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NORMAL VECTOR AS AN EIGENVECTOR OF THE WEINGARTEN MATRIX

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Abstract. In this paper, we study the condition under which the normal space (the linear span of the normal vector) becomes invariant subspace of the Weingarten matrix at preferred points (called pivotal points) on a surface represented as a level set of a function. This condition has been obtained in terms of the critical points of the norm of the gradient of the surface function.

1. Introduction

The Weingarten map at a point p on a 2-dimensional surface S in R^3 is a 3×3 matrix acting as a linear operator $L_p : S_p \to S_p$, where S_p denotes the tangent space to S at p. We consider a preferred point p on S and view the Weingarten map extended as a linear map defined on the affine space $R^3(p)$, having image in $R^3(p)$ with S_p being its invariant subspace (i.e., $L_p : R^3(p) \to R^3(p)$ and $L_p(S_p) \subseteq S_p$). Following the level set representation of S we get a 3×3 matrix representation of L_p at a preferred point (a pivotal point) where the normal vector is directed along one of the coordinate directions. The main purpose of this paper is to study the condition under which the normal space (spanned by the normal vector) becomes invariant subspace of L_p. As the normal space is unidimensional, this amounts to studying the condition under which the normal vector is eigenvector of L_p.

2. Pivot Point And Pivot Vector

Let $f^{-1}(c) = \{(x_1, x_2, x_3) \in U \subset R^3 : f(x_1, x_2, x_3) = c\}$, where $f : U \to R$ and $c \in R$. Thus $f^{-1}(c)$ is a level set. If $\nabla f \neq 0$ at each point, then the level set is called a 2-surface S in R^3. The vector field ∇f is normal to S and the unit normal vector field $N = \frac{\nabla f}{|\nabla f|}$ determines an orientation on S.

The Weingarten map L_p of S at a point p transforms the tangent vector v as the tangent vector $L_p v = -\nabla_v N = -(N \dot{o} \alpha)(t_0)$, where α is a parametrized curve

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$\alpha : I \to S$ such that $\alpha(t_0) = p, \bar{\alpha}(t_o) = v$. It is known that L_p is self-adjoint and

$$(L_p v).w = -\frac{1}{|(\nabla f)(p)|}\sum_{i,j=1}^{n+1}\frac{\partial^2 f}{\partial x_i \partial x_j}(p)v_i w_j \tag{2.1}$$

for any two tangent vectors v, w.

The following lemma given in [1] suggests the idea of a pivot point.

Lemma 2.1. *Let $S = f^{-1}(c)$ be an n-surface in $\Re^{n+1}$, oriented by $\nabla f/\|\nabla f\|$. Suppose $p \in S$ is such that $\nabla f/\|\nabla f\| = e_{n+1}$, and $e_i = (p, 0, \dots, 1, \dots, 0)$ with the one in the $(i+1)^{th}$ spot (i spots after p) for $i \in \{1, \dots, n+1\}$. Then the matrix for L_p with respect to the basis $\{e_1, e_2, \dots, e_n\}$ for S_p is*

$$L_{ij}(p) = \left(-\frac{1}{|\nabla f(p)|}\frac{\partial^2 f}{\partial x_i \partial x_j}(p)\right) \tag{2.2}$$

The lemma can be proven easily by using equation (2.1).

Let us consider a point $p \in S$ such that $N(p) = \frac{\nabla f}{|\nabla f|}(p) = e_3 = (p, 0, 0, 1)$ and so $e_1 = (p, 1, 0, 0), e_2 = (p, 0, 1, 0)$ are two basic tangent vectors at p, and call such a point a *pivot point* on the surface S. Actually, the normal $N(p)$ at a pivot point could be $(p, 1, 0, 0)$ or $(p, 0, 1, 0)$. More generally, we state

Definition 2.1. A point p on the surface S is said to be a pivot point if the normal vector at p is parallel to one of the co-ordinate axes.

Hence, in view of Equation (2.2), the matrix representation of L_p can be written out explicitly as the matrix:

$$L_{ij}(p) = -\frac{1}{|(\nabla f)(p)|}\begin{bmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{bmatrix} \tag{2.3}$$

where f_{ij} is the Hessian $\frac{\partial^2 f}{\partial x_i \partial x_j}$. Henceforth, the matrix $L_{ij}(p)$ will be called *the Weingarten matrix.*

Definition 2.2. The normal vector to the surface S at a pivot point p is said to be a pivot vector if it is an eigenvector of the Weigarten matrix $L_{ij}(p)$ at p.

Let us now seek a condition for N to be a pivot vector, i.e.

$$(L_{ij}N_j)(p) = \lambda N_i(p) \tag{2.4}$$

for a constant λ depending on p. Equation (2.4) is rewritten as the system of equations:

$$\begin{aligned} f_{11}f_1 + f_{12}f_2 + f_{13}f_3 &= -\lambda|\nabla f|f_1 \\ f_{12}f_1 + f_{22}f_2 + f_{23}f_3 &= -\lambda|\nabla f|f_2 \\ f_{13}f_1 + f_{23}f_2 + f_{33}f_3 &= -\lambda|\nabla f|f_3 \end{aligned}$$

and more compactly as

$$\nabla(|\nabla f|^2) = -2\lambda|\nabla f|(\nabla f)$$

holding at p. As $\nabla f(p) \neq 0$, the above equation reduces to:

$$\nabla(|\nabla f|) = -\lambda(\nabla f) \tag{2.5}$$

at p. But this condition means that the point p is a critical point of the function $|\nabla f|$ on S, i.e. $|\nabla f|$ restricted to S is stationary at p. Consequently, we obtain the following result.

Theorem 2.1. *Let S be a 2-surface in R^3, defined as $f^{-1}(c)$ for $f : U \to R$ and $c \in R$. Let p be a pivot point in S. Then $N(p)$ is a pivot vector if and only if $|\nabla f|$ restricted to S is stationary at p.*

Remark: We know that $|\nabla f|$ at a point of S determines the maximum rate of change of f at that point. Theorem 2.1 provides additional interpretation of $|\nabla f|$ at a pivot point.

Finally, we provide some examples of the case when N is a pivot vector and an example when it is not a pivot vector at a pivot point.

3. Examples

Example 1. The unit sphere $S^2 : x_1^2 + x_2^2 + x_3^2 = 1$. Here $f(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2$ and $c = 1$. The point $p = (0, 0, 1)$ is a pivot point because $\nabla f = 2(x_1, x_2, x_3)$ and hence $N(p) = \frac{(\nabla f)(p)}{|\nabla f|(p)} = (0, 0, 1)$. We also have

$$L_{ij}(p) = -\frac{1}{2}\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix},$$

As $N_j(p) = (0, 0, 1)$, we find that $L_{ij}N_j(p) = -N_i(p)$, i.e. $N(p)$ is a pivot vector. Further, we find that $|\nabla f| = 2$ on S^2, i.e. $|\nabla f|$ is trivially stationary at p. Hence, theorem 2.1 re-confirms that $N(p)$ is a pivot vector.

The case of the cylinder $x_1^2 + x_2^2 = 1$ is similar to the sphere.

Example 2. The torus T: $x_3^2 + (\sqrt{x_1^2 + x_2^2} - a)^2 = b^2 (a > b > 0)$. Take the point $p(0, 0, 2b)$ on T. Direct calculation shows that $\nabla f(p) = (0, 0, 2b)$ and hence $N(p) = (0, 0, 1)$, i.e. p is a pivot point. As the computation of the Weingarten matrix is a little tedious, we would like to apply theorem 2.1 directly. Here $|\nabla f| = 2b$ at any point of T, therefore, $|\nabla f|$ restricted to T is constant, hence trivially stationary at p, and $N(p)$ is a pivot vector.

We remark that though $|\nabla f|$ is constant on the surfaces of both Examples 1 and 2, nevertheless not all points in Example 2 are pivot points, whereas all points of Example 1 are pivot points.

Example 3. The paraboloid: $x_3 - x_1^2 - x_2^2 = 0$. Here, $f(x_1, x_2, x_3) = x_3 - x_1^2 - x_2^2$. $\nabla f = (-2x_1, -2x_2, 1)$. At the vertex $p(0, 0, 0)$, $N(p) = (0, 0, 1)$, and hence $p(0, 0, 0)$ is a pivot point. In this case we find that

$$L_{ij}(p) = 2\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

and hence $L_{ij}(p)N_j(p) = 2(0,0,0) = 0N_i(p)$. So, $N(p)$ is a pivot vector. Here $|\nabla f| = \sqrt{1+4x_1^2+4x_2^2}$ which is stationary at $p(0,0,0)$ and hence theorem 2.1 re-confirms that $N(p)$ is a pivot vector.

Example 4. The hyperboloid of 1 sheet: $x_1^2 + x_3^2 - x_2^2 = 1$. $f(x_1, x_2, x_3) = x_1^2 + x_3^2 - x_2^2$. At $p = (0,0,1)$, $N(p) = (0,0,1)$ and $L_{ij}(p)N_j(p) = (0,0,1) = N_i(p)$. Thus $N(p)$ is a pivot vector. Straight computation shows $|\nabla f| = 2\sqrt{x_1^2 + x_2^2 + x_3^2} = 2\sqrt{1+2x_2^2}$ and hence is stationary at $p(0,0,1)$. Thus, theorem 2.1 re-confirms the pivotality of $N(p)$. We may note here that any point p on the neck ($x_2 = 0$) is a pivot point (This can be viewed by suitably rotating the plane $x_3 = 0$ about the central x_3 axis, however, retaining the function f as it is) and $N(p)$, the pivot vector.

Example 5. The surface defined by: $x_1x_3 + x_2x_3 + x_1x_2 = -\frac{1}{4}$ over $R^3 - (0,0,0)$. Here, $f(x_1, x_2, x_3) = x_1x_3 + x_2x_3 + x_1x_2$. $\nabla f = (x_2 + x_3, x_3 + x_1, x_1 + x_2)$. At the point $p = (1/2, 1/2, -1/2)$, $(\nabla f)(p) = (0,0,1)$. Hence p is a pivot point. A straightforward calculation shows that $L_{ij}(p)N_j(p) = -(1,1,0)$ and so, $N(p)$ *is not a pivot vector*. Furthermore,

$$\begin{aligned}
|\nabla f|^2 &= 2(x_1^2 + x_2^2 + x_3^2 + x_1x_2 + x_2x_3 + x_3x_1) \\
&= 2((x_1 + x_2 + x_3)^2 + \frac{1}{4})
\end{aligned}$$

on the surface. Hence $|\nabla f|$ restricted to the surface is minimum (hence stationary) when $x_1 + x_2 + x_3 = 0$. As the pivot point $p = (1/2, 1/2, -1/2)$ does not satisfy this condition, $|\nabla f|$ restricted to the surface is not stationary at p. Hence theorem 2.1 re-confirms that $N(p)$ *is not a pivot vector*.

Concluding Remark: For surfaces of Examples 1 (including, of course, cylinder and plane) and 2, $|\nabla f|$ is constant on the surfaces. This raises the following question "Does there exist surfaces other than these surfaces for which $|\nabla f|$ is constant on them?"

References

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