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Generalized B-Curvature Tensor of a Normal Paracontact Metric Manifold

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Abstract
The aim of present paper is to study the generalized B-curvature tensor of a normal paracontact metric manifold satisfying the conditions generalized B-flatness, generalized B-semi-symmetric, $B.\widetilde{Z}=0$, $B.S=0$ and $B.P=0$, where $B,\widetilde{Z},P,S$ denotes the generalized B-curvature tensor, concircular curvature tensor, projective curvature tensor and Ricci tensor, respectively.

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1. Introduction

The study of paracontact geometry was initiated by Kenayuki and Williams [8]. Zamkovoy studied paracontact metric manifolds and their subclasses [9]. Recently Welyczko studied curvature and torsion of Frenet Legendre curves in 3-dimensional normal paracontact metric manifolds [3, 4]. In the recent years, (para) contact metric manifolds and their curvature properties have been studied by many authors [2, 10, 11].

A $n-$dimensional differentiable manifold (M,g) is said to be an almost paracontact metric manifold if there exist on M a $(1,1)$ tensor field ϕ, a contravariant vector ξ and a 1-form η-such that

$$\phi^2X = X-\eta(X)\xi, \quad \phi\xi = 0, \quad \eta(\phi X)=0, \quad \eta(\xi)=1 \tag{1}$$

and

$$g(\phi X,\phi Y) = g(X,Y)-\eta(X)\eta(Y), \quad \eta(X)=g(X,\xi), \tag{2}$$

for any $X,Y\in\chi(M)$. If the covariant derivative of ϕ satisfies

$$(\nabla_X\phi)Y = -g(X,Y)\xi-\eta(Y)X+2\eta(X)\eta(Y)\xi \tag{3}$$

then, M is called a normal paracontact metric manifold, where ∇ is Levi-Civita connection.
From (3), we can easily to see that

$$\phi X = \nabla_X\xi \tag{4}$$

for any $X\in\chi(M)$ [8].

Moreover, if such a manifold has constant sectional curvature equal to c, then its the Riemannian curvature tensor is R given by

$$\begin{aligned} R(X,Y)Z &= \frac{c+3}{4}\{g(Y,Z)X - g(X,Z)Y\} \\ &+ \frac{c-1}{4}\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\xi \\ &- g(Y,Z)\eta(X)\xi + g(\phi Y,Z)\phi X - g(\phi X,Z)\phi Y - 2g(\phi X,Y)\phi Z\}, \end{aligned} \tag{5}$$

for any vector fields $X,Y,Z \in \chi(M)$ [2].

In 2014, Shaikh and Kundu [1] to imported and studied a type of tensor field, called generalized B curvature tensor on a Riemannian manifold. It count the structures of Quasi-conformal, Weyl-conformal, Conharmonic and Concircular curvature tensors and is spell out just as

$$\begin{aligned} B(X,Y)Z &= p_0 R(X,Y)Z \\ &+ p_1\big[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY\big] \\ &+ 2p_2 r\big[g(Y,Z)X - g(X,Z)Y\big] \end{aligned} \tag{6}$$

wehere R, S, Q and r are the Riemannian curvature tensor, the Ricci tensor, the Ricci operator and the scalar curvature, respectively.

In particular, the B-curvature tensor is reduced to:

1. The quasi-conformal curvature tensor C [5] if

$$p_0 = a, p_1 = b \ \ and \ \ p_2 = -\frac{1}{2n}\big[\frac{a}{n-1} + 2b\big].$$

2. The Weyl-conformal curvature tensor $\widetilde{C}$ [7] if

$$p_0 = 1, p_1 = -\frac{1}{n-1} \ \ and \ \ p_2 = -\frac{1}{2(n-1)(n-2)}.$$

3. The concircular curvature tensor $\widetilde{Z}$ [6] if

$$p_0 = 1, p_1 = 0 \ \ and \ \ p_2 = -\frac{1}{n(n-1)}.$$

4. The conharmonic curvature tensor H [12] if

$$p_0 = 1, p_1 = -\frac{1}{n-1} \ \ and \ \ p_2 = 0.$$

The projective curvature tensor P and the concircular curvature tensor $\widetilde{Z}$ of n-dimensional Riemann manifold are defined by

$$P(X,Y)Z = R(X,Y)Z - \frac{1}{(n-1)}\big[S(Y,Z)X - S(X,Z)Y\big], \tag{7}$$

and

$$\widetilde{Z}(X,Y)Z = R(X,Y)Z - \frac{r}{n(n-1)}\big[g(Y,Z)X - g(X,Z)Y\big], \tag{8}$$

where S is the Ricci tensor and r is the scalar curvature of the manifold [6].

In a normal paracontact metric space form by direct calculations, we can easily to see that

$$S(X,Y)=(\frac{c(n-5)+3n+1}{4})g(X,Y)+(\frac{(c-1)(5-n)}{4})\eta(X)\eta(Y) \tag{9}$$

from which

$$QX=(\frac{c(n-5)+3n+1}{4})X+(\frac{(c-1)(5-n)}{4})\eta(X)\xi \tag{10}$$

for any $X,Y\in\chi(M)$, where Q is the Ricci operator and S is the Ricci tensor of M.

Corollary 1. *A normal paracontact metric space form is always an η-Einstein manifold.*

From (9) and (10), we can easily see

$$S(X,\xi)=(n-1)\eta(X), \tag{11}$$

$$Q\xi=(n-1)\xi, \tag{12}$$

and

$$r=\frac{n-1}{4}\left[c(n-5)+3n+5\right]. \tag{13}$$

Let M be n-dimensional normal paracontact metric space form and we denote the Riemannian curvature tensor of R, then we have from (5), for $X=\xi$

$$R(\xi,Y)Z=g(Y,Z)\xi-\eta(Z)Y, \tag{14}$$

for $Z=\xi$

$$R(X,Y)\xi=\eta(Y)X-\eta(X)Y. \tag{15}$$

In (15) choosing $Y=\xi$, we get

$$R(X,\xi)\xi=X-\eta(X)\xi. \tag{16}$$

Taking the inner product both of the sides (5) with $\xi\in\chi(M)$, we obtain

$$\eta(R(X,Y)Z)=g(Y,Z)\eta(X)-g(X,Z)\eta(Y). \tag{17}$$

In the same way we obtain from (7) and (8),

$$P(\xi,Y)Z=g(Y,Z)\xi-\frac{1}{n-1}S(Y,Z)\xi, \tag{18}$$

$$P(\xi,Y)\xi=0, \tag{19}$$

and

$$\widetilde{Z}(\xi,Y)Z=\left[1-\frac{r}{n(n-1)}\right]\left[g(Y,Z)\xi-\eta(Z)Y\right], \tag{20}$$

$$\widetilde{Z}(\xi,Y)\xi=\left[1-\frac{r}{n(n-1)}\right]\left[\eta(Y)\xi-Y\right]. \tag{21}$$

Also from (6), we obtain

$$B(\xi,Y)Z=\left[p_0+\frac{p_1}{4}\left[c(n-5)+7n-1\right]+2p_2r\right]\left[g(Y,Z)\xi-\eta(Z)Y\right], \tag{22}$$

and

$$B(Y,Z)\xi=\left[p_0+\frac{p_1}{4}\left[c(n-5)+7n-1\right]+2p_2r\right]\left[\eta(Z)Y-\eta(Y)Z\right]. \tag{23}$$

If a normal paracontact metric space form M^n is a generalized B-flat, then from (6) we obatin

$$\begin{aligned} & p_0R(X,Y)Z + p_1\left[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY\right] \\ + & \; 2p_2r\left[g(Y,Z)X - g(X,Z)Y\right] \\ = & \; 0, \end{aligned} \tag{24}$$

for all $X,Y,Z \in \chi(M)$, where Q is the Ricci operator and S is the Ricci tensor of M.

Choosing $Z = \xi$ and using (1), (11),(15) in (24), we obtain

$$\left[p_0 + p_1(n-1) + 2p_2r\right]\left[\eta(Y)X - \eta(X)Y\right] + p_1\left[\eta(Y)QX - \eta(X)QY\right] = 0. \tag{25}$$

We choosing $Y = \xi$ in (25) and taking into account (10), we have

$$-p_1QX = \left[p_0 + p_1(n-1) + 2p_2r\right]X - \left[p_0 + 2p_1(n-1) + 2p_2r\right]\eta(X)\xi. \tag{26}$$

Inner product both sides of the equation by $W \in \chi(M)$ in (26), we conclude

$$S(X,W) = -\frac{1}{p_1}\left[p_0 + p_1(n-1) + 2p_2r\right]g(X,W) + \frac{1}{p_1}\left[p_0 + 2p_1(n-1) + 2p_2r\right]\eta(X)\eta(W).$$

We are able to state the following theorem

Theorem 2. *An n-dimensional $(n \geq 3)$ normal paracontact metric manifold M is generalized B-flat if and only if M reduce an Einstein manifold provided that $(p_1 \neq 0)$.*

2. Generalized B-Semi-Symmetric Normal Paracontact Metric Manifold

Theorem 3. *Let M be n-dimensional a normal paracontact metric manifold. Then, M is generalized B-semi symmetric if and only if the scalar curvature of M is $r = \frac{(1-n)\left[2p_0+p_1(2n-3)\right]}{2p_1+4p_2(n-1)}$.*

Proof. Let $(R(X,Y)B)(U,W)Z = 0$ be on M for any $X,Y,Z,U,W \in \chi(M)$, then we get

$$\begin{aligned} (R(X,Y)B)(U,W)Z &= R(X,Y)B(U,W)Z - B(R(X,Y)U,W)Z \\ &- B(U,R(X,Y)W)Z - B(U,W)R(X,Y)Z. \end{aligned} \tag{27}$$

In (27), choosing $X = \xi$ and from the hypothesis,we have

$$\begin{aligned} (R(\xi,Y)B)(U,W)Z &= R(\xi,Y)B(U,W)Z - B(R(\xi,Y)U,W)Z \\ &- B(U,R(\xi,Y)W)Z - B(U,W)R(\xi,Y)Z \\ &= 0. \end{aligned} \tag{28}$$

Using (14) in (28), we obtain

$$\begin{aligned} & g(Y,B(U,W)Z)\xi - \eta(B(U,W)Z)Y \\ - & \; g(Y,U)B(\xi,W)Z + \eta(U)B(Y,W)Z \\ - & \; g(Y,W)B(\xi,U)Z + \eta(W)B(U,Y)Z \\ - & \; g(Y,Z)B(U,W)\xi + \eta(Z)B(U,W)Y \\ = & \; 0. \end{aligned} \tag{29}$$

In (29), putting $U = \xi$ and using (22) and (23), we obtain

$$B(Y,W)Z - \left[p_0 + \frac{p_1}{4}\left[c(n-5) + 7n - 1\right] + 2p_2r\right]\left[g(W,Z)Y - g(Y,Z)W\right] = 0.$$

Now, choosing $Z=\xi$ and using (23) in the last equation, we conclude

$$r=\frac{(1-n)\left[2p_0+p_1(2n-3)\right]}{2p_1+4p_2(n-1)}. \tag{30}$$

The converse obvious. The achieve the proof. ∎

3. Curvature Conditions $B.\widetilde{Z}=0$, $B.S=0$ and $B.P=0$

Now, we theorize that the manifold bearing the curvature condition, that is, $B.\widetilde{Z}=0$, $B.S=0$ and $B.P=0$, where B, $\widetilde{Z}$, P and S are the the generalized B-curvature tensor, concircular curvature tensor and projective curvature tensor and the Ricci tensor, respectively. Now, in this position we show the theorem.

Theorem 4. *Let M be n-dimensional a normal paracontact metric manifold. Then, $B.\widetilde{Z}=0$ if and only if M either is a real space form with sectional curvature $c=1$ or the scalar curvature $r=\frac{(1-n)\left[2p_0+p_1(2n-3)\right]}{2p_1+4p_2(n-1)}$.*

Proof. Suppose that $B(\xi,Y)\widetilde{Z}=0$, we have

$$\begin{aligned} & B(\xi,Y)\widetilde{Z}(U,W)Z-\widetilde{Z}(B(\xi,Y)U,W)Z \\ - & \widetilde{Z}(U,B(\xi,Y)W)Z-\widetilde{Z}(U,W)B(\xi,Y)Z \\ = & 0, \end{aligned} \tag{31}$$

for all $Y,U,W,Z\in\chi(M)$. In (31), using (22) and putting $U=\xi$, we obtain

$$\begin{aligned} & \left[p_0+\frac{p_1}{4}\left[c(n-5)+7n-1\right]+2p_2r\right]\left[g(Y,\widetilde{Z}(\xi,W)Z)\xi\right. \\ - & \eta(\widetilde{Z}(\xi,W)Z)Y-\eta(Y)\widetilde{Z}(\xi,W)Z \\ + & \widetilde{Z}(Y,W)Z+\eta(W)\widetilde{Z}(\xi,Y)Z \\ - & \left. g(Y,Z)\widetilde{Z}(\xi,W)\xi+\eta(Z)\widetilde{Z}(\xi,W)Y\right] \\ = & 0. \end{aligned} \tag{32}$$

In (32), using the equations (20) and (21), we conclude

$$\left[p_0+\frac{p_1}{4}\left[c(n-5)+7n-1\right]+2p_2r\right]\left[\widetilde{Z}(Y,W)Z-\left(1-\frac{r}{n(n-1)}\right)\left[g(W,Z)Y-g(Y,Z)W\right]\right]=0.$$

Taking into account (8), in the last equation we result

$$\left[p_0+\frac{p_1}{4}\left[c(n-5)+7n-1\right]+2p_2r\right]\left[R(Y,W)Z-\left[g(W,Z)Y-g(Y,Z)W\right]\right]=0.$$

This tell us that M is a real space form with constant sectional curvature $c=1$ or the scalar curvature $r=\frac{(1-n)\left[2p_0+p_1(2n-3)\right]}{2p_1+4p_2(n-1)}$ of the manifold.

The converse is obvious. The proof is complete. ∎

Theorem 5. *Let M be n-dimensional a normal paracontact metric manifold. Then, $B.P=0$ if and only if M either reduce an Einstein manifold or the scalar curvature $r=\frac{(1-n)\left[2p_0+p_1(2n-3)\right]}{2p_1+4p_2(n-1)}$.*

Proof. Assume that $B(\xi,Y)P=0$, we have

$$\begin{aligned} & B(\xi,Y)P(U,W)Z-P(B(\xi,Y)U,W)Z \\ - & P(U,B(\xi,Y)W)Z-P(U,W)B(\xi,Y)Z \\ = & 0, \end{aligned} \tag{33}$$

for all $Y,U,W,Z \in \chi(M)$. In (33), using (22) and putting $U=\xi$, we obtain

$$\begin{aligned} 0 &= \big[p_0+\frac{p_1}{4}\big[c(n-5)+7n-1\big]+2p_2r\big]\big[g(W,Z)\eta(Y)\xi-\frac{1}{n-1}S(W,Z)\eta(Y)\xi \\ &- g(W,Z)Y+\frac{1}{n-1}S(W,Z)Y \\ &- \eta(Y)\big[g(W,Z)\xi-\frac{1}{n-1}S(W,Z)\xi\big]+P(Y,W)Z \\ &+ \eta(W)\big[g(Y,Z)\xi-\frac{1}{n-1}S(Y,Z)\xi\big] \\ &+ \eta(Z)\big[g(W,Y)\xi-\frac{1}{n-1}S(W,Y)\xi\big]\big]. \end{aligned} \tag{34}$$

When the equation (34) is shortened, we have

$$\begin{aligned} 0 &= \big[p_0+\frac{p_1}{4}\big[c(n-5)+7n-1\big]+2p_2r\big]\big[\frac{1}{n-1}S(W,Z)Y-\frac{1}{n-1}S(Y,Z)W \\ &- \frac{1}{n-1}S(W,Y)\eta(Z)\xi+P(Y,W)Z-g(W,Z)Y+g(Y,Z)\eta(W)\xi \\ &+ g(W,Y)\eta(Z)\xi\big]. \end{aligned} \tag{35}$$

In (35), choosing $Z=\xi$ and inner product both sides of the equation by $\xi \in \chi(M)$, we conclude

$$\big[p_0+\frac{p_1}{4}\big[c(n-5)+7n-1\big]+2p_2r\big]\big[S(Y,W)-(n-1)g(Y,W)\big]=0 \tag{36}$$

This show that, either M is an Einstein manifold or the scalar curvature $r=\frac{(1-n)\left[2p_0+p_1(2n-3)\right]}{2p_1+4p_2(n-1)}$ of the manifold. This proves our assertion. The converse is obvious. ■

Theorem 6. *Let M be n-dimensional a normal paracontact metric manifold. Then, $B.S=0$ if and only if M either reduce an Einstein manifold or the scalar curvature $r=\frac{(1-n)\left[2p_0+p_1(2n-3)\right]}{2p_1+4p_2(n-1)}$.*

Proof. Let the condition $B.S=0$ holds on M, which implies that $(B(Y,X)S(U,W))=0$ for all vector fields $X,Y,U,W\in\chi(M)$. Then we have

$$S(B(Y,X)U,W)+S(U,B(Y,X)W)=0. \tag{37}$$

Substituting $Y=U=\xi$ in (37), we have

$$S(B(\xi,X)\xi,W)+S(\xi,B(\xi,X)W)=0. \tag{38}$$

By the use of (1), (11), (22) we get from (38) that

$$\big[p_0+\frac{p_1}{4}\big[c(n-5)+7n-1\big]+2p_2r\big]\big[S(X,W)-(n-1)g(X,W)\big]=0. \tag{39}$$

This tell us either M is an Einstein manifold or the scalar curvature $r=\frac{(1-n)\left[2p_0+p_1(2n-3)\right]}{2p_1+4p_2(n-1)}$ of the M.
This completes of the proof. The Converse is obvious. ■

4. Conclusion

In this paper, we study the generalized B-curvature tensor of a normal paracontact metric manifold. Necessary and sufficient conditions are given for a normal paracontact metric manifold satisfying the conditions, generalized B-flatness, generalized B-semi-symmetric,$B.\widetilde{Z}=0$, $B.S=0$ and $B.P=0$. According these cases, we classified normal paracontact metric manifolds. The same classification can be made for other curvature tensors.

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