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On a New Metric Space

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Murat Kirisci[1,] Necip Simsek[2]*

[1] *Department of Mathematical Education, Istanbul University- Cerrahpasa, Istanbul, Turkey, ORCID:0000-0003-4938-5207*
[2] *Department of Mathematics, Istanbul Commerce University, Istanbul, Turkey, ORCID:000000*
** Corresponding Author E-mail: mkirisci@hotmail.com*

Abstract: In this presentations, the definition of new metric space with neutrosophic numbers is given. Several topological and structural properties have been investigated. The analogue of Baire Category Theorem is given for Neutrosophic metric spaces.

Keywords: Baire category theorem, Completeness, Hausdorffness, Neutrosophic metric space, Nowhere dense.

1 Introduction

Fuzzy Sets (FSs) put forward by Zadeh [1] has influenced deeply all the scientific fields since the publication of the paper. It is seen that this concept, which is very important for real-life situations, had not enough solution to some problems in time. New quests for such problems have been coming up. Atanassov [2] initiated Intuitionistic fuzzy sets (IFSs) for such cases. Neutrosophic set (NS) is a new version of the idea of the classical set which is defined by Smarandache [3]. Examples of other generalizations are FS [1] interval-valued FS [4], IFS [2], interval-valued IFS [5], the sets paraconsistent, dialetheist, paradoxist, and tautological [6], Pythagorean fuzzy sets [7] .

Using the concepts Probabilistic metric space and fuzzy, fuzzy metric space (FMS) is introduced in [8]. Kaleva and Seikkala [9] have defined the FMS as a distance between two points to be a non-negative fuzzy number. In [10] some basic properties of FMS studied and the Baire Category Theorem for FMS proved. Further, some properties such as separability, countability are given and Uniform Limit Theorem is proved in [11]. Afterward, FMS has used in the applied sciences such as fixed point theory, image and signal processing, medical imaging, decision-making et al. After defined of the intuitionistic fuzzy set (IFS), it was used in all areas where FS theory was studied. Park [12] defined IF metric space (IFMS), which is a generalization of FMSs. Park used George and Veeramani's [10] idea of applying t-norm and t-conorm to the FMS meanwhile defining IFMS and studying its basic features.

Bera and Mahapatra defined the neutrosophic soft linear spaces (NSLSs) [13]. Later, neutrosophic soft normed linear spaces(NSNLS) has been defined by Bera and Mahapatra [14]. In [14], neutrosophic norm, Cauchy sequence in NSNLS, convexity of NSNLS, metric in NSNLS were studied.

In present study, from the idea of neutrosophic sets, new metric space was defined which is called Neutrosophic metric Spaces (NMS). We investigate some properties of NMS such as open set, Hausdorff, neutrosophic bounded, compactness, completeness, nowhere dense. Also we give Baire Category Theorem and Uniform Convergence Theorem for NMSs.

2 Preliminaries

Let's consider that K is a space of points(objects). Denote the $T_U(a)$ is a truth-MF, $I_U(a)$ is an indeterminacy-MF and $F_U(a)$ is a falsity-MF, where U is a set in K with $a \in K$. Then, if we take $J =]0^-, 1^+[$

$$\begin{aligned} T_U(a) &: K \to J, \\ I_U(a) &: K \to J, \\ F_U(a) &: K \to J. \end{aligned}$$

There is no restriction on the sum of $T_U(a)$, $I_U(a)$ and $F_U(a)$. Therefore,

$$0^- \leq \sup T_U(a) + \sup I_U(a) + \sup F_U(a) \leq 3^+.$$

The set U which consist of with $T_U(a)$, $I_U(a)$ and $F_U(a)$ in K is called a neutrosophic sets(NS) and can be denoted by

$$U = \{< a, (T_U(a), I_U(a), F_U(a)) >: a \in K, T_U(a), I_U(a), F_U(a) \in J\} \tag{1}$$

Clearly, NS is an enhancement of $[0, 1]$ of IFSs.

An NS U is included in another NS V, ($U \subseteq V$), if and only if,

$$\begin{aligned}
&\inf T_U(a) \leq \inf T_V(a), \quad \sup T_U(a) \leq \sup T_V(a),\\
&\inf I_U(a) \geq \inf I_V(a), \quad \sup I_U(a) \geq \sup I_V(a),\\
&\inf F_U(a) \geq \inf F_V(a), \quad \sup F_U(a) \geq \sup F_V(a).
\end{aligned}$$

for any $a \in K$. However, NSs are inconvenient to practice in real problems. To cope with this inconvenient situation, Wang et al [15] customized NS's definition and single-valued NSs (SVNSs) suggested. Ye [16], described the notion of simplified NSs, which may be characterized by three real numbers in the $[0, 1]$. At the same time, the simplified NSs' operations may be impractical, in some cases [16]. Hence, the operations and comparison way between SNSs and the aggregation operators for simplified NSs are redefined in [17].

According to the Ye [16], a simplification of an NS U, in (1), is

$$U = \{< a, (T_U(a), I_U(a), F_U(a)) >: a \in K\},$$

which called an simplified NS. Especially, if K has only one element $< G_U(a), B_U(a), Y_U(a) >$ is said to be an simplified NN. Expressly, we may see simplified NSs as a subclass of NSs.

An simplified NS U is comprised in another simplified NS V ($U \subseteq V$), iff $G_U(a) \leq G_V(a)$, $B_U(a) \geq B_V(a)$ and $Y_U(a) \geq Y_V(a)$ for any $a \in K$. Then, the following operations are given by Ye[16]:

$$\begin{aligned}
U + V &= \langle G_U(a) + G_V(a) - G_U(a).G_V(a), B_U(a) + B_V(a) - B_U(a).B_V(a), Y_U(a) + Y_V(a) - Y_U(a).Y_V(a)\rangle,\\
U.V &= \langle G_U(a).G_V(a), B_U(a).B_V(a), Y_U(a).Y_V(a)\rangle,\\
\alpha.U &= \langle 1 - (1 - G_U(a))^\alpha, 1 - (1 - B_U(a))^\alpha, 1 - (1 - Y_U(a))^\alpha\rangle \qquad for \quad \alpha > 0,\\
U^\alpha &= \langle G_U^\alpha(a), B_U^\alpha(a), Y_U^\alpha(a)\rangle \qquad for \quad \alpha > 0.
\end{aligned}$$

Definition 1. *Give an operation* $\circ : [0, 1] \times [0, 1] \to [0, 1]$. *If the operation* $\circ$ *is satisfying the following conditions, then it is called that the operation* $\circ$ *is* continuous TN*: For* $s, t, u, v \in [0, 1]$,

i. $s \circ 1 = s$,
ii. If $s \leq u$ *and* $t \leq v$, *then* $s \circ t \leq u \circ v$,
iii. $\circ$ *is continuous,*
iv. $\circ$ *is commutative and associative.*

Definition 2. *Give an operation* $\bullet : [0, 1] \times [0, 1] \to [0, 1]$. *If the operation* $\bullet$ *is satisfying the following conditions, then it is called that the operation* $\bullet$ *is* continuous TC*:*

i. $s \bullet 0 = s$,
ii. If $s \leq u$ *and* $t \leq v$, *then* $s \bullet t \leq u \bullet v$,
iii. $\bullet$ *is continuous,*
iv. $\bullet$ *is commutative and associative.*

Form above definitions, we note that if we choose $0 < \varepsilon_1, \varepsilon_2 < 1$ for $\varepsilon_1 > \varepsilon_2$, then there exist $0 < \varepsilon_3, \varepsilon_4 < 0, 1$ such that $\varepsilon_1 \circ \varepsilon_3 \geq \varepsilon_2$, $\varepsilon_1 \geq \varepsilon_4 \bullet \varepsilon_2$. Further, if we choose $\varepsilon_5 \in (0, 1)$, then there exist $\varepsilon_6, \varepsilon_7 \in (0, 1)$ such that $\varepsilon_6 \circ \varepsilon_6 \geq \varepsilon_5$ and $\varepsilon_7 \bullet \varepsilon_7 \leq \varepsilon_5$.

3 New metric spaces

Definition 3. *Take* K *be an arbitrary set,* $\mathcal{N} = \{< a, T(a), I(a), F(a) >: a \in K\}$ *be a NS such that* $\mathcal{N} : K \times K \times \mathbb{R}^+ \to [0, 1]$. *Let* $\circ$ *and* $\bullet$ *show the continuous TN and continuous TC, respectively. The four-tuple* $(K, \mathcal{N}, \circ, \bullet)$ *is called neutrosophic metric space(NMS) when the following conditions are satisfied.* $\forall a, b, c \in K$,

i. $0 \leq T(a, b, \lambda) \leq 1, \quad 0 \leq I(a, b, \lambda) \leq 1, \quad 0 \leq F(a, b, \lambda) \leq 1 \quad \forall \lambda \in \mathbb{R}^+$,
ii. $T(a, b, \lambda) + I(a, b, \lambda) + F(a, b, \lambda) \leq 3$, *(for* $\lambda \in \mathbb{R}^+$*),*
iii. $T(a, b, \lambda) = 1$ *(for* $\lambda > 0$*) if and only if* $a = b$,
iv. $T(a, b, \lambda) = T(b, a, \lambda)$ *(for* $\lambda > 0$*),*
v. $T(a, b, \lambda) \circ T(b, c, \mu) \leq T(a, c, \lambda + \mu) \quad (\forall \lambda, \mu > 0)$,
vi. $T(a, b, .) : [0, \infty) \to [0, 1]$ *is continuous,*
vii. $lim_{\lambda \to \infty} T(a, b, \lambda) = 1 \quad (\forall \lambda > 0)$,
viii. $I(a, b, \lambda) = 0$ *(for* $\lambda > 0$*) if and only if* $a = b$,
ix. $I(a, b, \lambda) = I(b, a, \lambda)$ *(for* $\lambda > 0$*),*
x. $I(a, b, \lambda) \bullet I(b, c, \mu) \geq I(a, c, \lambda + \mu) \quad (\forall \lambda, \mu > 0)$,
xi. $I(a, b, .) : [0, \infty) \to [0, 1]$ *is continuous,*
xii. $lim_{\lambda \to \infty} I(a, b, \lambda) = 0 \quad (\forall \lambda > 0)$,
xiii. $F(a, b, \lambda) = 0$ *(for* $\lambda > 0$*) if and only if* $a = b$,
xiv. $F(a, b, \lambda) = F(b, a, \lambda) \quad (\forall \lambda > 0)$,
xv. $F(a, b, \lambda) \bullet F(b, c, \mu) \geq F(a, c, \lambda + \mu) \quad (\forall \lambda, \mu > 0)$,

xvi. $F(a,b,.):[0,\infty)\to[0,1]$ *is continuous,*
xvii. $lim_{\lambda\to\infty}F(a,b,\lambda)=0$ *(for $\lambda>0$),*
xviii. If $\lambda\leq 0$*, then* $T(a,b,\lambda)=0$*,* $I(a,b,\lambda)=1$ *and* $F(a,b,\lambda)=1$.

Then $\mathcal{N}=(T,I,F)$ *is called Neutrosophic metric(NM) on* K.

The functions $T(a,b,\lambda), I(a,b,\lambda), F(a,b,\lambda)$ denote the degree of nearness, the degree of neutralness and the degree of non-nearness between a and b with respect to λ, respectively.

Example 1. *Let* $(K,\boldsymbol{d})$ *be a MS. Give the operations* $\circ$ *and* $\bullet$ *as default (min) TN* $a\circ b=\min\{a,b\}$ *and default(max) TC* $a\bullet b=\max\{a,b\}$.

$$T(a,b,\lambda)=\frac{\lambda}{\lambda+d(a,b)},\quad I(a,b,\lambda)=\frac{d(a,b)}{\lambda+d(a,b)},\quad F(a,b,\lambda)=\frac{d(a,b)}{\lambda},$$

$\forall a,b\in K$ *and* $\lambda>0$*. Then,* $(K,\mathcal{N},\circ,\bullet)$ *is NMS such that* $\mathcal{N}:K\times K\times\mathbb{R}^+\to[0,1]$*. This NMS is expressed as produced by a metric* ***d*** *the NM.*

Example 2. *Choose* K *as natural numbers set. Give the operations* $\circ$ *and* $\bullet$ *as TN* $a\circ b=\max\{0,a+b-1\}$ *and TC* $a\bullet b=a+b-ab$. $\forall a,b\in F,\quad\lambda>0$

$$T(a,b,\lambda)=\begin{cases}\frac{a}{b} & , \quad (a\leq b),\\ \frac{b}{a} & , \quad (b\leq a),\end{cases}$$

$$I(a,b,\lambda)=\begin{cases}\frac{b-a}{y} & , \quad (a\leq b),\\ \frac{a-b}{x} & , \quad (b\leq a),\end{cases}$$

$$F(a,b,\lambda)=\begin{cases}b-a & , \quad (a\leq b)\\ a-b & , \quad (b\leq a).\end{cases}$$

Then, $(K,\mathcal{N},\circ,\bullet)$ *is NMS such that* $\mathcal{N}:K\times K\times\mathbb{R}^+\to[0,1]$.

Example 3. $\mathcal{N}=\{<a,G(a),B(a),Y(a)>: a\in K\}$ *defined in Example* 1 *is not a NM with TN* $a\circ b=\max\{0,a+b-1\}$ *and TC* $a\bullet b=a+b-ab$.

Example 4. $\mathcal{N}=\{<a,G(a),B(a),Y(a)>: a\in K\}$ *defined in Example* 2 *is not a NM with TN* $a\circ b=\min\{a,b\}$ *and TC* $a\bullet b=\max\{a,b\}$.

Definition 4. *Give* $(K,\mathcal{N},\circ,\bullet)$ *be a NMS,* $0<\varepsilon<1$*,* $\lambda>0$ *and* $a\in K$*. The set* $O(a,\varepsilon,\lambda)=\{b\in K: T(a,b,\lambda)>1-\varepsilon,\quad I(a,b,\lambda)<\varepsilon,\quad F(a,b,\lambda)<\varepsilon\}$ *is said to be the open ball (OB) (center* a *and radius* ε *with respect to* λ*).*

Theorem 1. *Every OB* $O(a,\varepsilon,\lambda)$ *is an open set (OS).*

Theorem 2. *Every NMS is Hausdorff.*

Definition 5. *Let* $(K,\mathcal{N},\circ,\bullet)$ *be a NMS. A subset* A *of* K *is called Neutrosophic-bounded (NB), if there exist* $\lambda>0$ *and* $\varepsilon\in(0,1)$ *such that* $T(a,b,\lambda)>1-\varepsilon$*,* $I(a,b,\lambda)<\varepsilon$ *and* $F(a,b,\lambda)<\varepsilon\quad(\forall a,b\in A)$.

Theorem 3. *Every compact subset* A *of a NMS is NB.*

If $(K,\mathcal{N},\circ,\bullet)$ is NMS produces by a metric **d** on K and $A\subset K$, then A is NB if and only if it is bounded. Consequently, with Theorems 2 and 3, we can write:

Corollary 1. *In a NMS, every compact set is closed and bounded.*

Definition 6. *Take* $(K,\mathcal{N},\circ,\bullet)$ *to be a NMS. A sequence* (a_n) *in* K *is called* ***Cauchy*** *if for each* $\varepsilon>0$ *and each* $\lambda>0$*, there exist* $N\in\mathbb{N}$ *such that* $T(a_n,a_m,\lambda)>1-\varepsilon$*,* $I(a_n,a_m,\lambda)<\varepsilon$*,* $F(a_n,a_m,\lambda)<\varepsilon\quad\forall n,m\geq N$*.* $(K,\mathcal{N},\circ,\bullet)$ *is called* ***complete*** *if every Cauchy sequence is convergent with respect to* $\tau_{\mathcal{N}}$.

Theorem 4. *Take* $(K,\mathcal{N},\circ,\bullet)$ *to be a NMS. Let's every Cauchy sequence in* K *has a convergent subsequences. Then the NMS* $(K,\mathcal{N},\circ,\bullet)$ *is complete.*

Theorem 5. *Let* $(K,\mathcal{N},\circ,\bullet)$ *is NMS and let* A *be a subset of* K *with the subspace NM* $(T_A,I_A,F_A)=(T|_{A^2\times\mathbb{R}^+},I|_{A^2\times\mathbb{R}^+},F|_{A^2\times\mathbb{R}^+})$*. Then* $(A,\mathcal{N}_A,\circ,\bullet)$ *is complete if and only if* A *is closed subset of* F.

Theorem 6. *(Baire Category Theorem) Let* $\{\gamma_n:n\in\mathbb{N}\}$ *be a sequence of dense open subsets of a complete NMS* $(K,\mathcal{N},\circ,\bullet)$*. Then* $\cap_{n\in\mathbb{N}}\gamma_n$ *is also dense in* K.

4 Conclusion

The aim of this presentations is to define a neutrosophic metric spaces and examine some properties. The structural characteristic properties of NMSs such as open ball, open set, Hausdorffness, compactness, completeness, nowhere dense in NMS have been established. Analogue of Baire Category Theorem is given for NMS.

This new concept can also be studied to the fixed point theory, as in metric fixed metric theory and so it can constructed the NMS fixed point theory. As is well known, in recent years, the study of metric fixed point theory has been widely researched because of the this theory has a fundamental role in various areas of mathematics, science and economic studies.

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