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Research Article

Strongly minimal complex lightlike hypersurfaces

Mehmet Gülbahar*[1], Erol Kılıç[2]

[1] *Department of Mathematics, Faculty of Science and Art, Harran University, Şanlıurfa, Turkey*
[2] *Department of Mathematics, Faculty of Science and Art, İnönü University, Malatya, Turkey*

Abstract

In this paper, complex lightlike hypersurfaces of an indefinite Kähler manifold are studied. An optimal inequality characterized to strongly minimality for coisotropic lightlike submanifolds is proved. Strongly minimal Monge-type hypersurfaces in $\mathbb{C}_1^4$ are examined and some examples of these hypersurfaces are given.

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1. Introduction

The theory of minimal surfaces dates back to Lagrange (1762) who is seeking an answer that 'which surfaces are locally minimized their areas'. These surfaces have a significant role in various branches of mathematics, science, and engineering since they are the solutions to the celebrated Euler-Lagrange equation. In differential geometry, especially in non-degenerate manifolds theory, a surface is called minimal if it has a vanishing mean curvature at every point.

Taking into consideration of famous J. F. Nash embedding theorem, S. S. Chern [8] stated the following natural problem in 1968:

Problem 1.1. Let (M, g) be a Riemannian manifold with a $(0, 2)$ type metric tensor g. What are the necessary conditions for M to admit a minimal immersion into a Euclidean space.

For many years, it was known that the only necessary condition for M to admit a minimal immersion is that its Ricci tensor is negative semi-definite (cf. [23]). For these reasons, B. Y. Chen [5] introduced a new Riemannian curvature invariant so-called delta curvature and established some optimal inequalities involving this curvature. In this way, he presented solutions of Problem 1.1 for submanifolds of Riemannian manifolds. In later years, many authors have been looking for a solution to this problem into various space forms and proved significant relations (cf. [1, 2, 7, 9, 18–21, 26]).

*Corresponding Author.
Email addresses: mehmetgulbahar@harran.edu.tr (M. Gülbahar), erol.kilic@inonu.edu.tr (E. Kılıç)
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In Kählerian settings, B.-Y. Chen [6] introduced the delta curvature δ^{λ} on Kähler manifold (M, g) for each λ real number and $p \in M$ as follows:

$$\delta^{\lambda}(p) = r(p) - \lambda \inf K(\Pi),$$

where r denotes the scalar curvature, K denotes the sectional curvature and Π runs over all totally real plane sections in the tangent space T_pM.

Furthermore, B.-Y. Chen [6] proved the following inequality for a $2n$-real dimensional Kähler submanifold of complex space forms $\widetilde{M}(4c)$ with codimension 2ρ:

$$\delta^4(p) \leq (2n^2 + 2n - 4)c. \tag{1.1}$$

The equality case of (1.1) holds if and only if there exists an orthonormal basis such that the shape operator take forms with respect to a suitable basis as follows:

$$A_{\xi_\alpha} = \begin{pmatrix} a_\alpha & b_\alpha & 0 \\ b_\alpha & -a_\alpha & \\ & 0 & 0 \end{pmatrix} \quad \text{and} \quad A_{J\xi_\alpha} = \begin{pmatrix} c_\alpha & d_\alpha & 0 \\ d_\alpha & -c_\alpha & \\ & 0 & 0 \end{pmatrix}, \tag{1.2}$$

$\alpha \in \{1, \ldots, \rho\}$, where the normal space of T_pM is spanned by $\xi_1, \ldots, \xi_\rho, J\xi_1, \ldots, J\xi_\rho$.

Inspired by the equality case of (1.1) inequality, B.-Y. Chen defined the notion of strongly minimality for Kähler submanifolds. In [22], B. D. Suceavă proved several characterizations of strongly minimal complex surfaces in the complex three-dimensional space and presented some nice examples of strongly minimal Kähler surfaces.

Motivated by these facts, we study the notion of strongly minimality into degenerate manifolds. We showed that every complex lightlike hypersurface of an indefinite Kähler manifold is minimal. We investigate the delta curvature on the screen distribution of screen conformal coisotropic lightlike submanifolds and prove an optimal inequality for these submanifolds. With the help of this inequality, we introduce strongly minimal complex lightlike hypersurfaces and we focus on these hypersurfaces in $\mathbb{C}_1^4$. By the way, we shall answer to the problem of Suceavă suggested in the Conclusion section of [22] that '*examine the strong minimality condition on various classes of submanifolds*'.

2. Preliminaries

Let $(\widetilde{M}, \widetilde{g})$ be a $2\widetilde{m}$-dimensional semi-Riemannian manifold with a semi-Riemannian metric $\widetilde{g}$ of constant index $\widetilde{q}$. Denote the Riemannian connection on $\widetilde{M}$ with respect to the $\widetilde{g}$ by $\widetilde{\nabla}$. The manifold $(\widetilde{M}, \widetilde{g})$ is called an indefinite Kähler manifold if there exists a tensor field J of type $(1,1)$ on $\widetilde{M}$ satisfying

$$J^2X = -X, \tag{2.1}$$

$$\widetilde{\nabla}_X J = 0, \tag{2.2}$$

$$\widetilde{g}(JX, JY) = \widetilde{g}(X, Y) \tag{2.3}$$

for all $X, Y \in \Gamma(T\widetilde{M})$.

Let $\widetilde{R}$ denotes the Riemannian curvature tensor with respect to $\widetilde{\nabla}$. Then the following relations hold on any Kaehler manifold $(\widetilde{M}, \widetilde{g}, J)$:

$$\widetilde{R}(JX, JY)Z = \widetilde{R}(X, Y)Z \tag{2.4}$$

and

$$\widetilde{R}(X, Y)JZ = J\widetilde{R}(X, Y)Z \tag{2.5}$$

for $X, Y, Z \in \Gamma(T\widetilde{M})$.

In complex geometry, any 2-dimensional non-degenerate tangent plane section is identified with the action of J. A plane section Π on $T\widetilde{M}$ is called as

i) holomorphic plane if $J\Pi \subset \Pi$, i.e., J is invariant on Π,
ii) anti-holomorphic plane if $J\Pi \subset \Pi^{\perp}$, where $\Pi^{\perp}$ is the complementary space of Π in $T\widetilde{M}$.

Now, let Π be a holomorphic plane section spanned by any orthonormal vector pair $\{X, Y\}$. From (2.4) and (2.5), we get the following two equalities:

$$\widetilde{K}(JX, JY) = \widetilde{K}(X, Y) \tag{2.6}$$

and

$$\widetilde{K}(X, JY) = \widetilde{K}(JX, Y). \tag{2.7}$$

Here, we note that $\widetilde{K}(\Pi) \equiv \widetilde{K}(X, Y)$ is called the holomorphic sectional curvature of Π cf.[24, 25].

Let (M, g) be a submanifold of $(\widetilde{M}, \widetilde{g})$. The induced metric g of M might be non-degenerate or degenerate on the tangent bundle $T\widetilde{M}$. If g is degenerate, then M becomes a lightlike submanifold. In this case, the vectors lie in the normal bundle of M intersects with the tangent bundle TM along a non-zero smooth distribution so-called radical distribution while trivial intersection in the non-degenerate case. The radical distribution at $p \in M$ is given by

$$\mathrm{Rad}\,(T_pM) = \{\xi \in T_pM : g_p(\xi, X) = 0,\ \forall X \in \Gamma(TM)\}. \tag{2.8}$$

The complementary non-degenerate vector bundle $\mathrm{S}(TM)$ of $\mathrm{Rad}\,(TM)$ onto TM is called the screen distribution. Therefore, we always have

$$TM = \mathrm{Rad}\,(TM) \oplus_{\mathrm{orth}} \mathrm{S}(TM), \tag{2.9}$$

where $\oplus_{\mathrm{orth}}$ denotes the orthogonal direct sum.

A lightlike submanifold (M, g) is called as coisotropic if the rank of $\mathrm{Rad}\,(TM)$ is equal to the co-dimension. In this case, the normal space of (M, g) becomes a null space. We will examine coisotropic submanifolds with codimension 2 throughout the study.

Now let $(M, g, \mathrm{S}(TM))$ be a coisotropic lightlike submanifold and $\{\xi_1, \xi_2\}$ be a local basis of $\mathrm{Rad}(TM)$. Then there exists a local null frame $\{N_1, N_2\}$ such that

$$\widetilde{g}(N_i, \xi_j) = \delta_{ij}, \quad \forall i, j \in \{1, 2\}, \tag{2.10}$$

where δ_{ij} is the Kronecker delta function. The vector bundle spanned by N_1 and N_2 is called lightlike transversal bundle, denoted by $\mathrm{ltr}(TM)$.

Let P be the projection on $\Gamma(TM)$ onto $\Gamma(\mathrm{S}(TM))$. The Gauss and Weingarten type formulas are given by

$$\widetilde{\nabla}_X Y = \nabla_X Y + \sigma(X, Y), \tag{2.11}$$

$$\widetilde{\nabla}_X N_\ell = -A_{N_\ell} X + \nabla^{\mathrm{t}}_X N_\ell, \tag{2.12}$$

$$\nabla_X Y = \nabla^*_X PY + \sigma^*(X, PY), \tag{2.13}$$

$$\nabla_X \xi_\ell = -A^*_{\xi_\ell} X + \nabla^{\mathrm{t}^*}_X \xi_\ell \tag{2.14}$$

for all $X, Y \in \Gamma(TM)$ and $\ell \in \{1, 2\}$. Here, ∇ and ∇^* are the induced linear connections, σ and σ^* are the second fundamental forms, A and A^* are the shape operators on TM and $\mathrm{S}(TM)$ respectively. It is known that σ and σ^* are related to A and A^* respectively by

$$\widetilde{g}(\sigma(X, PY), \xi_\ell) = g(A^*_{\xi_\ell} X, PY) \tag{2.15}$$

and

$$\widetilde{g}(\sigma^*(X, PY), N_\ell) = g(A_{N_\ell} X, PY). \tag{2.16}$$

For more details, we refer to [10, 12, 13, 15].

A coistropic lightlike submanifold $(M, g, \mathrm{S}(TM))$ is called screen locally conformal [3] if there exists non-vanishing smooth functions φ_1 and φ_2 on a neighborhood in M satisfying

$$A_{N_\ell} = \varphi_\ell A^*_{\xi_\ell}, \quad \ell \in \{1, 2\}. \tag{2.17}$$

The submanifold $(M, g, \mathrm{S}(TM))$ is called irrotational if σ vanishes on $\mathrm{Rad}(TM)$ [17] and it is called totally geodesic if σ vanishes identically. If there exists a smooth transversal vector field H satisfying

$$\sigma(X, Y) = g(X, Y)H, \quad \forall X, Y \in \Gamma(TM), \tag{2.18}$$

then the submanifold is called totally umbilical [11]. Furthermore, $(M, g, \mathrm{S}(TM))$ is called minimal [4] if it is irrotational and

$$\mathrm{trace}_{\mathrm{S}(TM)}[A^*_\ell] = 0, \quad \forall \ell \in \{1, 2\}. \tag{2.19}$$

Here, $\mathrm{trace}_{\mathrm{S}(TM)}$ is the trace restricted to $\mathrm{S}(TM)$.

Suppose that $\{e_1, \ldots, e_{n-1}, e_n = X\}$ be an orthonormal basis of $\Gamma(\mathrm{S}(TM))$. The screen Ricci curvature at a vector field X is given by

$$\mathrm{Ric}_{\mathrm{S}(TM)}(X) = \sum_{j=1}^{n-1} g((R(X, e_j)e_j, X)\,. \tag{2.20}$$

The screen scalar curvature at a point $p \in M$ is defined by [16]

$$r_{\mathrm{S}(TM)}(p) = \frac{1}{2} \sum_{i,j=1}^{n} K(e_i, e_j). \tag{2.21}$$

3. Complex lightlike hypersurfaces

Let $(\widetilde{M}, \widetilde{g}, J)$ be an indefinite Kähler manifold and $(M, g, \mathrm{S}(TM))$ be an $(n+2)$-dimensional coisotropic lightlike submanifold of $\widetilde{M}$ with codimension 2. The manifold M is called a complex lightlike hypersurface if both $\mathrm{S}(TM)$ and $\mathrm{Rad}\,(TM)$ remain invariant under the action J.

From (2.9) and (2.10), one can choose the following quasi-orthonormal basis on $T\widetilde{M}$:

$$\{\xi, J\xi, e_1, \ldots, e_n, e_1^* = Je_1, \ldots, e_n^* = Je_n, N, JN\}, \tag{3.1}$$

where $\mathrm{Rad}(TM) = \mathrm{Span}\{\xi, J\xi\}$, $\mathrm{S}(TM) = \mathrm{Span}\{e_1, \ldots, e_n\}$ and $\mathrm{ltr}(TM) = \mathrm{Span}\{N, JN\}$. Therefore, we may write the following equalities from (2.11) and (2.13) for any $X, Y \in \Gamma(TM)$:

$$\begin{aligned} \widetilde{\nabla}_X Y &= \nabla_X Y + h(X, Y)N + k(X, Y)JN, & (3.2)\\ \nabla_X Y &= \nabla^*_X Y + h^*(X, Y)\xi + k^*(X, Y)J\xi. & (3.3) \end{aligned}$$

Let us write the coefficients of second fundamental forms with respect to the basis $\{\xi, J\xi, N, JN\}$ of $\mathrm{S}(TM)^\perp$ as follows:

$$\begin{aligned} \sigma(X, Y) &= h(X, Y)N + k(X, Y)JN, & (3.4)\\ \sigma^*(X, Y) &= h^*(X, Y)\xi + k^*(X, Y)J\xi. & (3.5) \end{aligned}$$

Then the following relation between curvature tensors holds:

$$\begin{aligned} \widetilde{g}\left(\widetilde{R}(X, Y)PZ, PW\right) &= g\left(\widetilde{R}(X, Y)PZ, PW\right) - h(X, PZ)h^*(Y, PW)\\ &\quad -h(Y, PZ)h^*(Y, PW) - k(X, PZ)k^*(Y, PW)\\ &\quad -k(Y, PZ)k^*(X, PW) \end{aligned} \tag{3.6}$$

for any $X, Y, Z, W \in \Gamma(TM)$. The Eq.(3.6) is known as the Gauss-Codazzi type equation for coisotropic lightlike submanifolds.

Taking into consideration (2.1), (2.2) in (3.2) and (3.3), we get the following proposition immediately.

Proposition 3.1. *Let $(M, g, S(TM))$ be a complex lightlike hypersurface. Then we have the following equalities for all $X, Y \in \Gamma(TM)$.*

$$\begin{aligned} h(JX, Y) &= h(X, JY) = -k(X, Y), \\ h^*(JX, Y) &= h^*(X, JY) = -k^*(X, Y), \\ k(JX, Y) &= k(X, JY) = h(X, Y), \\ k^*(JX, Y) &= k^*(X, JY) = h^*(X, Y). \end{aligned}$$

Corollary 3.2. *If M is a complex lightlike hypersurface of indefinite Kähler manifold then*

$$trace_{S(TM)}[A^*_\xi] = trace_{S(TM)}[A^*_{J\xi}] = 0. \tag{3.7}$$

Proof. From Proposition 3.1, we write

$$h(JX, JY) = -h(X, Y) \quad \text{and} \quad h^*(JX, JY) = -h^*(X, Y). \tag{3.8}$$

If we consider (3.8) in (2.15), we have (3.7). □

Taking into consideration of Proposition 3.1 and Corollary 3.2, it is easy to see that the shape operators take the forms as

$$A^*_\xi = \begin{pmatrix} A^*_1 & A^*_2 \\ A^*_2 & -A^*_1 \end{pmatrix} \tag{3.9}$$

and

$$A^*_{J\xi} = \begin{pmatrix} -A^*_2 & A^*_1 \\ A^*_1 & A^*_2 \end{pmatrix}, \tag{3.10}$$

where A^*_1 and A^*_2 are $n \times n$ matrices.

Remark 3.3. In [14], K. L. Duggal and B. Sahin proved that any irrotational or totally umbilical invariant lightlike submanifold of an indefinite Kähler manifold is minimal (cf. [14, Corollary 3]). We note that Corollary 3.2 is also a special case of this result.

Corollary 3.4. *Let $(M, g, S(TM))$ be a complex lightlike hypersurface. The following assertions are equivalent for a complex lightlike hypersurface $(M, g, S(TM))$:*

i. *M is totally umbilical.*
ii. *M is totally geodesic.*
iii *At least one of h and k vanishes on TM identically.*

Corollary 3.5. *The following assertions are equivalent for a complex lightlike hypersurface $(M, g, S(TM))$:*

i. *$S(TM)$ is totally umbilical.*
ii. *$S(TM)$ is totally geodesic.*
iii *At least one of h^* and k^* vanishes on $S(TM)$ identically.*

From (3.6) and Proposition 3.1, we get the followings by a straightforward computations:

Theorem 3.6. *Let $(M, g, S(TM))$ be a complex lightlike hypersurface of $(\widetilde{M}, \widetilde{g})$. Let X be a unit vector and $\Pi = Span\{X, JX\}$ be a holomorphic plane section of $\Gamma(S(TM))$. Then we have*

$$K(X, JX) = \widetilde{K}(X, JX) - 2h(X, X)h^*(X, X) - 2k(X, X)k^*(X, X). \tag{3.11}$$

Theorem 3.7. *Let $(M, g, S(TM))$ be a complex lightlike hypersurface of $(\widetilde{M}, \widetilde{g})$. Suppose that X and Y are any two linearly independent vector fields on $\Gamma(S(TM))$ and $\Pi = Span\{X, JY\}$ is a 2-dimensional totally real plane section of $\Gamma(S(TM))$. Then we have*

$$K(X, JX) = \widetilde{K}(X, JY) - 2h(Y, Y), h^*(X, X) - 2k(X, Y), k^*(Y, X). \tag{3.12}$$

Now we shall introduce the notion of strongly minimality for complex lightlike hypersurfaces inspired by [6].

Definition 3.8. A complex lightlike hypersurface of an indefinite Kaehler manifold is called strongly minimal if at each point of p, there exists an orthonormal basis $\{e_1, \ldots, e_n, e_1^* = Je_1, \ldots, e_n^* = Je_n\}$ of $\Gamma(\mathrm{S}(TM))$ such that

$$\mathrm{trace} A_1^* = \mathrm{trace} A_2^* = 0. \tag{3.13}$$

We note that this definition is independent of $S(TM)$ and $S(TM)^{\perp}$ (cf. Proposition 3.1 and Definition 2 in [4]).

Following the δ-curvature definition of B.-Y. Chen in Riemannian and Kahlerian settings, we can give the notion of δ-curvature for a complex lightlike hypersurfaces admitting an integrable distribution.

Definition 3.9. Let $(M, g, \mathrm{S}(TM))$ be a complex lightlike hypersurface. Then for each real number λ, δ^{λ}-curvature at $p \in M$ is defined by

$$\delta^{\lambda}(p) = r_{\mathrm{S}(TM)}(p) - \lambda \inf \left(K(\Pi) \right), \tag{3.14}$$

where Π runs over all anti holomorphic plane sections in $\Gamma(\mathrm{S}(TM))$.

Remark 3.10. We note that the sectional curvature map does not need to be symmetric for any lighlike submanifold of a semi-Riemannian manifold. As a result of [10, Theorem 2.2], one can only introduce δ-curvature for lightlike submanifolds whose screen distributions are integrable.

Now, we shall present an optimal inequality involving the δ^4 curvature:

Theorem 3.11. *Let $(M, g, S(TM))$ be a $2(n+1)$-dimensional screen conformal complex lightlike hypersurface with conformal factors $\varphi_\ell > 0, \ell \in \{1, 2\}$ of co-dimension 2 in an indefinite complex space form $\widetilde{M}(4c)$. Then we have*

$$\delta^4(p) \leq (2n^2 + 2n - 4)c. \tag{3.15}$$

The equality case of (3.15) satisfies if and only if there exists an orthonormal basis

$$e_1, \ldots, e_n, e_{n+1} = Je_1, \ldots, e_{2n} = Je_n$$

of $\Gamma(S(TM))$ such that the shape operator on $S(TM)$ becomes as (3.9) and (3.10), where

$$A^*_{\xi_\alpha} = \begin{pmatrix} \begin{matrix} a & b \\ b & -a \end{matrix} & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad A^*_{J\xi_\alpha} = \begin{pmatrix} \begin{matrix} c & d \\ c & -c \end{matrix} & 0 \\ 0 & 0 \end{pmatrix} \tag{3.16}$$

for $\alpha \in \{1, 2\}$, that is, M is strongly minimal.

Proof. Let us choose an orthonormal basis $\{e_1, \ldots, e_n, e_{n+1}, \ldots, e_{2n}\}$ for $\Gamma(\mathrm{S}(TM))$. Using the fact that M is screen conformal, we have from (3.6) that

$$\begin{aligned} g\left(R(e_i, e_j, e_j, e_i)\right) &= \sum_{\ell=1}^{2} \varphi_\ell [h^*(e_i, e_i) h^*(e_j, e_j) + k^*(e_i, e_i) k^*(e_j, e_j) \\ &\quad - h^*(e_i, e_j)^2 - k^*(e_i, e_j)^2] + c \left\{ 1 + 3\widetilde{g}\left(Je_i, e_j\right)^2 \right\}. \end{aligned} \tag{3.17}$$

From Corollary 3.2 and Eq.(3.17), we get

$$2r_{\mathrm{S}(TM)}(p) = 4n(n+1)c - \sum_{\ell=1}^{2} \sum_{i,j=1}^{2n} \varphi_\ell [h^*(e_i, e_j)^2 + k^*(e_i, e_j)^2]. \tag{3.18}$$

Now, suppose that $\Pi = \text{Span}\{e_1, e_2\}$. Then the shape operator on $\mathrm{S}(TM)$ becomes as (3.9) and (3.10) with respect to such basis. By a straightforward computation, we have

$$\begin{aligned} 4n(n+1)c - 2r_{\mathrm{S}(TM)}(p) &\geq 4\sum_{\ell=1}^{2}\varphi_\ell[h^*(e_1,e_1)^2 + h^*(e_2,e_2)^2 + 2h^*(e_1,e_2)^2 \\ &\quad + k^*(e_1,e_1)^2 + k^*(e_2,e_2)^2 + 2k^*(e_1,e_2)^2] \\ &\geq -8\sum_{\ell=1}^{2}\varphi_\ell[h^*(e_1,e_1)h^*(e_2,e_2) - h^*(e_1,e_2)^2 \\ &\quad + k^*(e_1,e_1)k^*(e_2,e_2) - k^*(e_1,e_2)^2], \end{aligned}$$

which implies that

$$r_{\mathrm{S}(TM)} - 4\inf K(\Pi) \leq (2n^2 + 2n - 4)c. \tag{3.19}$$

The proof of the converse part is straightforward. □

Following the proof way of [6, Proposition 6], we immediately have the following proposition:

Proposition 3.12. *Let $(M, g, S(TM))$ be a 6-dimensional complex lightlike hypersurface of an indefinite complex space form $\widetilde{M}(4c)$. Then there exists an orthonormal basis $\{e_1, \dots, e_{2n}\}$ such that*

$$Ric_{S(TM)}(e_i) = 6c - 2(a^2 + b^2 + c^2 + d^2). \tag{3.20}$$

Corollary 3.13. *Let $(M, g, S(TM))$ be a 6-dimensional complex lightlike hypersurface of an indefinite almost complex manifold $\widetilde{M}$. Then we have*

$$Ric_{S(TM)}(e_i) - \widetilde{Ric}_{S(TM)}(e_i) \leq 0. \tag{3.21}$$

Equality case (3.21) satisfies for every unit tangent vector on M if and only if M is tottaly geodesic.

Corollary 3.14. *The screen Ricci curvature of every complex lightlike hypersurface into semi-Euclidean space is non-positive.*

Now we shall introduce the screen-framed Einstein manifolds in complex lightlike hypersurface considering Proposition 3.12 and following the definition of framed Einstein manifolds of B.-Y. Chen [6] which is a generalization of Einstein manifolds.

Definition 3.15. An $(2n+2)$-dimensional complex lightlike hypersurface $(M, g, \mathrm{S}(TM))$ is called screen framed if there exists a function γ and orthonormal frame $\{e_1, e_2, \dots, e_{2n}\}$ on $\mathrm{S}(TM)$ such that

$$\mathrm{Ric}_{\mathrm{S}(TM)}(e_i) = \gamma g(e_i, e_i), \quad i \in \{1, \dots, 2n\}. \tag{3.22}$$

From Proposition 3.12 and Definition 3.15, we get the following corollary:

Corollary 3.16. *Every 6-dimensional complex lightlike hypersurface of an indefinite complex space form is framed Einstein manifold.*

4. Monge type complex lightlike hypersurfaces in $\mathbb{C}_1^4$

A hypersurface in $\mathbb{C}_1^{n+1}$ is defined with the aid of a holomorphic function ϕ by

$$\{z = (z_1, z_2, \dots, z_{n+1}) \in \mathbb{C}_1^{n+1} : \ \phi(z) = 0\}, \tag{4.1}$$

where $\frac{\partial\phi}{\partial z} = \left(\frac{\partial\phi}{\partial z_1}, \dots, \frac{\partial\phi}{\partial z_n}\right)$ never vanishes. Suppose that M is a Monge-type hypersurface of $\mathbb{C}_1^{n+1}$. Then there exists a smooth function $F : D \to \mathbb{C}$ such that

$$M = \{(z_1, z_2, \dots, z_{n+1}) \in \mathbb{C}_1^{n+1} : \ z_{n+1} = F(z_1, \dots, z_n)\}, \tag{4.2}$$

where D is an open set of $\mathbb{C}_1^{n+1}$. In this case, the natural frame fields on $\Gamma(TM)$ are given by

$$e_\alpha = \frac{\partial}{\partial z_\alpha} + F'_{z_\alpha}\frac{\partial}{\partial z_{n+1}}, \qquad \alpha \in \{1, \dots n\}$$

and the natural frame field on $\Gamma(TM)^\perp$ is given by

$$\xi = \sum_{\alpha=1}^{n} F'_{z_\alpha}\frac{\partial}{\partial z_\alpha} + \frac{\partial}{\partial z_{n+1}}.$$

Hence, we state the following proposition:

Proposition 4.1. *A Monge hypersurface M in $\mathbb{C}_1^{n+1}$ is lightlike if and only if F is a solution of the following partial differential equation:*

$$\sum_{i=1}^{n+1} F_{z_i}\overline{F_{z_i}} = 1. \tag{4.3}$$

Here, $\overline{F'_{z_i}}$ denotes the complex conjugate of F'_{z_i}.

In view of Theorem 6.4, Corollary 6.5 and Proposition 6.2 of K. L. Duggal and A. Bejancu in [10], the followings could be given:

Theorem 4.2 ([10])**.** *Let M be a lightlike Monge type hypersurface of $\mathbb{C}_1^{n+1}$. Then the following statements are satisfied:*

i. *$S(TM)$ is integrable.*
ii. *The second fundamental form σ^* is symmetric and $\sigma^* = \frac{1}{2}\sigma$.*
iii. *A_N is symmetric with respect to the induced metric g.*

Proposition 4.3 ([10])**.** *The Ricci tensor of a lightlike Monge type hypersurface of $\mathbb{C}_1^{n+1}$ is symmetric.*

Now, let M be a Monge-type complex lightlike hypersurface of $\mathbb{C}_1^4$ and V be an open set on $\mathbb{C}^3$. Then we may define a function $\omega : V \to \mathbb{C}_1^4$ such that

$$\omega(z_1, z_2, z_3) = (z_1, z_2, z_3, F(z_1, z_2, z_3)). \tag{4.4}$$

Define the function F with the aid of real and imaginary parts as follows:

$$F(z_1, z_2, z_3) = u(x_1, y_1, x_2, y_2, x_3, y_3) + iv(x_1, y_1, x_2, y_2, x_3, y_3). \tag{4.5}$$

Since F is a holomorphic function, we clearly have

$$u_{x_j} = v_{y_j} \text{ and } v_{x_j} = -u_{y_j}, \quad \forall j \in \{1, 2, 3\}. \tag{4.6}$$

In view of (4.4), there exists the following basis for T_pM at $p \in M$:

$$\begin{aligned}
\xi &= (u_{x_1}, -u_{y_1}, u_{x_2}, -u_{y_2}, u_{x_3}, -u_{y_3}, 0, 1), \\
e_1 &= (u_{x_2}, u_{y_2}, u_{x_1}, u_{y_1}, 0, 0, 0, 0), \\
e_2 &= (0, 0, 0, 0, 0, 1, u_{x_3}, u_{y_3}), \\
J\xi &= (u_{y_1}, u_{x_1}, u_{y_2}, u_{x_2}, u_{y_3}, u_{x_3}, -1, 0), \\
Je_1 &= (-u_{y_2}, -u_{x_2}, -u_{y_1}, -u_{x_1}, 0, 0, 0, 0), \\
Je_2 &= (0, 0, 0, 0, -1, 0, -u_{y_3}, u_{x_3}),
\end{aligned}$$

where $\mathrm{Rad}(TM) = \mathrm{Span}\{\xi\}$ and $\mathrm{S(TM)} = \mathrm{Span}\{e_1, e_2\}$. Here, the almost complex structure on R_2^8 is defined by

$$JX \equiv J(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8) = (-x_2, x_1, -x_4, x_3, -x_6, x_7, x_8, -x_7)$$

for any vector field $X = (x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8)$ on R_2^8.

Let $\widetilde{\nabla}$ be the Levi-Civita connection on $\mathbb{C}_1^4$. Then, we get

$$\begin{aligned}
\widetilde{\nabla}_\xi \xi &= (-u_{x_1x_1}u_{x_1} + u_{x_1y_1}u_{y_1} + u_{x_1x_2}u_{x_2} - u_{x_1y_2}u_{y_2} \\
&\quad +u_{x_1x_3}u_{x_3} - u_{x_1y_3}u_{y_3}, u_{y_1x_1}u_{x_1} - u_{y_1y_1}u_{y_1} \\
&\quad -u_{y_1x_2}u_{x_2} + u_{y_1y_2}u_{y_2} - u_{y_1x_3}u_{x_3} + u_{y_1y_3}u_{y_3}, \\
&\quad -u_{x_2x_1}u_{x_1} + u_{x_2y_1}u_{y_1} + u_{x_2x_2}u_{x_2} - u_{x_2y_2}u_{y_2} \\
&\quad +u_{x_2x_3}u_{x_3} - u_{x_2y_3}u_{y_3}, u_{y_2x_1}u_{x_1} - u_{y_2y_1}u_{y_1} \\
&\quad -u_{y_2x_2}u_{x_2} + u_{y_2y_2}u_{y_2} - u_{y_2x_3}u_{x_3} + u_{y_2y_3}u_{y_3}, \\
&\quad -u_{x_3x_1}u_{x_1} + u_{x_3y_1}u_{y_1} + u_{x_3x_2}u_{x_2} - u_{x_3y_2}u_{y_2} \\
&\quad +u_{x_3x_3}u_{x_3} - u_{x_3y_3}u_{y_3}, u_{y_3x_1}u_{x_1} - u_{y_3y_1}u_{y_1} \\
&\quad -u_{y_3x_2}u_{x_2} + u_{y_3y_2}u_{y_2} - u_{y_3x_3}u_{x_3} + u_{y_3y_3}u_{y_3}, 0, 0), \qquad (4.7) \\
\widetilde{\nabla}_{e_1} e_1 &= (u_{x_2x_1}u_{x_2} + u_{x_2y_1}u_{y_2} - u_{x_2x_2}u_{x_1} - u_{x_2y_2}u_{y_1}, \\
&\quad u_{y_2x_1}u_{x_2} + u_{y_2y_1}u_{y_2} - u_{y_2x_2}u_{x_1} - u_{y_2y_2}u_{y_1}, \\
&\quad -u_{x_1x_1}u_{x_2} - u_{x_1y_1}u_{y_2} + u_{x_1x_2}u_{x_1} + u_{x_1y_2}u_{y_1}, \\
&\quad -u_{y_1x_1}u_{x_2} - u_{y_1y_1}u_{y_2} + u_{y_1x_2}u_{x_1} + u_{y_1y_2}u_{y_1}, \\
&\quad 0, 0, 0, 0), \qquad (4.8) \\
\widetilde{\nabla}_{e_2} e_2 &= (0, 0, 0, 0, 0, 0, u_{x_3x_3}, u_{y_3x_3}), \qquad (4.9) \\
\widetilde{\nabla}_{e_1} e_2 &= 0. \qquad (4.10)
\end{aligned}$$

Now we shall give some examples of strongly minimal complex lightlike hypersurfaces in $\mathbb{C}_1^4$ given by a smooth function F as follows:

$$M = \{(z_1, z_2, z_3, F(z_1, z_2, z_3)) : \ z_1 z_2, z_3 \in \mathbb{C}\}. \qquad (4.11)$$

Example 4.4. Consider a hyperlane in $\mathbb{C}_1^4$ given by

$$F(z_1, z_2, z_3) = z_1.$$

It is an example of totally geodesic complex lightlike hypersurface.

Example 4.5. Consider a hyperplane in $\mathbb{C}_1^4$ given by

$$F(z) = \sqrt{3} z_1 + z_2 + z_3.$$

It is an example of totally geodesic complex lightlike hypersurface.

Example 4.6. Consider a Monge-type surface in $\mathbb{C}_1^4$ defined by

$$F(z_1 z_2, z_3) = \sqrt{3} e^{z_1} + e^{z_2} + e^{z_3}.$$

In this case, F is a harmonic function and

$$u(x_1, x_2, x_3, y_1, y_2, y_3) = \sqrt{3} e^{x_1} \cos y_1 + e^{x_1} \cos y_2 + e^{x_1} \cos y_3.$$

Therefore, we get

$$\begin{aligned}
\xi &= (e^{x_1} \cos y_1, e^{x_1} \sin y_1, e^{x_2} \cos y_2, e^{x_2} \sin y_2, e^{x_3} \cos y_3, e^{x_3} \sin y_3, 0, 1), \\
e_1 &= (e^{x_2} \cos y_2, -e^{x_2} \sin y_2, e^{x_1} \cos y_1, -e^{x_1} \sin y_1, 0, 0, 0, 0), \\
e_2 &= (0, 0, 0, 0, 0, 1, e^{x_2} \cos y_2, -e^{x_2} \sin y_2),
\end{aligned}$$

which imply that M is a complex lightlike hypersurface with Rad $(TM) = \mathrm{Span}\{\xi, J\xi\}$ and $\mathrm{S}(TM) = \mathrm{Span}\{e_1, e_2, Je_1, Je_2\}$ if

$$e^{2x_2} + e^{2x_3} = 3e^{2x_1} - 1.$$

By a straightforward computation, we obtain

$$h(e_1, e_1) = -\sqrt{3} e^{x_1 + x_2} \left[e^{x_1} \cos y_2 + e^{x_2} \cos y_1 \right]$$

and

$$h(e_2, e_2) = e^{x_3} [\cos y_3 + \cos y_3] .$$

Thus, the surface M is a strongly minimal at $p = (0,0,0,0)$ with $a = -1$ and $b = 0$.

Further examples could be given.

Proposition 4.7. *Let M is a complex Monge-type complex lightlike hypersurface of $\mathbb{C}_1^4$. Then there exists a frame field $\{\xi, J\xi, e_1, e_2, Je_1, Je_2\}$ of M satisfying the following equation:*

$$\sum_{i=1}^{3} (u_{x_i})^2 u_{x_i x_i} - (u_{y_i})^2 u_{y_i y_i} + \sum_{1 \le i \ne j \le 3} \varepsilon_i \varepsilon_j [u_{x_i x_j} u_{x_i} u_{x_j} - u_{y_i y_j} u_{y_i} u_{y_j}] = 0. \quad (4.12)$$

Proof. Using the facts that $\widetilde{\nabla}$ is the metric connection, $\sigma(\xi, \xi) = 0$, (2.11) and (4.7), the proof of proposition is straightforward. □

Proposition 4.8. *Any Monge-type complex lightlike hypersurface of $\mathbb{C}_1^4$ is totally geodesic if and only if there exists a frame field $\{\xi, J\xi, e_1, e_2, Je_1, Je_2\}$ satisfying the following differential equations:*

$$\begin{aligned} &u_{x_2 x_2} u_{x_1} u_{x_1} - 2u_{x_1 x_1} u_{x_2} u_{x_2} - u_{y_1 y_1} u_{y_2} u_{y_2} - u_{y_2 y_2} u_{y_1} u_{y_1} \\ &+2u_{x_1 y_2} u_{x_2} u_{y_1} - 2u_{x_2 y_1} u_{x_1} u_{y_2} = 0 \end{aligned} \quad (4.13)$$

and

$$u_{x_3 y_3} = 0. \quad (4.14)$$

Proof. Under the assumption, putting (4.8) and (4.9) in (2.11), we get (4.13) and (4.14), respectively. □

Let $(M, g, \mathrm{S}(TM))$ be a Monge-type complex lightlike hypersurface of $\mathbb{C}_1^4$. If M is strongly minimal, then the shape operator takes forms with respect to an orthonormal basis $\{X, Y, JX, JY\}$ on an open neighborhood U of $\mathrm{S}(TM)$ as

$$A_\xi^* = \begin{pmatrix} a(z) & b(z) & c(z) & d(z) \\ b(z) & -a(z) & d(z) & -c(z) \\ c(z) & d(z) & -a(z) & -b(z) \\ d(z) & -c(z) & -b(z) & a(z) \end{pmatrix} \quad (4.15)$$

and

$$A_{J\xi}^* = \begin{pmatrix} -c(z) & -d(z) & a(z) & b(z) \\ -d(z) & c(z) & b(z) & -a(z) \\ a(z) & b(z) & c(z) & d(z) \\ b(z) & -a(z) & d(z) & -c(z) \end{pmatrix}, \quad (4.16)$$

where a, b, c and d are real analytic functions on U.

Proposition 4.9. *Let $(M, g, S(TM)$ be a strongly minimal Monge-type complex lightlike hypersurface of $\mathbb{C}_1^4$. The shape operators A_ξ^* and $A_{J\xi}^*$ become as* (4.15) *and* (4.16) *if the frame field $\{\xi, J\xi, e_1, e_2, Je_1, Je_2\}$ of M satisfying the following equation:*

$$\begin{aligned} &u_{x_2 x_2} u_{x_1} u_{x_1} - 2u_{x_1 x_1} u_{x_2} u_{x_2} - u_{y_1 y_1} u_{y_2} u_{y_2} - u_{y_2 y_2} u_{y_1} u_{y_1} + 2u_{x_1 y_2} u_{x_2} u_{y_1} \\ &-2u_{x_2 y_1} u_{x_1} u_{y_2} + u_{x_3 y_3} = 0. \end{aligned} \quad (4.17)$$

Proof. From (4.8) and (4.9), we have

$$\begin{aligned} \langle A_\xi^* e_1, e_1 \rangle &= \langle \widetilde{\nabla}_{e_1} e_1, \xi \rangle \\ &= u_{x_2 x_2} u_{x_1} u_{x_1} - 2u_{x_1 x_1} u_{x_2} u_{x_2} - u_{y_1 y_1} u_{y_2} u_{y_2} - u_{y_2 y_2} u_{y_1} u_{y_1} \\ &\quad +2u_{x_1 y_2} u_{x_2} u_{y_1} - 2u_{x_2 y_1} u_{x_1} u_{y_2} \end{aligned} \quad (4.18)$$

and

$$\langle A^*_\xi e_2, e_2\rangle = \langle \widetilde{\nabla}_{e_2} e_2, \xi\rangle = u_{x_3y_3}. \tag{4.19}$$

Therefore, it is clear that the condition of strongly minimality is satisfies with respect to the given basis if the differential equation given by (4.17) holds. Hence the proof is completed. □

Now, let us write

$$\begin{aligned} X &= \lambda_1 e_1 + \lambda_2 e_2, \\ Y &= \lambda_3 e_1 + \lambda_4 e_2, \end{aligned} \tag{4.20}$$

where $\lambda_i \in$, $i \in \{1,2,3,4\}$, are real numbers. Taking into account of (4.15) and (4.16) and Proposition 4.9, we have the following corollary:

Corollary 4.10. *For any strongly minimal Monge-type complex lightlike hypersurface of $\mathcal{C}^4_1$, we have the following differential equation:*

$$\begin{aligned} &\left(\lambda_1^2+\lambda_3^2\right)(u_{x_2x_2}u_{x_1}u_{x_1} - 2u_{x_1x_1}u_{x_2}u_{x_2} - u_{y_1y_1}u_{y_2}u_{y_2} - u_{y_2y_2}u_{y_1}u_{y_1} \\ &\qquad +2u_{x_1y_2}u_{x_2}u_{y_1} - 2u_{x_2y_1}u_{x_1}u_{y_2}) + \left(\lambda_2^2+\lambda_4^2\right)u_{x_3y_3} = 0. \end{aligned} \tag{4.21}$$

Corollary 4.11. *Let $\{X, Y, JX, JY\}$ be an orthonormal basis on an open set U of $S(TM)$ which satisfies the strongly minimality condition. Then we have*

$$Ric_{S(TM)}(X) = Ric_{S(TM)}(Y) = -2(a^2+b^2+c^2+d^2). \tag{4.22}$$

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