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Negative Coefficient of Starlike Functions of Order 1/2

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Abstract: A function $g(z)$ is said to be univalent in a domain D if it provides a one-to-one mapping onto its image, $g(D)$. Geometrically , this means that the representation of the image domain can be visualized as a suitable set of points in the complex plane. We are mainly interested in univalent functions that are also regular (analytic, holomorphik) in U . Without lost of generality we assume D to be unit disk $U = \{z : |z| < 1\}$. One of the most important events in the history of complex analysis is Riemann's mapping theorem, that any simply connected domain in the complex plane $\mathbb{C}$ which is not the whole complex plane, can be mapped by any analytic function univalently on the unit disk U. The investigation of analytic functions which are univalent in a simply connected region with more than one boundary point can be confined to the investigation of analytic functions which are univalent in U. The theory of univalent functions owes the modern development the amazing Riemann mapping theorem. In 1916, Bieberbach proved that for every $g(z) = z + \sum_{n=2}^{\infty} a_n z^n$ in class S , $|a_2| \leq 2$ with equality only for the rotation of Koebe function $k(z) = \frac{z}{(1-z)^2}$. We give an example of this univalent function with negative coefficients of order $\frac{1}{4}$ and we try to explain $B_{\frac{1}{4}}\left(1, \frac{\pi}{3}, -1\right)$ with convex functions.

Keywords: Class s, Convex functions, Univalent functions.

1 Introduction

Let A denote the class of functions $f(z)$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad (n \in \mathbb{N} = \{1, 2, 3, ...\})\,.$$

$f(z)$ is a function in unit disk $U = \{z : |z| < 1\}$and analytic.
Let $A(n)$ denote the subclass of A consisting of functions of form

$$f(z) = z - \sum k = n + 1^{\infty} a_k z^k \quad (a_k \geq 0,\ n \in \mathbb{N} = \{1, 2, 3, ...\})\,.$$

Let $T(n)$ denote the subclass of $A(n)$ consisting of functions which are univalent in U. Further a function in $T(n)$ is said to be starlike of order $\dfrac{1}{2}$ if and only if

$$\frac{zf'(z)}{f(z)} > \frac{1}{2}\,(z \in U)$$

and such a subclass of $A(n)$ consisting of all the starlike functions of order $\frac{1}{2}$ is denote by $T_{\frac{1}{2}}(n)$. Also, $f(z) \in T(n)$ is said to be convex of order $\dfrac{1}{2}$ if and only if satisfies

$$\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \frac{1}{2}\,(z \in U)$$

and the subclass by $C_{\frac{1}{2}}(n)$ [1][2][3][6].

For $n = 1$, these notations are usually used as $T_{\frac{1}{2}}(1) = T^*\left(\dfrac{1}{2}\right)$ and $C_{\frac{1}{2}}(n) = C^*\left(\dfrac{1}{2}\right)$ [5].

Theorem 1. *A function $f(z)$ in $A(n)$ is in $T_{\frac{1}{2}}(n)$ if and only if*

$$\sum_{k=n+1}^{\infty} \left(k - \tfrac{1}{2}\right) a_k \leq 1 - \frac{1}{2} = \frac{1}{2}[1].$$

Theorem 2. *A function $f(z)$ in $A(n)$ is in $C_{\frac{1}{2}}(n)$ if and only if*

$$\sum_{k=n+1}^{\infty}\left(k-\frac{1}{2}\right)a_k \le 1-\frac{1}{2}=\frac{1}{2}[1].$$

We introduced the subclass A(n,θ) of A, and the subclass $T^*_{\frac{1}{2}}(n,\theta)$ and $C_{\frac{1}{2}}(n,\theta)$of $A(n,\theta)$ in the following manner. Let $A(n,\theta)$ denote the subclass of A consisting of function of the form

$$f(z)=z-\sum_{k=n+1}^{\infty}e^{i(k-1)\theta}a_k z^k \quad (a_k\ge 0,\ n\in\mathbb{N})\,[4].$$

We note that $A(n,\theta)=A(n)$, that is $A(n,0)$ is the subclass of analytic functions with negative coefficients. We denote by $T^*_{\frac{1}{2}}(n,\theta)$ and $C_{\frac{1}{2}}(n,\theta)$ the subclass of $A(n,\theta)$ of starlike and convex functions of order $\frac{1}{2}$ in U.

Theorem 3. *A function $f(z)$ in $A(n,\theta)$ is in $T^*_{\frac{1}{2}}(n,\theta)$ if and only if*

$$\sum_{k=n+1}^{\infty}\left(k-\frac{1}{2}\right)a_k \le 1-\frac{1}{2}=\frac{1}{2}\textit{[4]}.$$

Theorem 4. *A function $f(z)$ in $A(n,\theta)$ is in $C_{\frac{1}{2}}(n,\theta)$ if and only if*

$$\sum_{k=n+1}^{\infty}k\left(k-\frac{1}{2}\right)a_k \le 1-\frac{1}{2}=\frac{1}{2}\textit{[4]}.$$

Theorem 5. *$f(z)\in A_{\frac{1}{2}}(n,\theta,h)$,then $f(z)\in T^*_{\frac{1}{2}}(n,\theta)$.*

Proof:

$$\begin{aligned}\sum_{k=n+1}^{\infty}\left(k-\frac{1}{2}\right)a_{k,h} &= \sum_{k=n+1}^{\infty}\left(k-\frac{1}{2}\right)\frac{\frac{1}{4}}{\left(k+h-\frac{1}{2}\right)\left(k+h+\frac{1}{2}\right)\left(k-\frac{1}{2}\right)}\\ &=\frac{1}{4}\sum_{k=n+1}^{\infty}\left(\frac{2}{2k+2h-1}-\frac{2}{2k+2h+1}\right)=\frac{1}{4}\frac{1}{n+h+\frac{1}{2}}=\frac{1}{4n+4h+2}\end{aligned}$$

$$=\begin{cases}\dfrac{(1/2)^2}{1/2}=\frac{1}{2}, & h=-n,\\ \dfrac{(1/2)^2}{n+h+1/2}<\dfrac{(1/2)^2}{1/2}=\frac{1}{2}, & h>-n.\end{cases}$$

Hence we know that $f(z)$ is an element of $T^*_{\frac{1}{2}}(n,\theta)$. □

Theorem 6. *(Main theorem) If $f(z)\in A_{\frac{1}{2}}\left(1,\frac{\pi}{3},0\right)$, then we have starlike function and $A_{\frac{1}{2}}\left(1,\frac{\pi}{3},0\right)\in S^*$.*

$$f(z)=z-\frac{1+i\sqrt{3}}{45}z^2+\frac{1-i\sqrt{3}}{45}z^3-\frac{2}{45}z^4-\frac{1+i\sqrt{3}}{45}z^5-...$$

Proof: Let $f(z)\in A_{\frac{1}{2}}\left(1,\frac{\pi}{3},0\right)$ denote the subclass of $A\left(1,\frac{\pi}{3}\right)$ consisting of functions of the form

$$f(z)=z-\sum_{k=n+1}^{\infty}e^{i(k-1)\theta}a_k z^k \quad (h\ge -n,\ n\in\mathbb{N}=\{1,2,3,...\})$$

where

$$a_{k,h}=a_{2,0}=\frac{\left(1-\frac{1}{2}\right)}{2\left(2+0-\frac{1}{2}\right)\left(2+0+1-\frac{1}{2}\right)\left(2-\frac{1}{2}\right)}=\frac{\frac{1}{4}}{\frac{45}{8}}=\frac{2}{45}.$$

$$\begin{aligned}f(z)&=z-\sum_{k=n+1}^{\infty}e^{i(k-1)\theta}\frac{2}{45}\quad (h\ge -n,\ n\in\mathbb{N},n\ge 1)\\ &=z-\frac{2\left(\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)}{45}z^2+\frac{2\left(-\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)}{45}z^3-\frac{-2}{45}z^4-...\end{aligned}$$

$$= z - \frac{2e^{i\frac{\pi}{3}}}{45}z^2 + \frac{2e^{i\frac{\pi}{3}}}{45}z^3 - \frac{2e^{i\frac{\pi}{3}}}{45}z^4 - \frac{2e^{i\frac{\pi}{3}}}{45}z^5 - ...$$
$$= z - \frac{1+i\sqrt{3}}{45}z^2 + \frac{1-i\sqrt{3}}{45}z^3 - \frac{2}{45}z^4 - \frac{1+i\sqrt{3}}{45}z^5 - ...$$

We show the results we've achieved our proof. □

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