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On Stability of Fractional Differential Equations with Lyapunov Functions

Erdal KORKMAZ[1], Abdulhamit ÖZDEMİR[1]🖉
[1]Department of Mathematics, Faculty of Science and Literature, Muş, Turkey
🖂: ayna_hamit@hotmail.com

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ABSTRACT

We discuss the asymptotic stability of autonomous nonlinear fractional order systems, in which the state equations contain integer derivative and fractional order. We use the Lyapunov's second method to derive some sufficient conditions to ensure asymptotic stability of nonlinear fractional order differential equations. We also give two examples in order to consolidate the obtained results.

Keywords: Asymptotic Stability, Fractional Differential Equations, Lyapunov Functional

Lyapunov Fonksiyonları ile Fraksiyonel Diferansiyel Denklemlerin Kararlılığı

ÖZ

Bu çalışmada, fraksiyonel ve tamsayı mertebe içeren lineer olmayan otonom diferansiyel denklem sistemlerinin asimptotik kararlılığı araştırıldı. Lineer olmayan otonom fraksiyonel sistemlerin asimptotik kararlılığını göstermek için bazı yeterli şartlar elde edilerek Lyapunov'un ikinci metodu kullanıldı. Ayrıca elde edilen sonuçları pekiştirmek için iki örnek verildi.

Anahtar Kelimeler: Asimptotik Kararlılık, Fraksiyonel Diferansiyel Denklemler, Lyapunov Fonksiyonları,

INTRODUCTION

Fractional calculus is known as the generalization of the traditional integer-order calculus. It has attracted the attention of many authors due to its increasing application fields in recent years in many areas of science and engineering [1-5]. Studies on fractional derivatives and fractional integrals have been particularly conducted via fractional calculus. Thus, widely accepted calculations and interesting results have been obtained [6-20]. We studied the stability of the solutions of these equations by establishing differential equation models from a new fractional-order motivated by these studies considering the Lyapunov's Second Method, to determine the stability behavior of solutions of certain nonlinear fractional differential equations. The major advantage of this method is that stability can be obtained without any prior knowledge of solutions. Although A. M. Lyapunov, who introduced this method in 1892, used it only to establish simple stability theorems, his basic ideas have been extensively exploited and effectively applied to entirely new problems in physics and engineering for 40 years [8].

In this paper, we consider the nonlinear fractional differential equatios

$$x' + {}_0D_t^{\alpha} x(t) = f\big(x(t)\big), \ t \geq 0 \quad (1)$$

$$x''(t) + g\big(x'(t)\big) + {}_0D_t^{\alpha} x'(t) = f\big(x(t)\big), \quad t \geq 0 \quad (2)$$

where $0 < \alpha < 1$, ${}_0D_t^{\alpha} x(t)$ denotes Caputo's fractional derivative with the lower limit 0 for the function $x(t)$, $f, g \in C^1(\Omega)$, with $f(0) = g(0) = 0$, $\Omega \subset R$ is a domain that contains the origin $x = 0$.

Definition 1.1. ([2]). Given an interval $[a, b]$ of R, the fractional order integral of a function $f \in L^1([a, b], R)$ of order $\alpha \in R^+$ is defined by

$$ {}_aI_t^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s) ds, \quad t \in [a, b], \quad \alpha > 0 \quad (3)$$

where Γ is the Gamma function.

Definition 1.2. ([2]). Suppose that a function f is defined on the interval $[a, b]$. Caputo's fractional derivative of order α with lower limit a for f is defined as

$$ {}_aD_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-s)^{n-\alpha-1} f^{(n)}(s) ds = I_t^{n-\alpha} f^{(n)}(t), \quad t \in [a, b] \quad (4)$$

where $0 < n-1 < \alpha \leq n$.

Particularly, when $0 < \alpha \leq 1$, it holds

$$ {}_aD_t^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \int_a^t (t-s)^{-\alpha} \dot{f}(s) ds = {}_aI_t^{1-\alpha} \dot{f}(t), \quad t \in [a, b] \quad (5)$$

Lemma 1.1. ([2]). Let $0 < \alpha < 1$, $p \geq \alpha$ and $f(t)$ be continuous on $[a, +\infty)$.Then it holds

$$ {}_aD^{p}\big({}_aI_t^{\alpha} f(t)\big) = {}_a D_t^{p-\alpha} f(t). \quad (6)$$

Lemma 1.2. ([19]). Let $0 < \alpha < 1$ and $f(t) \geq 0$ on $[a, b]$. Then it holds

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$${}_{a}I_t^{\alpha} f(t) \geq 0, \quad t \in [a,b]. \tag{7}$$

We quote a result of existence and uniqueness of the global solution for the nonlinear fractional dynamical system

$$\begin{cases} {}_{a}D_t^{\alpha} x(t) = f\big(t, x(t)\big) \\ x(0) = x_0 \end{cases} \tag{8}$$

which is a precondition in the development of this paper.

Lemma 1.3. ([19]). Consider the system (8). Suppose $0 < \alpha < 1$, $\Omega \subset R$ is a domain that contains the origin $x = 0, x_0 \in \Omega$. Suppose further $f(t,x): [0,+\infty) \times \Omega \to R$ is continuous and satisfies a Lipschitz condition in x with a Lipschitz constant $L > 0$. Then there exists a unique function $x(t) \in C[0,+\infty)$ satisfying the system (8).

Lemma 1.4. ([18]). Let $x(t) \in R$ be a continuous and derivable function. Then, for any time instant $t \geq t_0$

$$\frac{1}{2}\ {}_{t_0}D_t^{\alpha} x^2(t) \leq x(t)\, {}_{t_0}D_t^{\alpha} x(t), \quad \forall \alpha \in (0,1). \tag{9}$$

Proof. Proving that expression (9) is true, is equivalent to prove that

$$x(t)\, {}_{t_0}D_t^{\alpha} x(t) - \frac{1}{2}\ {}_{t_0}D_t^{\alpha} x^2(t) \geq 0, \quad \forall \alpha \in (0,1) \tag{10}$$

Using Definition 1.2. it can be written that

$${}_{t_0}^{c}D_t^{\alpha} x(t) = \frac{1}{\Gamma(1-\alpha)} \int_{t_0}^{t} \frac{x'(\tau)}{(t-\tau)^{\alpha}} d\tau. \tag{11}$$

And in the same way

$$\frac{1}{2}\ {}_{t_0}^{c}D_t^{\alpha} x^2(t) = \frac{1}{\Gamma(1-\alpha)} \int_{t_0}^{t} \frac{x(\tau)x'(\tau)}{(t-\tau)^{\alpha}} d\tau. \tag{12}$$

So, expression (10) can be written as

$$\frac{1}{\Gamma(1-\alpha)} \int_{t_0}^{t} \frac{|x(t)-x(\tau)|x'(\tau)}{(t-\tau)^{\alpha}} d\tau \geq 0. \tag{13}$$

Let us define the auxiliar variable $y(\tau) = x(t) - x(\tau)$, which implies that $y'(\tau) = \frac{dy(\tau)}{d\tau} = -\frac{dx(\tau)}{d\tau}$. In this way, expression (13) can be written as

$$\frac{1}{\Gamma(1-\alpha)} \int_{t_0}^{t} \frac{y(\tau)y'(\tau)}{(t-\tau)^{\alpha}} d\tau \leq 0. \tag{14}$$

Let us integrate by parts expression (14), defining

$$du = y(\tau)y'(\tau)d\tau \qquad u = \frac{1}{2}y^2$$

$$v = \frac{1}{\Gamma(1-\alpha)}(t-\tau)^{-\alpha} \quad dv = \frac{\alpha}{\Gamma(1-\alpha)}(t-\tau)^{-\alpha-1}$$

In that way, expression (14) can be written as

$$-\left[\frac{y^2(\tau)}{2\Gamma(1-\alpha)(t-\tau)^{\alpha}}\right]\Bigg|_{\tau=t} + \left[\frac{y_0^2}{2\Gamma(1-\alpha)(t-t_0)^{\alpha}}\right] + \\ + \frac{\alpha}{2\Gamma(1-\alpha)} \int_{t_0}^{t} \frac{y^2(\tau)}{(t-\tau)^{\alpha+1}} d\tau \geq 0 \tag{15}$$

Let us check the first term of expression (15), which has an indetermination at $\tau = t$ so let us analyze the corresponding limit

$$\lim_{\tau \to t} \frac{y^2(\tau)}{2\Gamma(1-\alpha)(t-\tau)^{\alpha}} = \frac{1}{2\Gamma(1-\alpha)} \lim_{\tau \to t} \frac{[x(t)-x(\tau)]^2}{(t-\tau)^{\alpha}} \\ = \frac{1}{2\Gamma(1-\alpha)} \lim_{\tau \to t} \frac{x^2(t)-2x(t)x(\tau)+x^2(\tau)}{(t-\tau)^{\alpha}} \tag{16}$$

Given that the function is derivable, L'Hospital rule can be applied

$$\frac{1}{2\Gamma(1-\alpha)} \lim_{\tau \to t} \frac{x^2(t)-2x(t)x(\tau)+x^2(\tau)}{(t-\tau)^{\alpha}} \\ = \frac{1}{2\Gamma(1-\alpha)} \lim_{\tau \to t} \frac{-2x(t)x'(\tau)+2x(\tau)x'(\tau)}{-\alpha(t-\tau)^{\alpha-1}} \\ = \frac{1}{2\Gamma(1-\alpha)} \lim_{\tau \to t} \frac{[-2x(t)x'(\tau)+2x(\tau)x'(\tau)](t-\tau)^{1-\alpha}}{-\alpha} = 0. \tag{17}$$

So, expression (15) is reduced to

$$\frac{y_0^2}{2\Gamma(1-\alpha)(t-t_0)^{\alpha}} + \frac{\alpha}{2\Gamma(1-\alpha)} \int_{t_0}^{t} \frac{y^2(\tau)}{(t-\tau)^{\alpha+1}} d\tau \geq 0 \tag{18}$$

Expression (18) is clearly true, and this concludes the proof. ■

RESULTS and DISCUSSION

Theorem 2.1. Consider the system (1). $\Omega \subset R$ is a domain that contains the origin $x = 0$. Suppose further that $f(x) \in C^1(\Omega)$ with $f(0) = 0$. If $x \cdot f(x) \leq 0$, then the equilibrium point $x = 0$ is stable. Further, if $x \neq 0$ implies $x \cdot f(x) < 0$, then the equilibrium point $x = 0$ is asymptotically stable.

Proof. As the basic tool for proof, we choose Lyapunov function

$$V\big(x(t)\big) = \frac{1}{2}\left[x^2(t) + {}_{0}I_t^{1-\alpha} x^2(t)\right] \tag{19}$$

where $0 < \alpha < 1$. It is clear that $V(0) = 0$ and $V\big(x(t)\big) \geq 0$ from Lemma 1.2. Therefore $V\big(x(t)\big)$ pozitive definite. By derivativing along trajectory (1) expression (19), we get

$$\dot{V}\big(x(t)\big)|_{(1)} = x'(t)x(t) + \frac{1}{2}\, {}_{0}D_t^{1}\big({}_{0}I_t^{1-\alpha} x^2(t)\big) \\ = x(t)\big[f(x) - {}_{t_0}D_t^{\alpha} x(t)\big] + \frac{1}{2}\, {}_{t_0}D_t^{\alpha} x^2(t) \\ = x(t)f(x) - x(t)\, {}_{t_0}D_t^{\alpha} x(t) \\ + \frac{1}{2}\ {}_{t_0}D_t^{\alpha} x^2(t). \tag{20}$$

where $x(t)$ satisfies the system (1). By Lemma 1.4 the equality (20), can be written as

$$\dot{V}(x(t))|_{(1)} \leq x(t)f(x) - x(t)_{t_0}D_t^{\alpha}x(t) + x(t)_{t_0}D_t^{\alpha}x(t) = x(t)f(x) \leq 0. \tag{21}$$

Thus the equilibrium point $x(t) = 0$ of the system (1) is stable. Obviously, if $x(t) \neq 0$ implies $x(t) \cdot f(x(t)) < 0$ then $f(x(t)) \not\equiv 0$. It follows that the state $x(t)$ of the system (1) satisfies $x(t) \not\equiv constant$. Therefore $x'(t) \not\equiv 0$. Thus, $x(t)f(x) < 0$. So the system (1) is asymptotically stable. This completes the proof. ■

Theorem 2.2. Consider the fractional differential equation (2). $\Omega \subset R$ is a domain that contains the origin $x = 0$. Suppose further that $f, g \in C^1(\Omega)$ with $f(0) = 0$, $g(0) = 0$. If $x \cdot f(x) \leq 0$ and $y \cdot g(y) \geq 0$, then the equilibrium point $x = 0$ is stable. Further, if $x \neq 0$ and $y \neq 0$ implies $x \cdot f(x) < 0$ and $y \cdot g(y) > 0$, then the equilibrium point $x = 0$ is asymptotically stable.

Proof. Fractional differential equation (2), we write as differential equation system

$$\begin{aligned} x'(t) &= y(t), \\ y'(t) &= -g(y(t)) - {}_0D_t^{\alpha}y(t) + f(x(t)). \end{aligned} \tag{22}$$

For proof, we choose Lyapunov function as

$$V(x(t), y(t)) = \frac{1}{2}[y(t)]^2 + \frac{1}{2}\, {}_0I_t^{1-\alpha}y^2(t) - \int_0^{x(t)} f(s)ds \tag{23}$$

Obviously, $V(0,0) = 0$ and $V(x(t), y(t)) \geq 0$ from Lemma 1.2. So $V(x(t), y(t))$ pozitive definite. By derivativing along trajectory (22) expression (23), we get

$$\begin{aligned} \dot{V}(x(t), y(t))|_{(22)} &= -x'(t)f(x) + y'(t)y(t) + \frac{1}{2}\, {}_0D_t^{1}\left({}_0I_t^{1-\alpha}y^2(t)\right) \\ &= -y(t)f(x) + y(t)[-g(y(t)) - {}_0D_t^{\alpha}y(t) + f(x)] + \frac{1}{2}\, {}_0D_t^{\alpha}y^2(t) \\ &= -y(t)f(x) - y(t)g(y(t)) - y(t)_0D_t^{\alpha}y(t) + y(t)f(x) + \frac{1}{2}\, {}_0D_t^{\alpha}y^2(t) \end{aligned} \tag{24}$$

By Lemma 1.4 the equality (24), can be written as

$$\begin{aligned} \dot{V}(x(t), y(t))|_{(22)} &\leq -y(t)g(y(t)) - y(t)_0D_t^{\alpha}y(t) + y(t)_0D_t^{\alpha}y(t) \\ &= -y(t)g(y(t)) \end{aligned} \tag{25}$$

So $\dot{V}(x(t), y(t))|_{(22)} \leq 0$. Therefore the equilibrium point $x(t) = 0$ of the system (22) is stable. Obviously, if $x(t) \neq 0$ and $y(t) \neq 0$ implies $x(t) \cdot f(x(t)) < 0$ and $y(t) \cdot g(y(t)) > 0$ then $f(x(t)) \not\equiv 0$ and $g(y(t)) \not\equiv 0$. It follows that the state $(x(t), y(t))$ of the system (22) satisfies $y(t) \not\equiv constant$. So solution of the system (22) is asymptotically stable. This completes the proof. ■

Example 2.1. We consider fractional differential equation

$$x'(t) + {}_0D_t^{\alpha}x(t) = -x^5(t) \tag{26}$$

where $0 < \alpha < 1$ and $\Omega = R$. Denote $f(x) = -x^5(t)$. Because of $xf(x) = -x^6(t) \leq 0$, by Theorem 2.1 the equilibrium point $x = 0$ of (26) is stable. Obviously, $x \neq 0$ implies $xf(x) < 0$. By Theorem 2.1 the equilibrium point $x = 0$ of (26) is asymptotically stable.

Example 2.2. We consider fractional differential equation

$$x''(t) + [x'(t)]^3(2 + \sin(x'(t))) + {}_0D_t^{\alpha}x(t) = -x(t)e^{x(t)} \tag{27}$$

where $0 < \alpha < 1$ and $\Omega = R$. Denote $f(x) = -x(t)e^{x(t)}$ and $g(y(t)) = y^3(t)(2 + \sin(y(t)))$. Because of $xf(x) = -x^2(t)e^{x(t)} \leq 0$ and $y(t)g(y(t)) = y^4(t)(2 + \sin(y(t))) \geq 0$, by Theorem 2.2 the equilibrium point $x = 0$ of (27) is stable. Obviously, $x \neq 0$ and $y \neq 0$ implies $xf(x) < 0$ and $yg(y) > 0$. By Theorem 2.2 the equilibrium point $x = 0$ of (27) is asymptotically stable.

ORCID

Erdal KORKMAZ https://orcid.org/0000-0002-6647-9312
Abdulhamit ÖZDEMİR https://orcid.org/0000-0002-5310-6285

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