [4].

3. **Constructions of Hessian Manifolds of constant Hessian sectional curvature**

In this section we shall construct, for each constant c, a Hessian manifold with constant Hessian sectional curvature c. We now recall the following result due to Yagi [3]. Let $\left(M^{n+1}, D, g\right)$ be a simply connected Hessian manifold. If g is complete, then $\left(M^{n+1}, D, g\right)$ is isomorphic to $\left(\Omega, \widetilde{D}, \widetilde{D}^2\varphi\right)$ where Ω is a convex domain in $\mathbb{R}^{n+1}$, $\widetilde{D}$ is the canonical flat connection on $\mathbb{R}^{n+1}$ and φ is a smooth convex function on Ω.

A. The case $c = 0$. It is obvious that the Euclidean space

$$\left(\mathbb{R}^{n+1},\ \widetilde{D},\ g = (1/2)\,\widetilde{D}^2\left\{\left(x^A\right)^2\right\}\right)$$

is a simply connected Hessian manifold of constant Hessian sectional curvature 0.

B. The case $c > 0$.

Theorem 3.1. *Let Ω be a domain in $\mathbb{R}^{n+1}$ given by*

$$x^{n+1} > \frac{c}{2}\,{}^{\;n}_{A=1}\left(x^A\right)^2,$$

where c is a positive constant, and let φ be a smooth function on Ω defined by

$$\varphi = -\frac{1}{c}\log\left\{x^{n+1} - \frac{c}{2}\,{}^{\;n}_{A=1}\left(x^A\right)^2\right\}.$$

Then $\left(\Omega, \widetilde{D},\ g = \widetilde{D}^2\varphi\right)$ is a simply connected Hessian manifold of positive constant Hessian sectional curvature c. As Riemannian manifold (Ω, g) is isometric to the hyperbolic space $\left(H\left(-\frac{c}{4}\right), g\right)$ of constant sectional curvature $-c/4$;

$$\begin{aligned} H &= \left\{\left(\xi^1, ..., \xi^n, \xi^{n+1}\right) \in \mathbb{R}^{n+1} \left|\xi^{n+1} > 0\right.\right\}, \\ g &= \frac{1}{\left(\xi^{n+1}\right)^2}\left\{{}^{\;n}_{A=1}\left(d\xi^A\right)^2 + \frac{4}{c}\left(d\xi^{n+1}\right)^2\right\}. \end{aligned}$$

C. The case $c < 0$ [4].

Theorem 3.2. *Let φ be a smooth function on $\mathbb{R}^{n+1}$ defined by*

$$\varphi = -\frac{1}{c}\log\left(\sum_{A=1}^{n+1} e^{-cx^A} + 1\right),$$

where c is a negative constant. Then $\left(\mathbb{R}^{n+1}, \widetilde{D},\ g = \widetilde{D}^2\varphi\right)$ *is a simply connected Hessian manifold of negative constant Hessian sectional curvature c. The Riemannian manifold* $\left(\mathbb{R}^{n+1}, g\right)$ *is isometric a domain of the sphere* $\overset{n+2}{\underset{i=1}{}}\xi_A^2 = -\frac{4}{c}$ *defined by* $\xi_A > 0$ *for all A* [4].

For the proof of the theorems we refer to [4].

4. Basic concepts and equations on Riemannian hypersurfaces of Hessian manifolds of constant curvature

We consider $n-$dimensional hypersurface M isometrically immersed in an $(n+1)-$ dimensional Hessian manifold $M^{'}$of constant curvature $-\frac{c}{4}$ where c is a negative constant by an isometric immersion $\Phi : M \to M^{'}\left(-\frac{c}{4}\right)$, and denote g_{ji}, R^h_{kji} and H_{ji} components of the Riemannian metric tensor, the Riemannian curvature tensor and second fundamental tensor of M, respectively. Then the equation of Gauss for the hypersurface M and that of Codazzi are given below, respectively

$$R_{kjih} = -\frac{c}{4}\left(g_{kh}g_{ji} - g_{ki}g_{jh}\right) + H_{kh}H_{ji} - H_{ki}H_{jh} \tag{4.1}$$

$$\nabla_k H_{ji} - \nabla_j H_{ki} = 0 \tag{4.2}$$

where ∇ denotes the induced connection of M. Let us define the function f on M by

$$f = H_{ji}g^{ji} = H^j_j \tag{4.3}$$

which is globally defined on M up to the sign. Contracting the equation of Gauss with g^{kh}, we get

$$R_{ji} = -\frac{c}{4}\left(n-1\right)g_{ji} + fH_{ji} - H_{jr}H^r_i \tag{4.4}$$

where R_{ji} are conponents of the Ricci tensor. We denote by R the scalar curvature of M and calculate as follows

$$R = R_{ji}g^{ji}.$$

Considering this definition and equation (4.3) in (4.4) we obtain scalar curvature as

$$R = -\frac{c}{4}n\left(n-1\right) + f^2 - H_{ji}H^{ji}. \tag{4.5}$$

Then applying the Ricci identitiy to H_{ji} and taking account of equation (4.2) we have

$$\nabla_j\nabla_i f = \nabla_j\nabla_i H^k_k = \nabla^k\nabla_k H_{ji} - R_{jr}H^r_i - R^r_{jki}H^k_r$$

where $\nabla^k = g^{ki}\nabla_i$. Substituting (4.1) and (4.2) into the equation above, we obtain

$$\Delta H_{ji} = \nabla_j\nabla_i f - \frac{c}{4}\left(nH_{ji} - fg_{ji}\right) + fH_{jr}H^r_i - H_{rs}H^{rs}H_{ji}$$

where $\Delta = \nabla_r\nabla^r$ is the differential operator of Laplace and Beltrami. Then we have the following equation

$$H^{ji}\Delta H_{ji} = H^{ji}\nabla_j\nabla_i f - \frac{c}{4}\left(nH_{ji}H^{ji} - f^2\right) + fH_{jr}H^r_iH^{ji} - \left(H_{ji}H^ji\right)^2,$$

with analogous Chern, Do Cormo and Kobayashi [9]. Using the principal curvatures $k_1, k_2, ..., k_n$ of M we get

$$-\frac{c}{4}\left(nH_{ji}H^{ji}-f^2\right)+fH_{jr}H_i^rH^{ji}-\left(H_{ji}H^{ji}\right)^2=\sum_{i<j}\left(-\frac{c}{4}+k_ik_j\right)(k_i-k_j)^2 .$$

This shows that

$$H^{ji}\Delta H_{ji}=H^{ji}\nabla_j\nabla_i f+\sum_{i<j}\left(-\frac{c}{4}+k_ik_j\right)(k_i-k_j)^2 . \tag{4.6}$$

The familiar equation has been obtained by Nomizu and Smyth [10]. This equation can also be written as follows:

$$\begin{aligned}\frac{1}{2}\Delta\left(H_{ji}H^{ji}\right)-\nabla_j\left(H^{ji}f_i\right) &= \nabla_kH_{ji}\nabla^kH^{ji}-f_if^i\\ &\quad+\sum_{i<j}\left(-\frac{c}{4}+k_ik_j\right)(k_i-k_j)^2 , 4.7\end{aligned} \tag{4.1}$$

where $f_i=\nabla_i f$ and $f^i=g^{ji}f_j$.

5. Riemannian hypersurfaces of Hessian manifolds with constant scalar curvature

In this section, we consider M is a Riemannian hypersurface with constant scalar curvature R in Hessian manifold $M^{'}$. We shall investigate the sign of the right hand side of (4.7). First of all, we consider the first term and the second term of the right hand side of (4.7). By calculating the square of the norm of $f\nabla_kH_{ji}-f_kH_{ji}$, we get

$$\|f\nabla_kH_{ji}-f_kH_{ji}\|^2=f^2\nabla_kH_{ji}\nabla^kH_{ji}-2ff^kH^{ji}\nabla_kH_{ji}+f_kf^kH_{ji}H^{ji}.$$

Then we find from (4.5) that

$$H_{ji}H^{ji}=-\frac{c}{4}n(n-1)+f^2-R$$

and also we find

$$H^{ji}\nabla_kH_{ji}=\nabla_k\left(H_{ji}H^{ji}\right)/2=\nabla_kf^2/2=ff^k,$$

because the scalar curvature R is constant. Eliminating $H_{ij}H^{ji}$ and $H^{ji}\nabla_kH_{ji}$ from these equations, we obtain

$$\begin{aligned}\left\|f\nabla_kH_{ji}\nabla^kH^{ji}-f_kH_{ji}\right\|^2 &= f^2\left(\nabla_kH_{ji}\nabla^kH^{ji}-f_if^i\right)\\ &\quad-\left\{R+\frac{c}{4}n(n-1)\right\}f_if^i.5.1\end{aligned} \tag{5.1}$$

Then let us define a domain D in M as follows: D is the set of points x in M such that

$$\left(\nabla_kH_{ji}\nabla^kH^{ji}-f_if^i\right)(x)<0.$$

Making use of (5.1) following lemma can be proved. Also this lemma gives a sufficient condition for the algebraic sum of the first and second terms of the right hand side in (4.7) to be non-negative.

Lemma 5.1. *Let M be a hypersurface with constant scalar curvature R in $M^{'}$. If $R\geq-\frac{c}{4}n(n-1)$, then the domain D is empty.*

Proof. Suppose that D is not empty. By means of the domain D, follows from (5.1) that

$$0 \geq f^2 \left(\nabla_k H_{ji} \nabla^k H^{ji} - f_i f^i\right) \geq \left\{R + \frac{c}{4} n(n-1)\right\} f_i f^i$$

on D. From the assumption the scalar curvature R is equal to or greater than $-\frac{c}{4}\ n(n-1)$, the inequality above shows that $R = -\frac{c}{4} n(n-1)$ or f_i vanishes identically on D. Firstly we consider the first case in which the scalar curvature is equal to $-\frac{c}{4} n(n-1)$. the left hand side of (5.1) being equal to or greater than zero, we get

$$f^2 \left(\nabla_k H_{ji} \nabla^k H^{ji} - f_i f^i\right) \geq 0.$$

This means that f vanishes on D. Since the domain D is open, we obtain

$$f_i = 0$$

on D.This shows that $R = -\frac{c}{4} n(n-1)$ implies that $f_i = 0$ on D. When $f_i = 0$ on D, the scalar function $\nabla_k H_{ji} \nabla^k H^{ji} - f_i f^i$ is equal to or greater than zero on D. Thus the domain D must be empty. This concludes the proof. □

For each point x in M, let $X_1, X_2,..., X_n$ be an orthonormal frame of the tangent space M_x such that any X_j is an eigenvector of the second fundamental tensor that corresponds to an eigen value k_j. Then by remembering Gauss equation (4.1), the sectional curvature $K(X_i, X_j)$ of the plane spanned by X_i and X_j is given by

$$K(X_i, X_j) = -\frac{c}{4} + k_i k_j. \tag{5.2}$$

From this equation and remembering the right hand side of equation (4.7) we see that if M is non-negative curvature and with constant scalar curvature $R \geq -\frac{c}{4} n(n-1)$, then the right hand side is non-negative.

Lemma 5.2. *Let M be a compact orientable hypersurface of non-negative curvature and with constant scalar curvature R in M'. If $R \geq -\frac{c}{4} n(n-1)$, then there exist at most two distinct principal curvatures, say λ and μ, such that*

$$-\frac{c}{4} + \lambda\mu = 0. \tag{5.3}$$

Proof. Using Lemma 4.1 one can conclude that D is empty. This shows that

$$\nabla_k H_{ji} \nabla^k H^{ji} - f_i f^i \geq 0$$

on M. On the other hand, taking into account of Green's theorem and equation (4.7), we obtain

$$\int_M \left\{\nabla_k H_{ji} \nabla^k H^{ji} - f_i f^i + \sum_{i<j} \left(-\frac{c}{4} + k_i k_j\right)(k_i - k_j)^2\right\} dM = 0$$

dM being the volume element of M. Thus $\nabla_k H_{ji} \nabla^k H^{ji} - f_i f^i$ and $-\frac{c}{4} + k_i k_j$ being both non-negative, we find

$$\nabla_k H_{ji} \nabla^k H^{ji} - f_i f^i = 0 \tag{5.4}$$

and

$$\left(-\frac{c}{4} + k_i k_j\right)(k_i - k_j)^2 = 0 \tag{5.5}$$

for any indices at each point is at most two, and satisfy (5.3). □

Theorem 5.1. *Let $M^{'}$ be a complete and simply connected Hessian $(n+1)-$manifold of constant curvature $-\frac{c}{4}\,(c<0)$ and let M be a connected compact Riemannian $n-$ manifold. Let Φ be an isometric immersion of M into $M^{'}$. Suppose that M is of non-negative curvature and with constant scalar curvature R. If $R \geq -\frac{c}{4}n\,(n-1)$, then (M,Φ) is totally umbilic or there exist exactly two distinct and constant principal curvatures.*

Proof. Using the equations (5.1) and (5.4) and taking into account of the assumption $R \geq -\frac{c}{4}n\,(n-1)$, we conclude that the equation

$$\left\{R+\frac{c}{4}n\,(n-1)\right\} f_i f^i = 0$$

holds. We consider the case in which R is different from $-\frac{c}{4}n\,(n-1)$ and the case in which $R = -\frac{c}{4}n\,(n-1)$, separately. In the first case, f is constant. Accordingly a theorem Nomizu and Smyth [10], the assertion of Theorem 4.1 is true. In the other case, it follows from (4.5) that

$$f^2 - H_{ji}H^{ji} = 0.$$

This equation can be written by using principal curvatures as

$$\sum_{i<j} k_i k_j = 0. \tag{5.6}$$

Also considering the sectional curvature is non negative we see that the scalar curvature R is non-negative and so is the constant curvature $-\frac{c}{4}$. Considering c is equal to zero we conclude that the ambient space $M^{'}$ is an $(n+1)-$ dimensional Euclidean space E^{n+1}. We get by (5.6) and the assumption that the hypersurface is non-negative curvature

$$k_i k_j = 0$$

for any distinct indices i and j. This implies that the type number $t\,(x)$ at each point x in M is equal to 0 or 1. Since M is compact, it is seen that there exists a point in M at which all principal curvatures of M are positive or negative. This contradicts the fact that the type number is equal to 0 or 1. Thus the constant curvature $-\frac{c}{4}$ must be positive. Then let us suppose that there exists a non-umbilical point p in M, at which we have two distinct principal curvatures $\lambda\,(p)$ and $\mu\,(p)$. Then, by means of (5.3) they satisfy $-\frac{c}{4}+\lambda\,(p)\,\mu\,(p)=0$. Hence one is positive and other is negative. Under this situation, there exists a maximal connected open set U consisting of non-umbilic points, which contains p. At each point in U, (M,Φ) has exactly two distinct principal curvatures with constant multiplicities k and $n-k$, respectively. Then equation (5.6) is equivalent to

$$\frac{k\,(k-1)}{2}\lambda^2 + k\,(n-k)\,\lambda\mu + \frac{(n-k)\,(n-k-1)}{2}\mu^2 = 0. \tag{5.7}$$

Considering (5.3) and (5.7), we see that λ and μ are constant on U. This shows that λ is different from μ at the boundary point of U and then by the definition of U, the closure $U^{'}$ of U should be contained in $U <$ Thus U is closed. Since M is connected U is M itself. This completes the proof. □

If the ambient space is Euclidean as a direct consequence of Theorem 4.1 we obtain the following theorem that is similar to Nakagawa and Yokote's results.

Theorem 5.2. *Let M be a connected compact Riemannian $n-$ manifold of non-negative curvature and let Φ be an isometric immersion of M into E^{n+1}. If the scalar curvature R of M is constant, then M is isometric to a sphere S^n and Φ is an imbedding.*

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