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AN EXACT VARIANCE EXPRESSION OF THE k-PHD SINGLE-TONE FREQUENCY ESTIMATOR: RANDOM PHASE CASE

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Abstract: The k-PHD method, a generalization of the well-known Pisarenko harmonic decomposition (PHD) method, is considered for frequency estimation of a single real random-phased sinusoid in noise. With the use of a simple variance analysis technique, an exact expression of the k-PHD frequency variance is derived. An approximate k-PHD variance formula for sufficiently large data lengths and signal-to-noise ratios is also given. Computer simulations are included to validate the theoretical development.

Keywords: Frequency estimation, Real sinusoid, PHD method, k-PHD method, Variance analysis.

k-PHD Tek-Ton Frekans Kestiricisinin Bir Kesin Değişinti İfadesi: Rasgele Faz Durumu

Öz: İyi bilinen Pisarenko harmonik ayrışım (PHD) metodunun bir genelleştirilmesi olan k-PHD metodu gürültü içindeki bir reel rasgele fazlı sinüsün frekans kestirimi için ele alınmıştır. Basit bir değişinti analiz tekniği kullanılarak, k-PHD frekans değişintisinin bir kesin ifadesi çıkarılmıştır. Yeteri kadar geniş veri uzunlukları ve yüksek işaret gürültü oranları için bir yaklaşık k-PHD değişinti formülü de verilmektedir. Kuramsal sonuçları teyit eden bilgisayar benzetimleri dâhil edilmiştir.

Anahtar Kelimeler: Frekans kestirimi. Reel sinus, PHD metodu, k-PHD metodu, Değişinti analizi.

1. INTRODUCTION

The problem of estimating the frequency of a single real tone in noise has been frequently studied in the signal processing literature due to its wide range of applications. Among the various estimation techniques, the Piseranko harmonic decomposition (PHD) method has attracted people because of its ease of implementation. Several statistical analyses of the PHD method have been carried out in the literature (see, e.g., Sakai (1984), Chan and So (2003), and the references therein). In Chan and So (2003), an exact variance expression of the PHD frequency estimator has been derived, which holds for moderate data lengths and/or signal-to-noise ratios (SNRs). Despite the fact that the PHD estimator can be implemented in a very simple way, its variance is, in general, much larger than the Cramér-Rao lower bound (see, e.g., Chan and So, 2003).

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In order to reduce the variance of the PHD estimator, a two-step procedure called the k-PHD method has been proposed in De Sabata et. al. (2007). The k-PHD method consists of estimating an integer multiple (*k*-th multiple) of the frequency and using the PHD method in resolving the induced aliasing. For $k=1$ there is no ambiguity in the frequency estimates and the k-PHD reduces to the PHD. A variance expression of the k-PHD frequency estimator has also been given in De Sabata et. al. (2007).

In the above-mentioned analyses of the PHD and k-PHD methods the phase of the sinusoid is considered as a constant parameter which does not change from one realization to another. In this paper, it is assumed that the phase of the sinusoid is a random variable which is uniformly distributed in an interval of length 2π. By utilizing the variance analysis technique employed in Chan and So (2003) (and also in De Sabata et. al., 2007), and taking into account that the phase is now a random variable, we derive an exact closed-form expression for the frequency variance of the k-PHD method. An approximate variance formula that holds for sufficiently large data lengths and SNRs is also given.

The variance expressions derived in this paper can easily be adapted to the PHD estimator for the random-phased sinusoid case. The results for the PHD method have been presented, recently, in Keyta and Dilaveroğlu (2019).

The rest of the paper is organized as follows. The data model and the k-PHD frequency estimation method are described in Section 2. The k-PHD variance development for the random phase case considered herein is given in Section 3. Computer simulations are presented in Section 4 to validate the theoretical development. Finally, conclusions are drawn in Section 5.

2. DATA MODEL AND k-PHD METHOD

We consider the following data model consisting of a single real tone in noise:

$$x(n)=\alpha\cos(\omega n+\phi)+e(n)\,,\qquad n=1,2,\ldots,N \tag{1}$$

where the amplitude α and frequency $\omega\in(0,\pi)$ are unknown constants while phase ϕ is a random variable uniformly distributed in the interval $(-\pi,\pi]$. The noise $e(n)$ is assumed to be a zero-mean white Gaussian process with unknown variance σ^2 and independent of the phase ϕ. N denotes the number of available data samples.

The k-PHD method is a two-step frequency estimation algorithm. In the first step of the method, a rough estimate $\hat{\omega}_1$ of the frequency ω is calculated using the PHD frequency estimator, probably with a small value of N. In the second step of the method, first the cosine of a *k*-th multiple, where $k>1$ is an integer, of the frequency ω, $\cos(k\omega)$, is estimated from the estimator ρ_k given by

$$\rho_k=\frac{r_{2k}+\sqrt{r_{2k}^2+8r_k^2}}{2r_k} \tag{2}$$

where r_m is the sample covariance that has the form

$$r_m=\frac{1}{N-m}\sum_{n=1}^{N-m}x(n)x(n+m)\,,\qquad m=k,2k\,. \tag{3}$$

Then, the estimate $\hat{\omega}_k$ of the frequency ω is calculated as

$$\hat{\omega}_k = \frac{1}{k}\left((-1)^{i-1} \arccos(\rho_k) + \left\lfloor \frac{i}{2} \right\rfloor 2\pi \right) \tag{4}$$

where

$$i = 1 + \left\lfloor \frac{k\hat{\omega}_1}{\pi} \right\rfloor \tag{5}$$

and $\lfloor \bullet \rfloor$ denotes the largest integer smaller than ($\bullet$). (See De Sabata et. al., 2007). Note that for $k = 1$ the k-PHD method is equivalent to the PHD method.

We remark here that the two-step procedure employed in the k-PHD method is a general frequency estimation process. In the first step of the process, a rough estimate of the frequency can be calculated from any low complexity method (such as from an estimator $\rho_1 = \cos(\hat{\omega}_1)$ using $\hat{\omega}_1 = \arccos(\rho_1)$; for a number of such estimators and a comparative study of their performances, see, e.g., Uz and Dilaveroğlu, 2019). In the second step of the process, any estimator which is designed for estimating an integer multiple of the frequency can be used (see, e.g., Toma et. al., 2007).

3. EXACT VARIANCE DEVELOPMENT

Define a second-order polynomial

$$f(\rho) = 2r_k\rho^2 - r_{2k}\rho - r_k \tag{6}$$

for which ρ_k equals to one of the roots. For sufficiently large N and/or SNR, ρ_k will be close to $\cos(k\omega)$ and we can use the following formula to derive the variance of ρ_k, which is denoted by $\mathrm{var}(\rho_k)$ (Chan and So, 2003) (also see So et. al., 2013):

$$\mathrm{var}(\rho_k) \approx \left. \frac{E\{f^2(\rho)\}}{(E\{f'(\rho)\})^2} \right|_{\rho=\cos(k\omega)} . \tag{7}$$

In addition, the relationship between the variance of $\hat{\omega}_k$, which is denoted by $\mathrm{var}(\hat{\omega}_k)$ and $\mathrm{var}(\rho_k)$ is given as (Papoulis, 1991)

$$\mathrm{var}(\hat{\omega}_k) \approx \frac{\mathrm{var}(\rho_k)}{k^2 \sin^2(k\omega)} . \tag{8}$$

Here E denotes the expectation operator and $f'(\rho)$ is the derivative of $f(\rho)$ with respect to ρ. The derivations of the expectations in (7) are given as follows. From (6) we have

$$E\{f^2(\rho)\}\Big|_{\rho=\cos(k\omega)} = \cos^2(2k\omega)E\{r_k^2\} - 2\cos(k\omega)\cos(2k\omega)E\{r_k r_{2k}\} + \cos^2(k\omega)E\{r_{2k}^2\} \quad (9)$$

and

$$E\{f'(\rho)\}\Big|_{\rho=\cos(k\omega)} = 4\cos(k\omega)E\{r_k\} - E\{r_{2k}\}. \quad (10)$$

We see that the values of $E\{r_k\}$, $E\{r_{2k}\}$, $E\{r_k^2\}$, $E\{r_k r_{2k}\}$ and $E\{r_{2k}^2\}$ are required. These are derived as

$$E\{r_k\} = \frac{\alpha^2\cos(k\omega)}{2} \quad (11)$$

$$E\{r_{2k}\} = \frac{\alpha^2\cos(2k\omega)}{2} \quad (12)$$

$$E\{r_k^2\} = \frac{\sigma^4}{N-k} + \frac{\alpha^4\cos^2(k\omega)}{4} + \frac{\alpha^4\sin^2((N-k)\omega)}{8(N-k)^2\sin^2(\omega)} + \frac{\alpha^2\sigma^2}{N-k} + \frac{\alpha^2\sigma^2(N-2k)\cos(2k\omega)}{(N-k)^2} \quad (13)$$

$$E\{r_k r_{2k}\} =$$
$$\frac{\alpha^4\cos(k\omega)\cos(2k\omega)}{4} + \frac{\alpha^4\sin((N-k)\omega)\sin((N-2k)\omega)}{8(N-k)(N-2k)\sin^2(\omega)} + \frac{\alpha^2\sigma^2\cos(k\omega)}{N-k} + \frac{\alpha^2\sigma^2(N-3k)\cos(3k\omega)}{(N-k)(N-2k)} \quad (14)$$

and

$$E\{r_{2k}^2\} = \frac{\sigma^4}{N-2k} + \frac{\alpha^4\cos^2(2k\omega)}{4} + \frac{\alpha^4\sin^2((N-2k)\omega)}{8(N-2k)^2\sin^2(\omega)} + \frac{\alpha^2\sigma^2}{N-2k} + \frac{\alpha^2\sigma^2(N-4k)\cos(4k\omega)}{(N-2k)^2}. \quad (15)$$

Substituting (11)-(15) into (9) and (10) and using (7) and (8), after simplifications, give

$$\mathrm{var}(\hat{\omega}_k) \approx \frac{A(N,k,\omega)\cdot\mathrm{SNR}^{-2} + B(N,k,\omega)\cdot\mathrm{SNR}^{-1} + C(N,k,\omega)}{D^2(k,\omega)k^2\sin^2(k\omega)} \quad (16)$$

where the terms A, B, C and D are given as

$$A(N,k,\omega) = \frac{(2N-3k) + (N-k)\cos(2k\omega) + (N-2k)\cos(4k\omega)}{8(N-k)(N-2k)} \quad (17)$$

$$B(N,k,\omega) =$$
$$\frac{4k(N-k)(N-2k) + k^2(N-4k)\cos(2k\omega) - 4k^2(N-k)\cos(4k\omega) - k^2N\cos(6k\omega)}{8(N-k)^2(N-2k)^2} \quad (18)$$

$$C(N,k,\omega) = \frac{((N-k)\sin((N-k)\omega) + k\sin((N-3k)\omega) - (N-2k)\sin((N+k)\omega))^2}{32(N-k)^2(N-2k)^2\sin^2(\omega)} \quad (19)$$

and

$$D(k,\omega)=\frac{2+\cos(2k\omega)}{2}\,. \tag{20}$$

The variance is a function of ω, N, k and SNR, which is defined as $\mathrm{SNR}=\alpha^2/(2\sigma^2)$. It can be seen that A, B and C are of order N^{-1}, N^{-2} and N^{-2}, respectively, while D is not a function of N.

The expression in (16) gives the variance of the k-PHD frequency estimator $\hat{\omega}_k$ as a simple function of the SNR. For low SNRs, the first term of (16) becomes dominant and $\mathrm{var}(\hat{\omega}_k)$ is proportional to $(N^{-1}\cdot\mathrm{SNR}^{-2})$; for high SNRs, the last term of (16) becomes dominant and the variance is proportional to N^{-2} and, interestingly, independent of SNR. This unusual constant behavior of the variance with respect to SNR at high SNRs has also been observed for the constant phase case (De Sabata et. al., 2007).

The variance expression in De Sabata et. al. (2007) derived for the constant phase case depends on the phase ϕ of the sinusoid in a very complicated way; see De Sabata et. al. (2007), equation (10). In contrast, the variance expression (16) derived for the random phase case is independent of ϕ, as expected; by assuming a random phase we got rid of all the possible phase dependent terms.

For $N>>1$, the terms A, B and C can be approximated as

$$\tilde{A}(N,k,\omega)=\frac{\cos^2(k\omega)+\cos^2(2k\omega)}{4N} \tag{21}$$

$$\tilde{B}(N,k,\omega)=\frac{k}{2N^2} \tag{22}$$

and

$$\tilde{C}(N,k,\omega)=\frac{\cos^2(N\omega)\sin^2(k\omega)}{8N^2\sin^2(\omega)} \tag{23}$$

and consequently a simple approximate expression of $\mathrm{var}(\hat{\omega}_k)$ in this case is

$$\mathrm{var}(\hat{\omega}_k)\approx\frac{\tilde{A}(N,k,\omega)\cdot\mathrm{SNR}^{-2}+\tilde{B}(N,k,\omega)\cdot\mathrm{SNR}^{-1}+\tilde{C}(N,k,\omega)}{D^2(k,\omega)k^2\sin^2(k\omega)}\,. \tag{24}$$

For $k=1$, (16) and (24) give, respectively, exact and approximate variance expressions of the PHD method; for a discussion of the results for the PHD case, see Keyta and Dilaveroğlu (2019).

Note that the variance in (16) and (24) is unbounded at frequencies

$$\omega=\frac{i\pi}{k}\,,\qquad i=0,1,\ldots,k\,. \tag{25}$$

In practice, a value of k can be chosen so that the first estimate $\hat{\omega}_1$ is not close to a value given by (25).

4. NUMERICAL EXAMPLES

Computer simulations had been performed in order to validate our theoretical results. The tone amplitude was set to 2 while different SNRs were obtained by properly scaling the noise samples. All simulation results were obtained by averaging 1000 independent runs.

Figure 1 shows the frequency variances of the k-PHD estimators for $k=1$ (which is the PHD estimator), $k=2$ and $k=3$ for $\omega \in [0.025\pi, 0.975\pi]$ at $\mathrm{SNR}=20$ dB and for $N=32$. It is seen that at every frequency the variance of the k-PHD for $k=2$ and/or $k=3$ was smaller than that of the PHD.

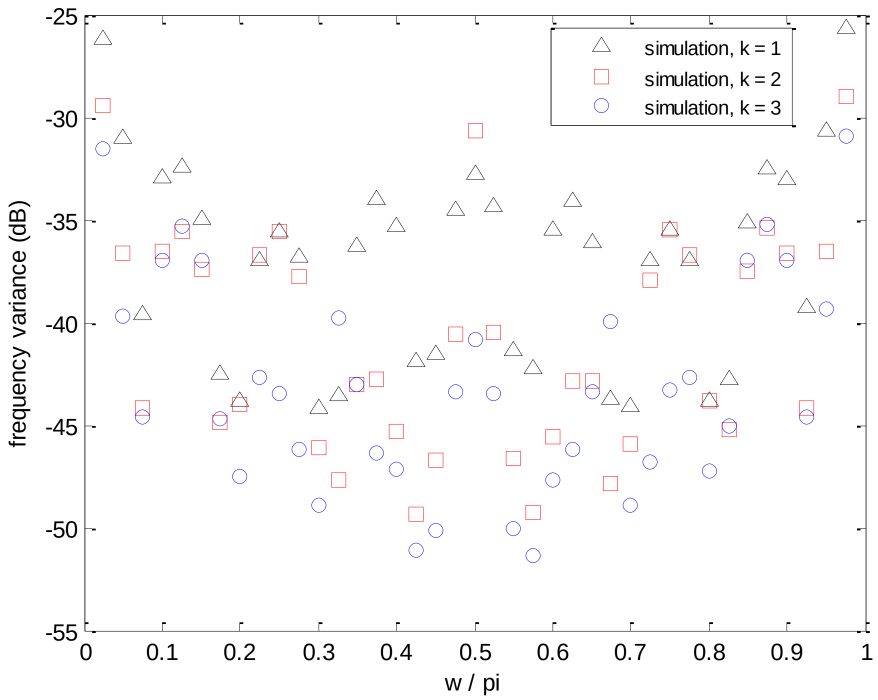

Figure 1:

Frequency variances versus ω at $\mathrm{SNR}=20$ dB and $N=32$.

Figure 2 shows the frequency variances of the PHD and the k-PHD for $k=3$ versus ω at $\mathrm{SNR}=20$ dB for $N=128$. The variance expressions of (16) and (24) were also included. We observe that the measured variances conformed the exact expression (16) provided that the frequency ω is not very near to 0 or π. Also, the simple approximate expression (24) was in good agreement with the measured ones. The decrease in variances for the k-PHD as compared to the PHD is apparent from the Figure for the whole range of the frequency except at $\omega=0.325\pi$ and $\omega=0.675\pi$ as predicted by (25).

Figure 3 shows frequency variances of the PHD method and the k-PHD method for $k=4$ versus SNR at $\omega=3\pi/8$ and for $N=128$. It can be observed that (16) agreed well with the simulation results while (24) was a good approximation provided that SNR is not too small. The variances of the k-PHD were smaller than those of the PHD for the whole range of SNR and the reduction was more than 9 dB for $\mathrm{SNR} \geq 20$ dB.

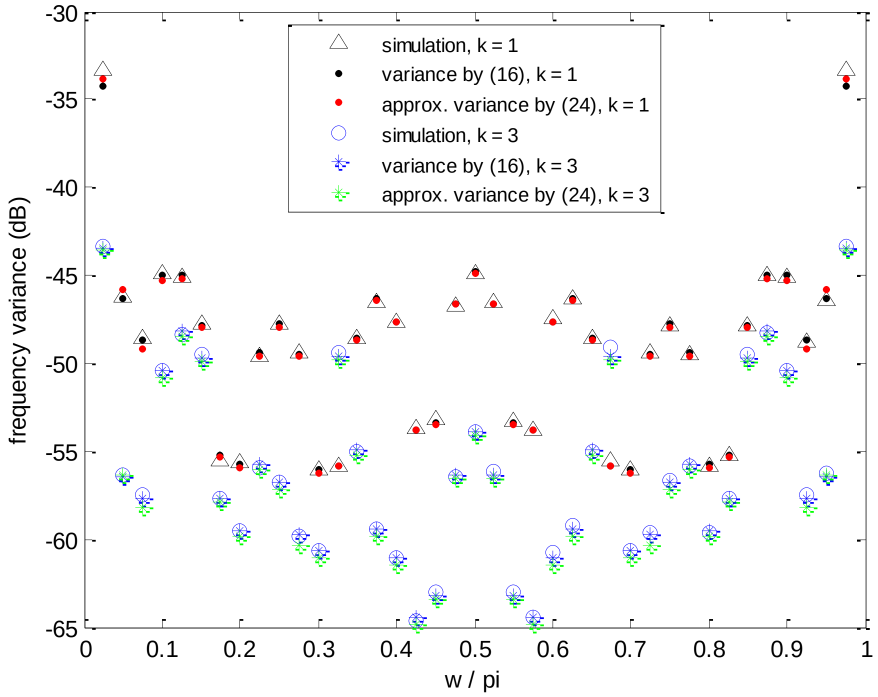

Figure 2:
Frequency variances versus ω *at* $\mathrm{SNR} = 20$ *dB and* $N = 128$.

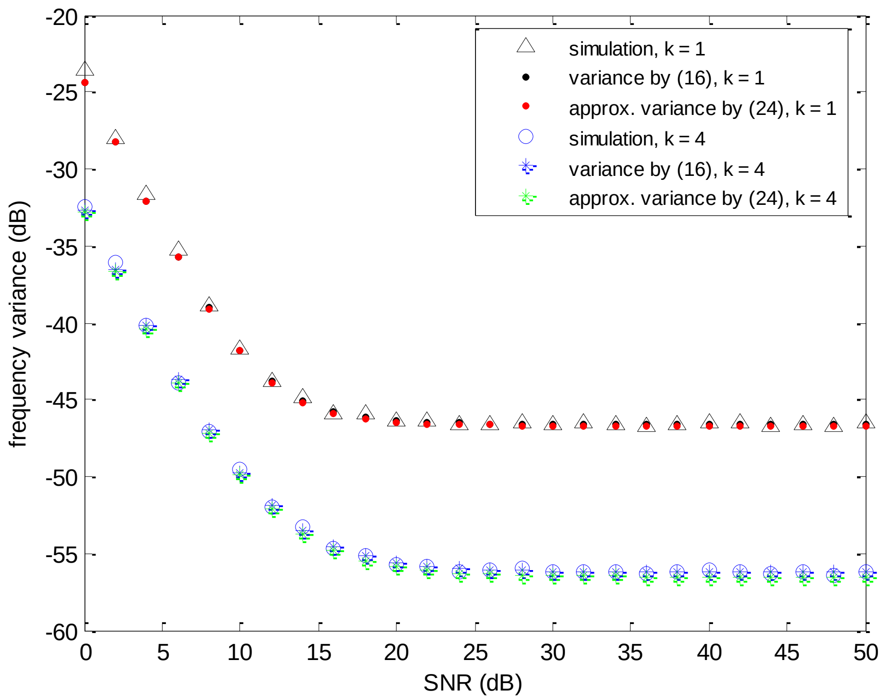

Figure 3:
Frequency variances versus SNR *at* $\omega = 3\pi/8$ *and* $N = 128$.

Table 1 shows frequency standard deviations of the PHD and the k-PHD for $k = 3$ for different values of the data length N at $\mathrm{SNR} = 20$ dB and $\omega = 3\pi/8$. It can be seen that (16) and (24) predicted the measured variances very well for all values of N and $N \geq 64$, respectively.

Table 1. Frequency standard deviations versus N at $\mathrm{SNR} = 20$ dB and $\omega = 3\pi/8$.

	Measured std. dev.	Std. dev. by (16)	Approx. std. dev. by (24)
$N = 16$, $k = 1$	4.1669×10^{-2}	4.1129×10^{-2}	3.7769×10^{-2}
$k = 3$	1.3547×10^{-2}	1.3158×10^{-2}	7.7352×10^{-3}
$N = 32$, $k = 1$	1.9294×10^{-2}	1.9703×10^{-2}	1.8903×10^{-2}
$k = 3$	4.6939×10^{-3}	4.9182×10^{-3}	3.8958×10^{-3}
$N = 64$, $k = 1$	9.8179×10^{-3}	9.6655×10^{-3}	9.4703×10^{-3}
$k = 3$	2.2516×10^{-3}	2.2024×10^{-3}	1.9758×10^{-3}
$N = 128$, $k = 1$	4.8010×10^{-3}	4.8020×10^{-3}	4.7538×10^{-3}
$k = 3$	1.0805×10^{-3}	1.0685×10^{-3}	1.0152×10^{-3}
$N = 256$, $k = 1$	2.4283×10^{-3}	2.4074×10^{-3}	2.3954×10^{-3}
$k = 3$	5.6440×10^{-4}	5.4674×10^{-4}	5.3390×10^{-4}

5. CONCLUSIONS

We have derived an exact closed-form variance expression of the k-PHD method for a single real sinusoid in additive white Gaussian noise for a random phase case. An approximate simple variance formula for sufficiently large data lengths and high signal-to-noise ratios has also been developed. Computer simulations have been provided to validate the theoretical results.

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