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Amalgam Spaces With Variable Exponent

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Abstract: Let $1 \leq s < \infty$ and $1 \leq r(.) \leq \infty$ where $r(.)$ is a variable exponent. In this study, we consider the variable exponent amalgam space $\left(L^{r(\cdot)}, \ell^s\right)$. Moreover, we present some examples about inclusion properties of this space. Finally, we obtain that the space $\left(L^{r(\cdot)}, \ell^s\right)$ is a Banach Function space.

Keywords: Amalgam space, Banach space, Variable exponent.

1 Introduction

The amalgam of L^p and l^q on the real line is the space $(L^p, l^q)(\mathbb{R})$ (or briefly (L^p, l^q)) consisting of functions f which are locally in L^p and have l^q behavior at infinity. Several authors studied special cases of amalgams on some sets including $\mathbb{R}$ and a locally compact abelian group G. The first appearance of amalgam spaces can be traced to Wiener [13]. A generalization Wiener's definition was given by Feichtinger in [6], and it can be found a good summary of some results about amalgam spaces in [10], [11]. For a historical background of classical amalgams we refer [7]. The variable exponent Lebesgue spaces $L^{p(\cdot)}$ and the classical Lebesgue spaces L^p have many common properties but a significant difference between these spaces is that $L^{p(\cdot)}$ is not invariant under translation in general, see [4], [12]. Recently, there are many interesting and important papers appeared in variable exponent amalgam space $\left(L^{r(\cdot)}, \ell^s\right)$ such as Aydin [1], Aydin and Gurkanli [3], Gurkanli and Aydin [9].

2 Main results

Definition 1. *For a measurable function $r(.) : \mathbb{R} \to [1, \infty)$ (called a variable exponent on $\mathbb{R}$), we put*

$$r^- = \underset{x\in\mathbb{R}}{essinf} r(x), \qquad r^+ = \underset{x\in\mathbb{R}}{esssup} r(x).$$

Also the convex modular function $\varrho_{r(.)}$ is defined as

$$\varrho_{r(.)}(f) = \int_{\mathbb{R}} |f(x)|^{r(x)}\, dx.$$

The variable exponent Lebesgue space $L^{r(\cdot)}(\mathbb{R})$ is defined as the set of all measurable functions f on $\mathbb{R}$ such that $\varrho_{r(.)}(\lambda f) < \infty$ for some $\lambda > 0$, equipped with the Luxemburg norm

$$\|f\|_{r(.)} = \inf\left\{\lambda > 0 : \varrho_{r(.)}\left(\frac{f}{\lambda}\right) \leq 1\right\}.$$

Let $r^+ < \infty$. Then $f \in L^{r(\cdot)}(\mathbb{R})$ if and only if $\varrho_{r(.)}(f) < \infty$, that is, the norm topology is equivalent to modular topology. The space $L^{r(\cdot)}(\mathbb{R})$ is a Banach space with respect to $\|.\|_{r(.)}$. Moreover, it is well known that if we take $r(.) = r$ (const.), then the space $L^{r(\cdot)}(\mathbb{R})$ coincides with the classical Lebesgue space $L^r(\mathbb{R})$, see [12]. In this paper, we will assume that $r^+ < \infty$.

Definition 2. *Let $1 \leq r(.), s < \infty$ and $J_k = [k, k+1)$, $k \in \mathbb{Z}$. The variable exponent amalgam space $\left(L^{r(\cdot)}, \ell^s\right)$ is a normed space defined as*

$$\left(L^{r(\cdot)}, \ell^s\right) = \left\{f \in L^{r(.)}_{loc}(\mathbb{R}) : \|f\|_{\left(L^{r(\cdot)}, \ell^s\right)} < \infty\right\},$$

where

$$\|f\|_{\left(L^{r(\cdot)}, \ell^s\right)} = \left(\sum_{k\in\mathbb{Z}} \|f\chi_{J_k}\|^s_{r(.)}\right)^{\frac{1}{s}}.$$

It is well known that $\left(L^{r(\cdot)}, \ell^s\right)$ *does not depend on the particular choice of* J_k*. This follows* J_k *can be equal to* $[k, k+1)$*,* $[k, k+1]$ *or* $(k, k+1)$*. Thus, we have same amalgam spaces* $\left(L^{r(\cdot)}, \ell^s\right)$*.*

Theorem 1. *The space* $\left(L^{r(\cdot)}, \ell^s\right)$ *is a Banach space with respect to the norm* $\|.\|_{\left(L^{r(\cdot)}, \ell^s\right)}$ *.*

Proof: Let $\{f_n\}_{n\in\mathbb{N}}$ be a Cauchy sequence in $\left(L^{r(\cdot)}, \ell^s\right)$. Then given $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that if $n, m \geq N$, then we have

$$\|f_n - f_m\|_{\left(L^{r(\cdot)}, \ell^s\right)} = \left(\sum_{k\in\mathbb{Z}} \|f_n - f_m\|^s_{r(.),J_k}\right)^{\frac{1}{s}} < \varepsilon. \tag{1}$$

Hence, for any fixed k, we get

$$\|f_n - f_m\|_{r(.),J_k} < \varepsilon \;\; (n, m \geq N)\,.$$

Thus $\{f_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $L^{r(\cdot)}(J_k)$ for $k \in \mathbb{Z}$. Let us define $f = \sum_{k\in\mathbb{Z}} f^k \chi_{J_k}$ where $f^k \in L^{r(\cdot)}(J_k)$. Now, we will show that $f \in \left(L^{r(\cdot)}, \ell^s\right)$. Using Fatou's Lemma (applied to the right-hand series viewed as integral over the integers),we obtain

$$\begin{aligned}
\|f\|^s_{\left(L^{r(\cdot)}, \ell^s\right)} &= \sum_{k\in\mathbb{Z}} \left\|f^k\right\|^s_{r(.),J_k} = \sum_{k\in\mathbb{Z}} \lim_{n\to\infty} \|f_n\|^s_{r(.),J_k} \\
&\leq \lim_{n\to\infty} \inf \|f_n\|^s_{\left(L^{r(\cdot)}, \ell^s\right)}\,.
\end{aligned} \tag{2}$$

Since $\{f_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence (hence $\{f_n\}_{n\in\mathbb{N}}$ is bounded in norm), the last quantity is finite. Therefore, the left side of (2) is finite, that is, $f \in \left(L^{r(\cdot)}, \ell^s\right)$. By (1), we have

$$\|f_m - f\|^s_{r(.),J_k} = \lim_{n\to\infty} \|f_m - f_n\|^s_{r(.),J_k}$$

and

$$\begin{aligned}
\|f_m - f\|^s_{\left(L^{r(\cdot)}, \ell^s\right)} &= \sum_{k\in\mathbb{Z}} \lim_{n\to\infty} \|f_m - f_n\|^s_{r(.),J_k} \\
&\leq \lim_{n\to\infty} \inf \sum_{k\in\mathbb{Z}} \|f_m - f_n\|^s_{r(.),J_k} \\
&< \varepsilon
\end{aligned}$$

for $m \geq N$. Thus the Cauchy sequence $\{f_n\}_{n\in\mathbb{N}}$ converges to f, which is desired result. □

Now, we will show that $L^{r(\cdot)} \neq \left(L^{r(\cdot)}, \ell^s\right)$ and that these two spaces are not translation invariant in general. Also, we will prove new two examples which are associated with this.

Example 1. *Let* $r(.) : \mathbb{R} \to [0, \infty)$ *be a function such that for* $k \in \mathbb{Z}$

$$r(x) = \left\{ \begin{array}{ll} 1, & x \in A_k = [2k-1, 2k) \\ 2, & x \in B_k = [2k-2, 2k-1) \end{array} \right. .$$

Hence, we have $r^+ < \infty$ *and* $A_k \cap B_k = \phi$ *for all* $k \in \mathbb{Z}$*. Also let us define a function* f *as*

$$f(x) = \left\{ \begin{array}{ll} 0, & x \in A_k, k \in \mathbb{N} \\ \frac{1}{k}, & x \in B_k, k \in \mathbb{N}, (k \neq 0) \\ 0, & x < 0 \;\; (x \notin A_k \cup B_k) \end{array} \right.$$

Therefore, we have

$$\begin{aligned}
\varrho_{r(.)}(f) &= \int_{\mathbb{R}} |f(x)|^{r(x)}\,dx = \sum_{k=1}^{\infty} \int_{J_k} |f(x)|^{r(x)}\,dx \\
&= \sum_{k=1}^{\infty} \int_{J_k\cap B_k} |f(x)|^{r(x)}\,dx \\
&= \sum_{k=1}^{\infty} \int_{B_k} \frac{1}{k^2}\,dx = \sum_{k=1}^{\infty} \frac{1}{k^2} < \infty.
\end{aligned}$$

This follows that $f \in L^{r(\cdot)}(\mathbb{R})$. Now, we will show that $f \notin \left(L^{r(\cdot)}, \ell^1\right)$. By using the definition of $\|.\|_{\left(L^{r(\cdot)}, \ell^1\right)}$, we obtain

$$\begin{aligned}
\|f\|_{\left(L^{r(\cdot)}, \ell^1\right)} &= \sum_{k \in Z} \left\| f\chi_{[k,k+1)} \right\|_{r(.)} = \sum_{k=1}^{\infty} \left\| f\chi_{[2k-2,2k-1)} \right\|_{r(.)} \\
&= \sum_{k=1}^{\infty} \left\| f\chi_{[2k-2,2k-1)} \right\|_{2} \\
&= \sum_{k=1}^{\infty} \left(\int_{2k-2}^{2k-1} \frac{1}{k^2} dx \right)^{\frac{1}{2}} = \sum_{k=1}^{\infty} \frac{1}{k} = \infty.
\end{aligned}$$

Therefore, we have $f \notin \left(L^{r(\cdot)}, \ell^1\right)$.

Example 2. *Let $r(.) : \mathbb{R} \to [0, \infty)$ be a function such that for $k \in \mathbb{Z}$*

$$r(x) = \begin{cases} 1, & x \in A_k = [2k+1, 2(k+1)) \\ 2, & x \in B_k = [2k, 2k+1) \end{cases}.$$

Then, we define the space as

$$L^{r(\cdot)}(\mathbb{R}) = \left\{ f : f = f_1 + f_2,\ f_1 \in L^1(\mathbb{R}), f_2 \in L^2(\mathbb{R}), \mathit{supp} f_1 = \cup_{k\in\mathbb{Z}} A_k \text{ and } \mathit{supp} f_2 = \cup_{k\in\mathbb{Z}} B_k \right\}.$$

If we denote $T_1 f$ as the translation of given any function $f \in L^{r(\cdot)}(\mathbb{R})$, then we obtain

$$T_1 f(x) = \begin{cases} f(x+1) = f_2(x), & x \in A_k \\ f(x+1) = f_1(x), & x \in B_k \end{cases}.$$

It is easy to see that $T_1 f \notin L^{r(\cdot)}(\mathbb{R})$ .That means the space $L^{r(\cdot)}(\mathbb{R})$ is not translation invariant. Now, we quote this idea to the amalgam space. To show this we take same function $r(.)$ and same space $L^{r(\cdot)}(\mathbb{R})$. Let $p > 1$. Let us define a function f as

$$f(x) = \begin{cases} 0, & x \in A_k \\ \frac{1}{(k+1)^p}, & x \in B_k \\ 0, & x < 0 \ (x \notin A_k \cup B_k) \end{cases}$$

Then, we obtain

$$\|f\|_{\left(L^{r(\cdot)}, \ell^1\right)} = \begin{cases} \sum\limits_{k\in Z} \left\{ \int\limits_k^{k+1} |f(x)|^2 dx \right\}^{\frac{1}{2}} = \sum\limits_{k=1}^{\infty} \frac{1}{(k+1)^p} < \infty, & x \in B_k \\ 0, & x \in A_k \end{cases}.$$

Therefore we have $f \in \left(L^{r(\cdot)}, \ell^1\right)$. By the definition of $T_1 f$, we get

$$T_1 f(x) = \begin{cases} f(x+1) = \frac{1}{(k+1)^p}, & x \in A_k \\ 0, & x \in B_k \end{cases}.$$

This follows that

$$\|T_1 f\|_{\left(L^{r(\cdot)}, \ell^1\right)} = \begin{cases} \sum\limits_{k\in Z} \left\{ \int\limits_k^{k+1} |f(x+1)| dx \right\} = \sum\limits_{k=1}^{\infty} \frac{1}{(k+1)^p} < \infty, & x \in A_k \\ 0, & x \in B_k \end{cases}.$$

Therefore, we have $T_1 f \in \left(L^{r(\cdot)}, \ell^1\right)$. This follows that the space $\left(L^{r(\cdot)}, \ell^1\right)$ is translation invariant. As an alternative method, it is easy to see that $\left(L^1, \ell^1\right) = L^1$ or $\left(L^2, \ell^1\right) \subset L^1$and the space L^1 is translation invariant. Therefore, the same result is satisfied.

Remark 1. *If we consider the Theorem 3.3 in [8], then $L^{r(\cdot)} = \left(L^{r(\cdot)}, \ell^s\right)$ holds for some special cases. Therefore, the amalgam space $\left(L^{r(\cdot)}, \ell^s\right)$ is not translation invariant in general.*

Definition 3. *$L_c^{r(\cdot)}(\mathbb{R})$ denotes the functions f in $L^{r(\cdot)}(\mathbb{R})$ such that $\mathit{supp} f \subset \mathbb{R}$ is compact, that is,*

$$L_c^{r(\cdot)}(\mathbb{R}) = \left\{ f \in L^{r(\cdot)}(\mathbb{R}) : \mathit{supp} f \ \mathit{compact} \right\}.$$

Now, let $K \subset \mathbb{R}$ be given. The cardinality of the set

$$S(K) = \{J_k : J_k \cap K \neq \varnothing\}$$

is denoted by $|S(K)|$ where $\{J_k\}_{k\in\mathbb{Z}}$ is a collection of intervals $J_k = [k, k+1] = k + [0, 1]$, and also cover $\mathbb{R}$.

The following proposition was proved by Aydin [2].

Proposition 1. *If $g \in L_c^{r(.)}(\mathbb{R})$ and K is the compact support of g, then we have*

(i) $\|g\|_{\left(L^{r(.)},\ell^s\right)} \leq |S(K)|^{\frac{1}{s}} \|g\|_{r(.)}$ *for* $1 \leq s < \infty$.
(ii) $\|g\|_{\left(L^{r(.)},\ell^\infty\right)} \leq |S(K)| \|g\|_{r(.)}$.

Moreover, we have $L_c^{r(.)}(\mathbb{R}) \subset \left(L^{r(.)}(\mathbb{R}), \ell^s\right)$ *for* $1 \leq s \leq \infty$.

The main result of this study is to show that the space $\left(L^{r(.)}, \ell^s\right)$ be a special case of Banach function space, in other words, the norm of $\left(L^{r(.)}, \ell^s\right)$ satisfies the following properties, where f, g, f_n in $\left(L^{r(.)}, \ell^s\right)$ for all $n \in \mathbb{N}$, $\lambda \geq 0$ and E is any measurable subset of $\mathbb{R}$ ($|E| < \infty$):

1. $\|f\|_{\left(L^{r(.)},\ell^s\right)} \geq 0$
2. $\|f\|_{\left(L^{r(.)},\ell^s\right)} = 0$ if and only if $f = 0$ a.e. in $\mathbb{R}$
3. $\|\lambda f\|_{\left(L^{r(.)},\ell^s\right)} = \lambda \|f\|_{\left(L^{r(.)},\ell^s\right)}$
4. $\|f+g\|_{\left(L^{r(.)},\ell^s\right)} \leq \|f\|_{\left(L^{r(.)},\ell^s\right)} + \|g\|_{\left(L^{r(.)},\ell^s\right)}$
5. If $|g| \leq |f|$ a.e. in $\mathbb{R}$, then $\|g\|_{\left(L^{r(.)},\ell^s\right)} \leq \|f\|_{\left(L^{r(.)},\ell^s\right)}$
6. If $0 \leq f_n \uparrow f$ a.e. in $\mathbb{R}$, then $\|f_n\|_{\left(L^{r(.)},\ell^s\right)} \uparrow \|f\|_{\left(L^{r(.)},\ell^s\right)}$
7. $\|\chi_E\|_{\left(L^{r(.)},\ell^s\right)} < \infty$
8. $\int_E |f|\,dx \leq C\left(r(.), E\right) \|f\|_{\left(L^{r(.)},\ell^s\right)}$ for some $C > 0$.

Theorem 2. *The space $\left(L^{r(.)}, \ell^s\right)$ is a Banach Function space with respect to the norm $\|.\|_{\left(L^{r(.)},\ell^s\right)}$.*

Proof: We have to prove the properties (1)-(8). The first three properties follow directly from the definition of the norm $\|.\|_{\left(L^{r(.)},\ell^s\right)}$.

Proof of Property 4. Let $f, g \in \left(L^{r(.)}, \ell^s\right)$ be given. It is well known that $f, g \in \left(L^{r(.)}, \ell^s\right)$ if and only if $\left\{\|f\|_{r(.),J_k}\right\}_{k\in Z}, \left\{\|g\|_{r(.),J_k}\right\}_{k\in Z} \in \ell^s(\mathbb{Z})$. Then we have

$$\begin{aligned}
\|f+g\|_{\left(L^{r(.)},\ell^s\right)} &= \left\|\|f+g\|_{r(.),J_k}\right\|_{\ell^s} \\
&\leq \left\|\|f\|_{r(.),J_k} + \|g\|_{r(.),J_k}\right\|_{\ell^s} \\
&\leq \left\|\|f\|_{r(.),J_k}\right\|_{\ell^s} + \left\|\|g\|_{r(.),J_k}\right\|_{\ell^s} \\
&= \|f\|_{\left(L^{r(.)},\ell^s\right)} + \|g\|_{\left(L^{r(.)},\ell^s\right)}.
\end{aligned}$$

Proof of Property 5. Let $|g| \leq |f|$. Then we obtain

$$\begin{aligned}
\|g\|_{\left(L^{r(.)},\ell^s\right)} &= \left\|\|g\|_{r(.),J_k}\right\|_{\ell^s} \\
&\leq \left\|\|f\|_{r(.),J_k}\right\|_{\ell^s} = \|f\|_{\left(L^{r(.)},\ell^s\right)}.
\end{aligned}$$

Proof of Property 6. It is well known that $L^{r(.)}$ is a BF-space by Proposition 1.3 in [5]. Since $0 \leq f_n \uparrow f$ a.e. in $\mathbb{R}$, then $\|f_n\|_{r(.),J_k} \uparrow \|f\|_{r(.),J_k}$. If we consider this property for ℓ^s, we have

$$\|f_n\|_{\left(L^{r(.)},\ell^s\right)} = \left\|\|f_n\|_{r(.),J_k}\right\|_{\ell^s} \uparrow \left\|\|f\|_{r(.),J_k}\right\|_{\ell^s} = \|f\|_{\left(L^{r(.)},\ell^s\right)}.$$

Proof of Property 7. Since $|E| < \infty$ and $\text{supp}\chi_E = \overline{E} \subset \mathbb{R}$ is compact, then $\chi_E \in L_c^{r(.)}(\mathbb{R})$ and

$$\|\chi_E\|_{\left(L^{r(.)},\ell^s\right)} \leq |S(E)|^{\frac{1}{s}} \|\chi_E\|_{r(.),E} < \infty$$

by Proposition 1.

Proof of Property 8. By Hölder's inequality for variable exponent amalgam spaces (see, Corollary 2.4, [3]), we get

$$\begin{aligned}
\int_E |f|\,dx &= \int_{\mathbb{R}} |f\chi_E|\,dx \leq c\|f\|_{\left(L^{r(.)},\ell^s\right)} \|\chi_E\|_{\left(L^{r'(.)},\ell^{s'}\right)} \\
&\leq C\left(r(.), E\right) \|f\|_{\left(L^{r(.)},\ell^s\right)}
\end{aligned}$$

for some $C > 0$ where $\frac{1}{r(.)} + \frac{1}{r'(.)} = \frac{1}{s} + \frac{1}{s'} = 1$ and $C = C\left(r(.), E\right) = c\|\chi_E\|_{\left(L^{r'(.)},\ell^{s'}\right)}$. □

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