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Solutions of Emden-Fowler Type Equations by Variational Iteration Method

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Abstract: In this research article, a numerical approach to the solutions of different forms of Emden-Fowler is presented. The variational Iteration Method (VIM) has been applied to a wide class of problems in mathematical physics, biology and chemical reactions. The solutions provided in this research proved that the method converged rapidly with easily computable terms.

Keywords: Emden-Fowler equation, Lane-Emden equation, Variational Iteration Methods, singular initial value problem.

1. Introduction

The Variational Iteration Method [1-5] has been applied to many real life problems in theoretical physics, mathematical physics, biological sciences and in recent years, a great deal of attention has been devoted to the study of the method.

Many problems in applied mathematical physics that occur on semi-infinite interval, are related to Emden-Fowler equation [6-16]. The equation is of the form:

$$u''(t)+\frac{h'(t)}{h(t)}u'(t)+f(t,u(t))=g(t) \tag{1}$$

$$u(0)=A, \quad u'(0)=B$$

where $f(t,u(t))$ and $g(t)$ are continuous real valued functions $h(t)$ is a continuous and differentiable function with $h(t)\neq 0$.

Equations such as Emden-Fowler and Lane-Emden are special cases of equation (1).

The Emden-Fouler equation of the form:

$$u''(t)+\frac{k}{t}u_t + f(t)f(u)=0, t>0 \tag{2}$$

$$u(0)=1,\ u'(0)=0$$

is sometimes called the generalized Lane-Emden equation while the standard Lane-Emden equation:

$$u''(t)+\frac{k}{t}u_t + f(u)=0, t>0 \tag{3}$$

was first studied by the astrophysicists Jonathan Homer Lane and Robert Emden [6]. The equations had attracted many researchers in the field of applied mathematics and computation. A. M. Wazwaz [7] presented Adomian decomposition method for the analytical solution of the time dependent Emden-Fowler type equations. He also investigated the solution of Lane-Emden problem [8]. Aslanov [9] studies the Emden Fowler type equation and presented approximate solution to some numerical problems. Homotopy perturbation method was used to solve Lane-Emden problems [10]. Recently, Wazwaz [11] presented the Variational Iteration Method for solving the Volterra Integro differential forms of the Lane-Emden and the Emden-Fowler problems.

Different approaches were also presented for the Emden-Fowler and Lane-Emden problems in [12-16].

2. Variational Iteration Method

The basic idea of the He's Variational Iteration Method (VIM) [1-5] can be explained by considering the following nonlinear partial differential equations:

$$Lu(t)+Nu(t)=g(t) \tag{4}$$

where L is the linear operator, N is the nonlinear operator and $g(t)$ is the inhomogeneous term. According to the method, the corresponding variational iteration method for solving (4) is given as:

$$u_{n+1}(t) = u_n(t) + \int_0^t \lambda(\tau)\left[Lu_n(\tau) + N\bar{u}_n(\tau) - g(\tau) \right] d\tau \tag{5}$$

where λ is a Lagrange multiplier which can be identified optimally by variational iteration method. The subscript n denote the nth approximation, $\bar{u}_n$ is considered as a restricted variation i.e $\delta\bar{u}_n = 0$. The successive approximation $u_{n+1,} n \geq 0$ of the solution u can be easily obtained by determine the Lagrange multiplier and the initial guess u_0, consequently, the solution is given by $u = \lim_{n\to\infty} u_n$.

In order to solve the equation (1) by means of the method, a correction functional can be constructed as follows:

$$u_{n+1}(t) = u_n(t) + \int_0^t \lambda(t,\tau)(u_n''(\tau) + \frac{h'(\tau)}{h(\tau)}u_n'(\tau) + \tilde{f}(\tau, u_n(\tau)) - g(\tau))d\tau \tag{6}$$

Making the equation (6) correction functional stationary and following the stationary conditions, the Lagrange muiltipler can be readily identified as:

$$\lambda(t,\tau) = h(\tau)\left(\int \frac{d\tau}{h(\tau)} - \int \frac{dt}{h(t)} \right) \tag{7}$$

3. Numerical Examples

In this section, VIM will be applied to some cases of Emden-Fowler type problems.

Problem 1: Consider the following linear, non homogeneous singular initial value problem:

$$u''(t) + \frac{2}{t}u'(t) + u = t^4 + t^3 + 21t^2 + 12t + 6 \tag{8}$$

with initial conditions:

$$u(0) = u'(0) = 0$$

Here,

$$h(t) = t^2 \text{ and } \lambda(t,\tau) = \frac{\tau^2 - t\tau}{t}$$

The successive approximations are:

$$u_1(t) = \frac{1}{420}t^2(10t^4 + 14t^3 + 441t^2 + 420t + 420)$$
$$u_2(t) = -\frac{1}{15120}t^2(5t^6 + 9t^5 + 18t^4 - 15120t^2 - 15120t - 15120)$$
$$u_3(t) = \frac{1}{3326400}t^2(10t^8 + 22t^7 + 55t^6 + 3326400t^2 + 3326400t + 3326400)$$
$$u_4(t) = -\frac{1}{259459200}t^2(5t^{10} + 13t^9 + 39t^8 - 259459200t^2 - 259459200t - 259459200)$$
$$u_5(t) = -\frac{1}{21794572800}t^2(2t^{12} + 6t^{11} + 21t^{10} + 21794572800t^2 + 21794572800t + 21794572800)$$

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$$u_{15}(t) = t^2 + t^3 + t^4 - \frac{1}{21794572800}t^{14} + \ldots$$

$$u(t) = \lim_{n\to\infty} u_n(t) = t^2 + t^3 + t^4 \tag{9}$$

The exact solution is also given as:

$$u(t) = t^4 + t^3 + t^2 \tag{10}$$

Problem 2: Consider the following linear, singular initial value problem:

$$u''(t) + \frac{2}{t}u'(t) + 4t^2u(5 - 4t^2) = 0 \tag{11}$$

The initial conditions are:

$$u(0) = 1, u'(0) = 0$$

Here,

$$h(t)=t^2 \text{ and } \lambda(t,\tau)=\frac{\tau^2-t\tau}{t}$$

The successive approximations are:

$$u_1(t)=\frac{1}{9}(t^4-3)(2t^4-3)$$
$$u_2(t)=\frac{1}{2}t^8-t^4+1+\frac{2}{153}t^{16}-\frac{46}{351}t^{12}$$
$$u_3(t)=\frac{1}{2}t^8-t^4+1+\frac{233}{5967}t^{16}-\frac{1}{6}t^{12}+\frac{4}{11475}t^{24}-^{28}\frac{3518}{626535}t^{20}$$
$$u_4(t)=\frac{1}{2}t^8-t^4+1+\frac{1}{24}t^{16}-\frac{1}{6}t^{12}+\frac{2309}{1879605}t^{24}-\frac{5143}{626535}t^{20}+\frac{2}{378675}t^{32}-\frac{892}{7481565}t^{28}$$
$$u_5(t)=\frac{1}{2}t^8-t^4+1+\frac{1}{24}t^{16}-\frac{1}{6}t^{12}+\frac{10411}{7518420}t^{24}-\frac{1}{120}t^{20}+\frac{262796}{12591473895}t^{32}$$
$$-\frac{73261}{381559815}t^{28}+\frac{4}{77628375}t^{40}-\frac{704162}{465884534115}t^{36}$$

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$$u_{15}(t)=1-t^4+\frac{1}{2}t^8-\frac{1}{6}t^{12}+\dots$$
$$u(t)=\lim_{n\to\infty}u_n(t)=e^{-t^4} \qquad (12)$$

The exact solution is:

$$u(t)=e^{-t^4} \qquad (13)$$

Problem 3: Consider the following nonlinear singular initial value problem [13]:

$$u''+\frac{2}{t}u'(t)+2t^2u^2=0 \qquad (14)$$

with initial conditions:

$$u(0)=1,\ u'(0)=0$$

Similarly, the successive approximations can be obtained as:

$$u_1(t) = 1 - \frac{1}{12}t^4$$

$$u_2(t) = 1 - \frac{1}{12}t^4 - \frac{1}{12096}t^{12} + \frac{1}{240}t^8$$

$$u_3(t) = 1 - \frac{1}{12}t^4 - \frac{11}{60480}t^{12} + \frac{1}{240}t^8 - \frac{1}{61451550720}t^{28} + \frac{1}{452874240}t^{24}$$

$$- \frac{113}{798336000}t^{20} + \frac{13}{2177280}t^{16}$$

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$$u_{30}(t) = 1 - \frac{1}{12}t^4 + \frac{1}{240}t^8 - \frac{11}{60480}t^{12} + \frac{1}{136080}t^{16} - \frac{59}{217728000}t^{20}$$

$$+ \frac{10037}{1120863744000}t^{24} - \frac{96631}{366148823040000}t^{28} + 0(t^{32}) \qquad (15)$$

The problem has no closed form solution.

Above result is in agreement with results obtained in [13] after the 40th iteration.

Problem 4: Consider the following nonlinear, non-homogeneous singular initial value problem [13].

$$y'' + \frac{8}{t}y' + tu^2 = t^5 + t^4 \qquad (16)$$

$$y(0) = 1, \ y'(0) = 0$$

The following can be easily obtained:

$$h(t) = t^8 \text{ and } \lambda(t,\tau) = \frac{\tau^8 - t^7\tau}{t^7}$$

The successive results of the iteration are:

$$u_1(t)=1+\frac{1}{98}t^7+\frac{1}{78}t^6-\frac{1}{30}t^3$$

$$u_2(t)=1+\frac{1}{98}t^7+\frac{8}{585}t^6-\frac{1}{30}t^3-\frac{1}{3918432}t^{17}-\frac{1}{1406496}t^{16}-\frac{1}{2007720}t^{15}+\frac{1}{382200}t^{13}$$
$$+\frac{1}{266760}t^{12}-\frac{1}{8330}t^{10}-\frac{313}{1684800}t^9$$

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$$u_{20}(t)=1-\frac{1}{30}t^3+\frac{8}{585}t^6+\frac{1}{98}t^7-\frac{37}{187200}t^9-\frac{1}{8330}t^{10}+\frac{1081}{192067200}t^{12}+\frac{23}{6497400}t^{13}$$
$$-\frac{1291471}{2059920720000}t^{15}-\frac{407}{512366400}t^{16}$$
$$-\frac{1}{3918432}t^{17}+\frac{162919}{11587054050000}t^{18}+\frac{1399487}{79729335504000}t^{19}+\ldots \quad (17)$$

The problem has no closed form solution.

Above result is in agreement with results obtained in [13] the after 40th iteration.

Problem 5 Consider the following nonlinear Lane-Emden equation [16]:

$$u''+\frac{8}{t}u'+tu=t^5-t^4+44t^2-30t, 0<x\leq 1 \quad (18)$$

$$u(0)=u'(0)=0$$

which has the following analytical solution:

$$u(t)=t^4-t^3 \quad (19)$$

$$u_1(t) = \frac{1}{3922} t^3 (39t^4 - 49t^3 + 3822t - 3822)$$
$$u_2(t) = -\frac{1}{46781280} t^3 (2808t^7 - 4165t^6 - 46781280t + 46781280)$$
$$u_3(t) = \frac{1}{53330659200} t^3 (12312t^{10} - 20825t^9 + 53330659200t - 53330659200)$$
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$$u_{10}(t) = -\frac{1}{32962020801405696000} t^3 (65610t^{19} - 147407t^{18} - 32962020801405696000t + 32962020801405696000)$$

$$u(t) = \lim_{n \to \infty} u_n(t) = t^4 - t^3 \qquad (20)$$

4. Conclusions

In this research, the numerical and or analytical solutions of five singular initial value problem of Emden-Fowler type were calculated by using the variational iteration method. The method is a very fast convergent, effective and reliable through the Maple18 code generated and implemented for the numerical examples. It can be easily concluded that the VIM is a reliable tool for both the linear and nonlinear singular initial value problems of Emden-Fowler type.

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