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A Bound On The Spectral Radius of A Weighted Graph

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ABSTRACT

Let G be simple, connected weighted graphs, where the edge weights are positive definite matrices. In this paper, we will give an upper bound on the spectral radius of the adjacency matrix for a graph G and characterize graphs for which the bound is attained.

Key Words: *Weighted graph; Adjacency matrix; Spectral radius; Upper bound.*

1. INTRODUCTION

We consider simple graphs, that is, graphs which have no loops or parallel edges. Thus a graph $G=(V,E)$ consist of a finite set of vertices, V, and a set of edges, E, each of whose elements is an unordered pair of distinct vertices. We generally take $V=\{1,2,...,n\}$.

A weighted graph is a graph, each edge of which has been assigned a square matrix, called the weight of the edge. All the weight matrices will be assumed to be of the same order and will be assumed to be positive definite. In this paper, by "weighted graph" we will mean a "weighted graph with each of its edges bearing a positive definite matrix as weight", unless otherwise stated.

Let G be a weighted graph with *n* vertices. Denote by $w_{i,j}$ the positive definite weight matrix of order *p* of the edge *ij,* and assume that $w_{i,j}=w_{j,i}$.We write $i\sim j$ if vertices i and j are adjacent. For $i\in V$, the set of neighbors of i is denoted by N_i. Let $w_i=\sum_{j:i\sim j} w_{i,j}$.

The adjacency matrix of a graph G is a block matrix, defined as $A(G)=\left(a_{i,j}\right)$, where

$$a_{i,j}=\begin{cases} w_{i,j} & : i\sim j \\ 0 & :otherwise. \end{cases}$$

In the definition above, the zero denotes the pxp zero matrix. Thus *A(G)* is a square matrix of order np. On the other hand, the eigenvalues of a graph are the eigenvalues of its adjacency matrix. Thoroughly this paper, $|\rho_1|$ is called the spectral radius of a matrix.

In literature [1,4,5,6,7], there are a lot of studies deal with upper and lower bounds for the spectral radius of unweighted graphs. In this paper we obtain an upper bound for weighted graph, and this bound compare with Das and Bapat's bound in [2].

2. A BOUND ON SPECTRAL RADIUS OF WEIGHTED GRAPHS

The following is a consequence of the Cauchy-Schwarz inequality. The proof is omitted.

Lemma 2.1 (Horn and Johnson [3]) Let B be a Hermitian nxn matrix with ρ_1 as its largest eigenvalue, in modulus. Then for any $\overline{x}\in R^n\,(\overline{x}\neq 0), \overline{y}\in R^n\,(\overline{y}\neq 0)$, the spectral radius satisfies

$$\left|\overline{x}^T B\overline{y}\right|\le|\rho_1|\sqrt{\overline{x}^T\overline{x}}\sqrt{\overline{y}^T\overline{y}} \qquad (1)$$

Equality holds if and only if $\overline{x}$ is an eigenvector of B corresponding to ρ_1 and $\overline{y}=\alpha\overline{x}$ for some $\alpha\in R$.

Lemma 2.2 (Weyl, Horn and Johnson [3]) Let $A,B\in M_n$ be Hermitian and $\rho_i(A),\rho_i(B)$ and $\rho_i(A+B)$ be arranged in increasing order ($\rho_n\le\rho_{n-1}\le...\le\rho_2\le\rho_1$). For each $k=1,2,...,n$ we have

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$\rho_k(A)+\rho_n(B)\le\rho_k(A+B)\le\rho_k(A)+\rho_1(B)$.

Lemma 2.3 (Das and Bapat [2]) Let $B_1,B_2,...,B_k$ be positive definite matrices of order n and let $B=\sum_{i=1}^{k}B_i$. If $\bar{x}$ is an eigenvector of each B_i corresponding to largest eigenvalue $\rho_1(B_i)$ for all i , then $\bar{x}$ is also an eigenvector of B corresponding to largest eigenvalue $\rho_1(B)$.

Theorem 2.4 Let G be a weighted graph which is simple, connected and let ρ_1 be the largest eigenvalue (in modulus) of G , so that $|\rho_1|$ is the spectral radius of G . Then,

$$|\rho_1|\le\max_{i\sim j}\left\{\sqrt{\sum_{k:k\sim i}\rho_1(w_{i,k})\sum_{k:k\sim j}\rho_1(w_{j,k})}\right\} \qquad (2)$$

where $w_{i,j}$ is the positive definite weight matrix of order p of the edge ij . Moreover equality holds if and only if

i) G is a weight-regular graph or G is a weight-semiregular bipartite graph.

ii) $w_{i,j}$ has a common eigenvector corresponding to the largest eigenvalue $\rho_1(w_j)$ for all i,j [2].

Theorem 2.5. Let G be a weighted graph which is simple, connected and let ρ_1 be the largest eigenvalue (in modulus) of G , so that $|\rho_1|$ is the spectral radius of G . Then,

$$|\rho_1|\le\max_{i}\left\{\sqrt{\sum_{k:k\sim i}\rho_1(w_{i,k}^2)+\sum_{j}\sum_{k'\in N_i\cap N_j}\rho_1(w_{j,k'}w_{k',i})}\right\} \qquad (3)$$

where $w_{i,j}$ is the positive definite weight matrix of order p of the edge ij , $N_i\cap N_j$ is the set of common neighbors of i and j . Moreover, equality holds if and only if

i) G is a weight-regular graph or G is a weight-semiregular bipartite graph

ii) $w_{i,j}$ has a common eigenvector corresponding to the largest eigenvalue $\rho_1(w_j)$ for all i,j .

Proof. Let consider matrix $A^2(G)$ such that $A(G)$ is the adjacency matrix of graph G and $|\rho_1|$ the spectral radius of $A(G)$ adjacency matrix. So, $|\rho_1|^2$ is also the spectral radius of $A^2(G)$.

Thus, we get

Let $\bar{X}=\left(\bar{x}_1^T,\bar{x}_2^T,...,\bar{x}_n^T\right)^T$ be an eigenvector corresponding to the spectral radius $|\rho_1|^2$ for $A^2(G)$. We assume that $\bar{x}_i$ is the vector component of X such that

$$\bar{x}_i^T\bar{x}_i=\max_{k\in V}\left\{\bar{x}_k^T\bar{x}_k\right\} . \qquad (4)$$

Since X is nonzero, so is $\bar{x}_i$. We have

$$A^2(G)\bar{X}=\rho_1^2\bar{X} \qquad (5)$$

Since G is a simple, connected and $w_{i,j}=w_{j,i}$, the (i,j) th block of $A^2(G)$ matrix is defined by

$$\begin{cases}\sum_{k:k\sim i}w_{i,k}^2 & :if\ i=j\\ \sum_{k\in N_i\cap N_j}w_{j,k}w_{k,i} & :if\ N_i\cap N_j\neq\varnothing\\ 0 & :if\ N_i\cap N_j=\varnothing\end{cases} \qquad (6)$$

From the i-th equation of (5), we have

$$\rho_1^2\bar{x}_i=\sum_{k:k\sim i}w_{i,k}^2x_i+\sum_j\sum_{k:k\in N_i\cap N_j}w_{i,k}w_{k,j}\bar{x}_j \qquad (7)$$

i.e.,

$$\rho_1^2\bar{x}_i^T\bar{x}_i=\sum_{k:k\sim i}\bar{x}_i^Tw_{i,k}^2x_i+\sum_j\sum_{k:k\in N_i\cap N_j}\bar{x}_i^Tw_{i,k}w_{k,j}\bar{x}_j \qquad (8)$$

i.e.

$$\rho_1^2\bar{x}_i^T\bar{x}_i=\left|\sum_{k:k\sim i}\bar{x}_i^Tw_{i,k}^2x_i+\sum_j\sum_{k:k\in N_i\cap N_j}\bar{x}_i^Tw_{i,k}w_{k,j}\bar{x}_j\right| \qquad (9)$$

$$\le\sum_{k:k\sim i}\left|\bar{x}_i^Tw_{i,k}^2x_i\right|+\sum_j\sum_{k:k\in N_i\cap N_j}\left|\bar{x}_i^Tw_{i,k}w_{k,j}\bar{x}_j\right| \text{ by (1)} \qquad (10)$$

$$\le\left\{\begin{array}{l}\sqrt{\bar{x}_i^T\bar{x}_i}\sqrt{\bar{x}_i^T\bar{x}_i}\sum_{k:k\sim i}\rho_1\left(w_{i,k}^2\right)\\ +\sqrt{\bar{x}_i^T\bar{x}_i}\sqrt{\bar{x}_j^T\bar{x}_j}\sum_j\sum_{k:k\in N_i\cap N_j}\rho_1\left(w_{i,k}w_{k,j}\right)\end{array}\right\}\text{ by(4)} \qquad (11)$$

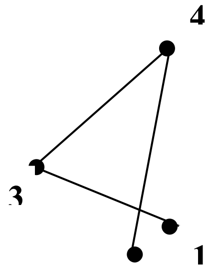
$$\le\sqrt{\bar{x}_i^T\bar{x}_i}\sqrt{\bar{x}_i^T\bar{x}_i}\sum_{k:k\sim i}\rho_1\left(w_{i,k}^2\right)+\sqrt{\bar{x}_i^T\bar{x}_i}\sqrt{\bar{x}_i^T\bar{x}_i}\sum_j\sum_{k:k\in N_i\cap N_j}\rho_1\left(w_{i,k}w_{k,j}\right) \qquad (12)$$

$$|\rho_1| \le \sqrt{\sum_{k:k\sim i} \rho_1(w_{i,k}^2) + \sum_j \sum_{k\in N_i \cap N_j} \rho_1(w_{i,k}w_{k,j})}$$

$$\le \max_i \left\{ \sqrt{\sum_{k:k\sim i} \rho_1(w_{i,k}^2) + \sum_j \sum_{k\in N_i \cap N_j} \rho_1(w_{i,k}w_{k,j})} \right\}$$

Hence this completes the proof of (4).

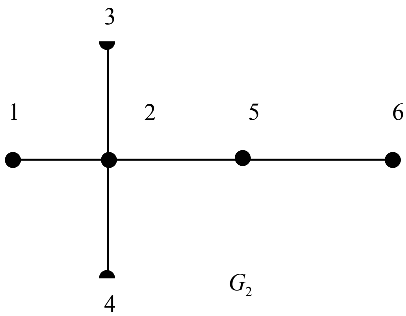
Example 2.5 Let G_1 and G_2 following graphs.



G_1

$$w_{3,4} = w_{4,3} = \begin{bmatrix} 6 & 2 & -2 \\ 2 & 6 & -2 \\ -2 & -2 & 10 \end{bmatrix},$$

$$w_{2,4} = w_{4,2} = \begin{bmatrix} 5 & 0 & 2 \\ 0 & 5 & 2 \\ 2 & 2 & 5 \end{bmatrix}, w_{1,3} = w_{3,1} = \begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & -1 \\ -1 & -1 & 5 \end{bmatrix}$$



G_2

$$w_{1,2} = w_{2,1} = \begin{bmatrix} 1 & 1 \\ 1 & 4 \end{bmatrix}, w_{2,3} = w_{3,2} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$w_{2,4} = w_{4,2} = \begin{bmatrix} 5 & 2 \\ 2 & 5 \end{bmatrix}, w_{2,5} = w_{5,2} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$w_{5,6} = w_{6,5} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

The bounds of spectral radius are following:

	ρ_1	(2)	(3)
G_1	13.63	18.88	16.64
G_2	8.63	10.34	8.91

Consequently, we see that the bound in (3) is better than the bound (2). But, this is an open problem for all weighted graphs.

Corollary 2.6 Let G be a weighted graph which is simple, connected in which the edge weight are positive number (i.e. $1x1$ matrices). Then,

$$|\rho_1| \leq \max_i \left\{ \sqrt{d_i + \sum_j |N_i \cap N_j|} \right\}$$

where d_i is the degree of vertex i and $|N_i \cap N_j|$ is the number of common neighbors of i and j vertices.

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