

D'ALEMBERT'S SOLUTION OF THE INITIAL VALUE PROBLEM FOR THE THIRD-ORDER LINEAR HYPERBOLIC EQUATION

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ABSTRACT

It is well known that the famous D'Alembert formula for solving the wave equation of second-order is a very important instrument in the study of the dynamics of waves. It is also obvious that D'Alembert's solutions for higher-order partial differential equations are of great importance. In this paper, the D'Alembert solutions of the Cauchy problem for linear partial differential equations with homogeneous constant coefficients of the third-order are obtained. Finally, using the obtained solutions, some computer tests on three distinct roots have been carried out. The results clearly indicate the dispersion dynamics of waves with some initial profile.

Keywords: *D'Alembert's solution, third-order hyperbolic equation, characteristic equation*

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ÜÇÜNCÜ MERTEBEYE GÖRE HOMOJEN DENKLEM İÇİN BAŞLANGIÇ DEĞER PROBLEMİNİN D'ALEMBERT ÇÖZÜMÜ

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ÖZ

İkinci mertebeden dalga denklemini çözümü için ünlü D'Alembert formülünün, dalgaların dinamiğini incelemek açısından çok önemli bir araç olduğu iyi bilinmektedir. Yüksek mertebeden kısmi türevli diferansiyel denklemler için de D'Alembert tipinden çözümlerin elde edilmesinin büyük önem taşıdığı açıktır. Bu makalede üçüncü mertebeye göre homojen sabit katsayılı lineer kısmi diferansiyel denklemler için Cauchy probleminin D'Alembert çözümleri ele alınmıştır. Son olarak, elde edilen çözümler kullanılarak, üç farklı kök durumunda bazı bilgisayar testleri yapılmıştır. Bulunan sonuçlar belli başlangıç profile sahip dalgaların dağılım dinamiklerini açıkça ifade etmektedir.

Anahtar Kelimeler: D'alembert tipli çözüm, üçüncü mertebeden hiperbolik denklem, karakteristik denklem

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1. Introduction

Many problems that are often encountered in engineering and science are reduced to the initial value problems for partial differential equations [3],[7],[9],[12]. Since the solution of the initial value problem has both theoretical and practical importance, the solution of the mentioned problems is thoroughly researched in the literature. In these studies, the existence and uniqueness of solutions of these problems are analyzed by functional analysis method [4],[5],[8],[11].

In the works of authors such as Courant, Petrovski, Tikhonov and Samarski, the solution of the Cauchy problem for higher-order partial differential equations is obtained by Fourier transformation [1],[9],[12]. However, it is difficult to calculate the special integrals in these expressions. For this reason, in order to solve the higher-order partial differential equations, it is necessary to obtain solutions that can be used easily from a practical point of view.

It is also known in the literature that there is a theory on the partial differential equations with constant coefficients [2],[3],[6],[9],[10].

In this article, D’Alembert’s solutions have been obtained for the solution of the Cauchy problem for the constant-coefficients linear homogeneous hyperbolic type equation with respect to the third-order derivative. D’Alembert’s solutions allow proving the existence, uniqueness and continuous dependence on the initial data of the solution of the problem as well as provide easy-to-use expressions from a practical point of view. In upper half plane of $R^2(x, t)$ Euclid space we consider for unknown function $u = u(x,t)$ the following third-order equation

$$a_0 \frac{\partial^3 u}{\partial t^3} + a_1 \frac{\partial^3 u}{\partial t^2 \partial x} + a_2 \frac{\partial^3 u}{\partial t \partial x^2} + a_3 \frac{\partial^3 u}{\partial x^3} = 0 \quad (1)$$

with the following initial conditions

$$\frac{\partial^k u(x,t)}{\partial t^k} \Big|_{t=0} = \phi_k(x), \quad (k = 0,1,2) \quad (2)$$

where the functions $\phi_k(x)$, $(k = 0,1,2)$ are known and the constants a_i , $(i = 0,1,2)$ are certain constants. In order to solve the problem (1), (2) we investigate a solution of the form $u(x,t) = \phi(x+\lambda t)$, where ϕ is a three times continuously differentiable function and λ is an unknown constant. The speed of traveling wave λ is obtained from the following equation

$$a_0 \lambda^3 + a_1 \lambda^2 + a_2 \lambda + a_3 = 0 \quad (3)$$

which is called as characteristic polynomial. Equation (3) is the third-order algebraic equation. According to Fundamental Theorem of Algebra, it has exactly three roots.

Hence, one of the following cases occurs:

- 1^o. $\lambda_1 \neq \lambda_2 \neq \lambda_3$ i.e., the roots are real and distinct,
- 2^o. $\lambda_1 = \lambda_2 = \lambda_3$ i.e., $\lambda = \lambda_1$ is a root of multiplicity 3,
- 3^o. $\lambda_1 \neq \lambda_2 = \lambda_3$ i.e., $\lambda = \lambda_1$ is a distinct root and λ_2 is a double root.

1.1 Three Distinct Roots Case

Suppose that equation (3) has three simple real roots λ_i , $(i = 1,2,3)$. In this case, it can easily be shown that the functions $\varphi_1(x + \lambda_1 t)$, $\varphi_2(x + \lambda_2 t)$ and $\varphi_3(x + \lambda_3 t)$ satisfy equation (1). Since equation (1) is linear, the function

$$u(x,t) = \varphi_1(x + \lambda_1 t) + \varphi_2(x + \lambda_2 t) + \varphi_3(x + \lambda_3 t) \quad (4)$$

is clearly a general solution of (1). By replacing (4) with (2), we have the following system of algebraic equations for the unknown functions φ_1 , φ_2 and φ_3

$$\begin{cases} \varphi_1(x) + \varphi_2(x) + \varphi_3(x) = \phi_0(x), \\ \lambda_1 \varphi_1'(x) + \lambda_2 \varphi_2'(x) + \lambda_3 \varphi_3'(x) = \phi_1(x), \\ \lambda_1^2 \varphi_1''(x) + \lambda_2^2 \varphi_2''(x) + \lambda_3^2 \varphi_3''(x) = \phi_2(x). \end{cases} \quad (5)$$

By integrating the second equation once and the third equation twice of system (5) with respect to x and taking into account the Cauchy formula, we obtain the following algebraic equations system for unknown functions φ_1 , φ_2 and φ_3

$$\begin{cases} \varphi_1(x) + \varphi_2(x) + \varphi_3(x) = \phi_0(x), \\ \lambda_1 \varphi_1(x) + \lambda_2 \varphi_2(x) + \lambda_3 \varphi_3(x) = \int_0^x \phi_1(\xi) d\xi + C_{20}, \\ \lambda_1^2 \varphi_1(x) + \lambda_2^2 \varphi_2(x) + \lambda_3^2 \varphi_3(x) = \int_0^x (\alpha - \xi) \phi_2(\xi) d\xi + C_{30}x + C_{31}. \end{cases}$$

Here, C_{20} , C_{30} and C_{31} are any constants. The following determinant consisting of the coefficients of the system is nonzero,

$$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ \lambda_1 & \lambda_2 & \lambda_3 \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 \end{vmatrix} = (\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)(\lambda_3 - \lambda_2)$$

because λ_i , $(i = 1,2,3)$ are different from each other. Therefore, the system has only one solution. By Cramer’s method we get

$$\varphi_1(x) = \frac{1}{\Delta} \begin{vmatrix} \phi_0(x) & 1 & 1 \\ \int_0^x \phi_1(\xi) d\xi & \lambda_2 & \lambda_3 \\ \int_0^x (\alpha - \xi) \phi_2(\xi) d\xi & \lambda_2^2 & \lambda_3^2 \end{vmatrix} + \frac{1}{\Delta} \begin{vmatrix} 0 & 1 & 1 \\ C_{20} & \lambda_2 & \lambda_3 \\ C_{30}x + C_{31} & \lambda_2^2 & \lambda_3^2 \end{vmatrix} \quad (6)$$

$$\varphi_2(x) = \frac{1}{\Delta} \begin{vmatrix} 1 & \phi_0(x) & 1 \\ \lambda_1 & \int_0^x \phi_1(\xi) d\xi & \lambda_3 \\ \lambda_1^2 & \int_0^x (\alpha - \xi) \phi_2(\xi) d\xi & \lambda_3^2 \end{vmatrix} + \frac{1}{\Delta} \begin{vmatrix} 1 & 0 & 1 \\ \lambda_1 & C_{20} & \lambda_3 \\ \lambda_1^2 & C_{30}x + C_{31} & \lambda_3^2 \end{vmatrix} \quad (7)$$

$$\varphi_3(x) = \frac{1}{\Delta} \begin{vmatrix} 1 & 1 & \phi_0(x) \\ \lambda_1 & \lambda_2 & \int_0^x \phi_1(\xi) d\xi \\ \lambda_1^2 & \lambda_2^2 & \int_0^x (\alpha - \xi) \phi_2(\xi) d\xi \end{vmatrix} + \frac{1}{\Delta} \begin{vmatrix} 1 & 1 & 0 \\ \lambda_1 & \lambda_2 & C_{20} \\ \lambda_1^2 & \lambda_2^2 & C_{30}x + C_{31} \end{vmatrix} \quad (8)$$

For the second determinants in formulas (6)-(8), it is easily seen that

$$\begin{vmatrix} 0 & 1 & 1 \\ C_{20} & \lambda_2 & \lambda_3 \\ C_{30}x + C_{31} & \lambda_2^2 & \lambda_3^2 \end{vmatrix} + \begin{vmatrix} 1 & 0 & 1 \\ \lambda_1 & C_{20} & \lambda_3 \\ \lambda_1^2 & C_{30}x + C_{31} & \lambda_3^2 \end{vmatrix} + \begin{vmatrix} 1 & 1 & 0 \\ \lambda_1 & \lambda_2 & C_{20} \\ \lambda_1^2 & \lambda_2^2 & C_{30}x + C_{31} \end{vmatrix} = 0.$$

In this case, taking into account (4) for the solution of problem (1), (2) we get

$$u(x, t) = \frac{1}{\Delta} \left\{ \begin{array}{l} \int_0^{x+\lambda_1 t} \phi_0(\xi) d\xi \\ \int_0^{x+\lambda_1 t} (x + \lambda_1 t - \xi) \phi_2(\xi) d\xi \end{array} \right. \begin{array}{l} 1 \\ \lambda_2^2 \end{array} \begin{array}{l} 1 \\ \lambda_3^2 \end{array} \left. + \begin{array}{l} 1 \\ \lambda_1 \end{array} \begin{array}{l} \int_0^{x+\lambda_2 t} \phi_1(\xi) d\xi \\ \int_0^{x+\lambda_2 t} (x + \lambda_2 t - \xi) \phi_2(\xi) d\xi \end{array} \right. \begin{array}{l} 1 \\ \lambda_3 \end{array} \left. + \begin{array}{l} 1 \\ \lambda_1 \end{array} \begin{array}{l} \int_0^{x+\lambda_3 t} \phi_1(\xi) d\xi \\ \int_0^{x+\lambda_3 t} (x + \lambda_3 t - \xi) \phi_2(\xi) d\xi \end{array} \right\} \quad (9)$$

Expression (9) indicates that the solution of problem (1), (2) is an expansion of D'Alembert's formula.

1.2 Three-Multiple Root Case

Now, we assume that the characteristic equation corresponding to equation (1) has three-multiple root denoted by λ_1 . In this case, the general solution of problem (1), (2) has the following form

$$u(x, t) = \varphi_1(x + \lambda_1 t) + t\varphi_2(x + \lambda_1 t) + t^2\varphi_3(x + \lambda_1 t).$$

The necessity of satisfying the initial conditions (2), we obtain the algebraic equations system for the unknown functions φ_1, φ_2 and φ_3 . From this system, we have

$$\begin{aligned} \varphi_1(x) &= \phi_0(x), & \varphi_2(x) &= \phi_1(x) - \lambda_1 \phi_0'(x), \\ \varphi_3(x) &= \frac{1}{2} [\phi_2(x) - 2\lambda_1 \phi_1'(x) + \lambda_1^2 \phi_0''(x)]. \end{aligned}$$

The calculated values of these functions are replaced by expression (10), we get

$$u(x, t) = \phi_0(x + \lambda_1 t) - t\lambda_1 \phi_0'(x + \lambda_1 t) + \frac{t^2 \lambda_1^2}{2} \phi_0''(x + \lambda_1 t) + t\phi_1(x + \lambda_1 t) - t^2 \lambda_1 \phi_1'(x + \lambda_1 t) + \frac{t^2}{2} \phi_2(x + \lambda_1 t). \quad (11)$$

It is clear that (11) demonstrates the superposition of simple waves scattered at $-\lambda$ speed.

1.3 A Repeated and A Distinct Root Case

In this section, we assume that the characteristic equation corresponding to equation (1) has one pair of repeated roots and a distinct root. Let λ_1 be a distinct root and λ_2 be a repeated root. In this case, the general solution of equation (1) has the form

$$u(x, t) = \varphi_{11}(x + \lambda_1 t) + \varphi_{21}(x + \lambda_2 t) + t\varphi_{22}(x + \lambda_2 t). \quad (12)$$

Using initial conditions (2) to find the unknown functions $\varphi_{11}, \varphi_{21}$ and φ_{22} in equation (12), the following system of algebraic equations is obtained

$$\begin{cases} \varphi_{11}(x) + \varphi_{21}(x) = \phi_0(x), \\ \lambda_1 \varphi'_{11}(x) + \lambda_2 \varphi'_{21}(x) + \varphi_{22}(x) = \phi_1(x), \\ \lambda_1^2 \varphi''_{11}(x) + \lambda_2^2 \varphi''_{21}(x) + 2\lambda_2 \varphi'_{22}(x) = \phi_2(x). \end{cases} \quad (13)$$

By differentiating the first equation and integrating the third equation of system (13) with respect to x , we obtain the following algebraic equations system for unknown functions $\varphi'_{11}(x), \varphi'_{21}(x)$ and $\varphi_{22}(x)$

$$\begin{cases} \varphi'_{11}(x) + \varphi'_{21}(x) = \phi_0'(x), \\ \lambda_1 \varphi'_{11}(x) + \lambda_2 \varphi'_{21}(x) + \varphi_{22}(x) = \phi_1(x), \\ \lambda_1^2 \varphi'_{11}(x) + \lambda_2^2 \varphi'_{21}(x) + 2\lambda_2 \varphi_{22}(x) = \int_0^x \phi_2(\xi) d\xi + C_{30}. \end{cases} \quad (14)$$

Here C_{30} is a constant. Calculating the determinant of the coefficients, we get

$$\Delta = \begin{vmatrix} 1 & 1 & 0 \\ \lambda_1 & \lambda_2 & 1 \\ \lambda_1^2 & \lambda_2^2 & 2\lambda_2 \end{vmatrix} = (\lambda_1 - \lambda_2)^2$$

Since $\lambda_1 \neq \lambda_2$, then $\Delta \neq 0$. Using Cramer's rule again, we have the solution of the system as follows:

$$\varphi'_{11}(x) = \frac{1}{\Delta} \begin{vmatrix} \phi_0'(x) & 1 & 0 \\ \phi_1(x) & \lambda_2 & 1 \\ \int_0^x \phi_2(\xi) d\xi + C_{30} & \lambda_2^2 & 2\lambda_2 \end{vmatrix} = \frac{1}{\Delta} [\lambda_2^2 \phi_0'(x) - 2\lambda_2 \phi_1(x) + \int_0^x \phi_2(\xi) d\xi + C_{30}], \quad (15)$$

$$\varphi'_{21}(x) = \frac{1}{\Delta} \begin{vmatrix} 1 & \phi_0'(x) & 0 \\ \lambda_1 & \phi_1(x) & 1 \\ \lambda_1^2 & \int_0^x \phi_2(\xi) d\xi + C_{30} & 2\lambda_2 \end{vmatrix} = \frac{1}{\Delta} [(\lambda_2^2 - 2\lambda_1 \lambda_2) \phi_0'(x) + 2\lambda_2 \phi_1(x) - \int_0^x \phi_2(\xi) d\xi - C_{30}], \quad (16)$$

$$\varphi_{22}(x) = \frac{1}{\Delta} \begin{vmatrix} 1 & 1 & \phi_0'(x) \\ \lambda_1 & \lambda_2 & \phi_1(x) \\ \lambda_1^2 & \lambda_2^2 & \int_0^x \phi_2(\xi)d\xi + C_{30} \end{vmatrix}$$

$$= \frac{1}{\Delta} [\lambda_1 \lambda_2 (\lambda_2 - \lambda_1) \phi_0'(x) - (\lambda_2^2 - \lambda_1^2) \phi_1(x) + (\lambda_2 - \lambda_1) \int_0^x \phi_2(\xi)d\xi + (\lambda_2 - \lambda_1) C_{30}] \quad (17)$$

Integrating the expressions in (15)-(17), we also find the functions $\varphi_{11}(x)$ and $\varphi_{21}(x)$

$$\varphi_{11}(x) = \frac{1}{\Delta} [\lambda_2^2 \phi_0(x) - 2\lambda_2 \int_0^x \phi_1(\xi)d\xi + \int_0^x (x-\xi)\phi_2(\xi)d\xi + C_{30}x + C_{11}]$$

$$\varphi_{21}(x) = \frac{1}{\Delta} [(\lambda_2^2 - 2\lambda_1\lambda_2)\phi_0(x) + 2\lambda_2 \int_0^x \phi_1(\xi)d\xi - \int_0^x (x-\xi)\phi_2(\xi)d\xi - C_{30}x + C_{21}]$$

respectively. When the obtained values of functions φ_{11} , φ_{21} and φ_{22} are replaced in (12), we take the following representation for the solution of problem (1),(2)

$$u(x,t) = \frac{1}{\Delta} [\lambda_2^2 \phi_0(x + \lambda_1 t) - 2\lambda_2 \int_0^{x+\lambda_1 t} \phi_1(\xi)d\xi + \int_0^{x+\lambda_1 t} (x + \lambda_1 t - \xi)\phi_2(\xi)d\xi + C_{30}x + C_{11} + \frac{1}{\Delta} (\lambda_2^2 - 2\lambda_1\lambda_2)\phi_0(x + \lambda_2 t) + 2\lambda_2 \int_0^{x+\lambda_2 t} \phi_1(\xi)d\xi - \int_0^{x+\lambda_2 t} (x + \lambda_2 t - \xi)\phi_2(\xi)d\xi - C_{30}x + C_{21}] + \frac{1}{\Delta} [\lambda_1 \lambda_2 (\lambda_2 - \lambda_1) \phi_0'(x + \lambda_2 t) - (\lambda_2^2 - \lambda_1^2) \phi_1(x + \lambda_2 t) + (\lambda_2 - \lambda_1) \int_0^{x+\lambda_2 t} \phi_2(\xi)d\xi + (\lambda_2 - \lambda_1) C_{30}]$$

2. Computer Tests

We now obtain the graph of the solution of problem (1), (2), which is the expansion of the D'Alembert's formula in (9). In order to solve equation (1), let the coefficients be $a_0 = 1$, $a_1 = -2$, $a_2 = -1$ and $a_3 = 2$. As easily seen, the roots of the characteristic polynomial corresponding to the equation are $\lambda_1 = 1$, $\lambda_2 = 2$ and $\lambda_3 = -1$. Thus, the determinant consisting of the coefficients of system $\Delta = 6$, is nonzero.

Similar to case of three distinct roots, we get the functions φ_1, φ_2 and φ_3

$$\varphi_1(x) = \frac{1}{6} \begin{vmatrix} \phi_0(x) & 1 & 1 \\ \int_0^x \phi_1(\xi)d\xi & 2 & -1 \\ \int_0^x (x-\xi)\phi_2(\xi)d\xi & 4 & 1 \end{vmatrix} + \frac{1}{6} \begin{vmatrix} 0 & 1 & 1 \\ C_{20} & 2 & -1 \\ C_{30}\xi + C_{31} & 4 & 1 \end{vmatrix}$$

$$\varphi_2(x) = \frac{1}{6} \begin{vmatrix} \phi_0(x) & 1 & 1 \\ \int_0^x \phi_1(\xi)d\xi & -1 & 1 \\ \int_0^x (x-\xi)\phi_2(\xi)d\xi & 1 & 1 \end{vmatrix} + \frac{1}{6} \begin{vmatrix} 0 & 1 & 1 \\ C_{20} & -1 & 1 \\ C_{30}\xi + C_{31} & 1 & 1 \end{vmatrix}$$

$$\varphi_3(x) = \frac{1}{6} \begin{vmatrix} 1 & 1 & \phi_0(x) \\ 1 & 2 & \int_0^x \phi_1(\xi)d\xi + C_{20} \\ 1 & 4 & \int_0^x (x-\xi)\phi_2(\xi)d\xi \end{vmatrix} + \frac{1}{6} \begin{vmatrix} 1 & 1 & 0 \\ 1 & 2 & C_{20} \\ 1 & 4 & C_{30}\xi + C_{31} \end{vmatrix}$$

respectively, where C_{20} , C_{30} and C_{31} are constants. Since

$$\begin{vmatrix} 0 & 1 & 1 \\ C_{20} & 2 & -1 \\ C_{30}\xi + C_{31} & 4 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 1 & \phi_0(x) \\ 1 & 2 & \int_0^x \phi_1(\xi)d\xi + C_{20} \\ 1 & 4 & \int_0^x (x-\xi)\phi_2(\xi)d\xi \end{vmatrix} + \begin{vmatrix} 1 & 1 & 0 \\ 1 & 2 & C_{20} \\ 1 & 4 & C_{30}\xi + C_{31} \end{vmatrix} = 3C_{20} - 3C_{30}\xi - 3C_{31} + 2C_{30}\xi + 2C_{31} + C_{30}\xi + C_{31} - 3C_{20} = 0$$

then the general solution of homogeneous equation

$$u(x,t) = \frac{1}{6} \left\{ \begin{vmatrix} \phi_0(x+t) & 1 & 1 \\ \int_0^{x+t} \phi_1(\xi)d\xi & 2 & -1 \\ \int_0^{x+t} (x+t-\xi)\phi_2(\xi)d\xi & 4 & 1 \end{vmatrix} + \begin{vmatrix} \phi_0(x-2t) & 1 & 1 \\ \int_0^{x-2t} \phi_1(\xi)d\xi & -1 & 1 \\ \int_0^{x-2t} (x+2t-\xi)\phi_2(\xi)d\xi & 1 & 1 \end{vmatrix} + \begin{vmatrix} \phi_0(x-t) & 1 & 1 \\ \int_0^{x-t} \phi_1(\xi)d\xi & 1 & 1 \\ \int_0^{x-t} (x-t-\xi)\phi_2(\xi)d\xi & 4 & 1 \end{vmatrix} \right\}$$

or more clearly

$$u(x,t) = \frac{1}{6} [6\phi_0(x+t) - 2\phi_0(x+2t) + 2\phi_0(x-t)] + \frac{1}{6} \left\{ 3 \int_0^{x+t} \phi_1(\xi)d\xi - 3 \int_0^{x-2t} \phi_1(\xi)d\xi + \int_0^{x-t} \phi_1(\xi)d\xi - 3 \int_0^{x-t} (x-t-\xi)\phi_2(\xi)d\xi + \int_0^{x-2t} (x+2t-\xi)\phi_2(\xi)d\xi + \int_0^{x-t} (x-t-\xi)\phi_2(\xi)d\xi \right\} \quad (18)$$

Now, we take into account equation (18) under the following conditions: The initial profile, velocity and acceleration are

$$\phi_0(x) = \begin{cases} 0, & x < -2, \\ \frac{x+2}{2}, & -2 \leq x \leq 0, \\ \frac{x-2}{-2}, & 0 \leq x \leq 2, \\ 0, & x > 2, \end{cases} \quad (19)$$

$$\phi_1(x) = \begin{cases} 0, & x < -2, \\ 2, & -2 \leq x \leq 2, \\ 0, & x > 2 \end{cases} \quad (20)$$

and $\phi_2(x) = c$ respectively, where c is a constant. In order to see that the solution of (18) consists of two waves travelling to right and to left, firstly, we take the functions $\phi_1(x)$

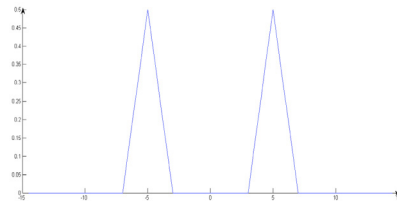


Figure 1. The solution of equation (18), $(\phi_1(x) = \phi_2(x) = 0)$

and $\phi_2(x)$ both identically zero. Figure 1 shows that the solution graph is the same as the D'Alembert formula for the classical vibration of string equation. Indeed, if the roots of characteristic equation corresponding to equation (3) are $\lambda_1 = a$ and $\lambda_2 = -a$, then $\Delta = \begin{vmatrix} 1 & 1 \\ a & -a \end{vmatrix} = -2a$. In that case, we put them in equation (9)

$$u(x,t) = -\frac{1}{2} \left\{ \begin{vmatrix} \phi_0(x+at) & 1 & 1 \\ \int_0^{x+at} \phi_1(\xi)d\xi & a & -a \\ \int_0^{x+at} (x+at-\xi)\phi_2(\xi)d\xi & 1 & 1 \end{vmatrix} + \begin{vmatrix} \phi_0(x-at) & 1 & 1 \\ \int_0^{x-at} \phi_1(\xi)d\xi & -a & a \\ \int_0^{x-at} (x-at-\xi)\phi_2(\xi)d\xi & 1 & 1 \end{vmatrix} \right\}$$

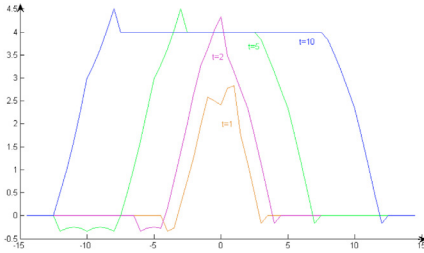


Figure 2. The solution of equation (18), $(\phi_2(x) = c < 0)$

so we get the classical D'Alembert formula, [1],[9]

$$u(x, t) = \frac{\phi_0(x + at) + \phi_0(x - at)}{2} + \frac{1}{2a} \int_{x-at}^{x+at} \phi_1(\xi) d\xi.$$

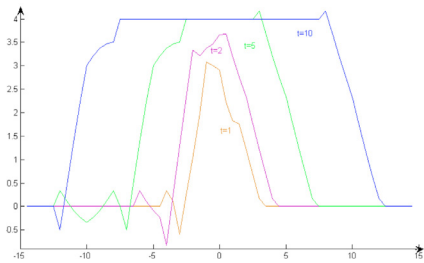


Figure 3. The solution of equation (18), $(\phi_2(x) = c > 0)$

When the initial acceleration function $\phi_2(x)$ is taken into account, no waves moving to the right and left will occur. When $c < 0$ and $c > 0$, Figure 2 and 3 are obtained, respectively.

3. Conclusion

The analytical solutions of the Cauchy problem written for the hyperbolic equation have been obtained by taking into account the simple and multiple of the roots of the characteristic equation. In the case of three distinct roots, it is shown that the solution graph of the equation is the same as the D'Alembert formula for the classical vibration of string equation. The solutions here allow the study of the propagation dynamics of waves in some mediums.

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