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Types of Generalized δ-Open Sets in Bitopological Spaces

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Abstract

In theoretical and applied areas of mathematics, one can work with sets endowed with several structures. A bitopological space is a set equipped with two topologies. In this paper, some types of open sets weaker than delta-open sets are generalized to bitopological spaces and their corresponding interior and closure operators are introduced. The relations between these sets and counter examples for the reverse relations are given. By using these sets, new types of continuous functions are defined and some of their properties are studied in bitopological spaces.

Keywords: *Bitopological spaces; δ-open sets; preopen sets; semiopen sets.*
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1. Introduction

In many areas of applications, the significance of topological spaces continues to rise. Some basic notions of the investigations in general topological spaces are closure and interior operators which widely used particularly in rough set theory and fuzzy set theory.
For the examining of topological concepts, the presence of two topologies ensures a substantial generalization. Kelly [12] initiated the study of bitopological spaces which are sets endowed with two topologies. Many others contributed to the development of the theory of such spaces by generalizing the topological concepts to bitopological spaces. For some examples one can see the papers [7, 8, 9, 10, 15] and references therein.
Veličko [20] introduced δ-open sets in topological spaces. These sets are stronger than open sets and the collection of all δ-open sets generates a topology. Later, as generalizations of δ-open sets, δ-preopen, δ-semiopen, a-open, e-open and e^*-open sets were introduced by Park et al. [17], Raychaudhuri and Mukherjee [18], Ekici [4], Ekici [5] and Ekici [6], respectively.
In this paper, we define (p,q)-δ-preopen, (p,q)-δ-semiopen, (p,q)-a-open, (p,q)-e-open and (p,q)-e^*-open sets in bitopological spaces and study the relation between them. Also we obtain counter examples for some of the reverse directions. Thamizharasi and Thangavelu [19] extended the results given in [1, 2] to bitopological spaces. We give analogues results for the interior and closure operators of sets mentioned above. In the last section, we introduce weaker forms of continuous functions between bitopological spaces by utilizing the notions of (p,q)-δ-preopen, (p,q)-δ-semiopen, (p,q)-a-open, (p,q)-e-open and (p,q)-e^*-open sets.

2. Preliminaries

In this section, some definitions and results are given and some notations are represented which will be used in the course of the paper.
A subset A of a topological space (X,τ) is regular open (resp. regular closed) if $A =$Int(Cl(A)) (resp. $A =$Cl(Int(A))), where Int(A) and Cl(A) are interior and closure of A, respectively. The union of all regular open sets of X contained in A is called δ-interior of A and A is δ-open if A is equal to the δ-interior of itself. The complement of a δ-open set is said to be δ-closed. For each regular open set U containing x, x is called a δ-cluster point of A if $A\cap U\neq\emptyset$. The set of all δ-cluster points of A forms the δ-closure of A.
By the triple (X,τ_1,τ_2) and (Y,ζ_1,ζ_2), where τ_1, τ_2 and ζ_1, ζ_2 are topologies on X and Y, respectively, we denote the bitopological spaces. τ_p-Int(A), τ_p-Int$_\delta$(A) and τ_q-Cl(A), τ_q-Cl$_\delta$(A) stand for interior of A, δ-interior of A and closure of A, δ-closure of A with respect to topologies τ_p and τ_q, respectively. τ_p-δ-open set means δ-open set with respect to the topology τ_p.
Using closure and interior operators, some generalized open sets in a bitopological space were defined as follows.

Definition 2.1. *A subset A of a bitopological space (X,τ_1,τ_2) is said to be:*

1. *(p,q)-preopen set if $A\subseteq\tau_p$-Int$(\tau_q$-Cl$(A))$ [11],*
2. *(p,q)-semiopen set if $A\subseteq\tau_q$-Cl$(\tau_p$-Int$(A))$ [16],*

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3. (p,q)-α-*open set if* $A \subseteq \tau_p$-*Int*$(\tau_q$-*Cl*$(\tau_p$-*Int*$(A)))$ [13],
4. (p,q)-*b-open set if* $A \subseteq \tau_p$-*Int*$(\tau_q$-*Cl*$(A)) \cup \tau_q$-*Cl*$(\tau_p$-*Int*$(A))$ [3],
5. (p,q)-β-*open set if* $A \subseteq \tau_q$-*Cl*$(\tau_p$-*Int*$(\tau_q$-*Cl*$(A)))$ [14],

where $p,q = 1,2$ *and* $p \neq q$.

3. Generalized δ-open sets

In this section, we define new types of generalized δ-open sets in bitopological spaces and study their properties.

Definition 3.1. *A subset A of a bitopological space* (X,τ_1,τ_2) *is said to be:*

1. (p,q)-δ-*preopen set if* $A \subseteq \tau_p$-*Int*$(\tau_q$-*Cl*$_\delta(A))$,
2. (p,q)-δ-*semiopen set if* $A \subseteq \tau_q$-*Cl*$(\tau_p$-*Int*$_\delta(A))$,
3. (p,q)-*a-open set if* $A \subseteq \tau_p$-*Int*$(\tau_q$-*Cl*$(\tau_p$-*Int*$_\delta(A)))$,
4. (p,q)-*e-open set if* $A \subseteq \tau_p$-*Int*$(\tau_q$-*Cl*$_\delta(A)) \cup \tau_q$-*Cl*$(\tau_p$-*Int*$_\delta(A))$,
5. (p,q)-e^*-*open set if* $A \subseteq \tau_q$-*Cl*$(\tau_p$-*Int*$(\tau_q$-*Cl*$_\delta(A)))$,

where $p,q = 1,2$ *and* $p \neq q$.

Theorem 3.2. *Let* (X,τ_1,τ_2) *be a bitopological space. Then the following properties hold:*

1. *Every* τ_p-δ-*open set is* (p,q)-*a-open.*
2. *Every* (p,q)-*a-open set is* (p,q)-δ-*preopen.*
3. *Every* (p,q)-*a-open set is* (p,q)-δ-*semiopen.*
4. *Every* (p,q)-δ-*preopen set is* (p,q)-*e-open.*
5. *Every* (p,q)-δ-*semiopen set is* (p,q)-*e-open.*
6. *Every* (p,q)-*e-open set is* (p,q)-e^*-*open.*

Proof.

1. Let A be a τ_p-δ-open set. Then we have
$$\tau_p\text{-Int}(A) \subseteq \tau_p\text{-Int}(\tau_q\text{-Cl}(\tau_p\text{-Int}_\delta(A)))$$
and this implies
$$A \subseteq \tau_p\text{-Int}(\tau_q\text{-Cl}(\tau_p\text{-Int}_\delta(A))).$$
Hence A is (p,q)-a-open.
2. Let A be (p,q)-a-open set. Then we have
$$\begin{aligned} A \subseteq \tau_p\text{-Int}(\tau_q\text{-Cl}(\tau_p\text{-Int}_\delta(A))) &\subseteq \tau_p\text{-Int}(\tau_q\text{-Cl}_\delta(\tau_p\text{-Int}_\delta(A))) \\ &\subseteq \tau_p\text{-Int}(\tau_q\text{-Cl}_\delta(A)). \end{aligned}$$
Hence A is (p,q)-δ-preopen.
3. Let A be (p,q)-a-open set. Then we have
$$A \subseteq \tau_p\text{-Int}(\tau_q\text{-Cl}(\tau_p\text{-Int}_\delta(A))) \subseteq \tau_q\text{-Cl}(\tau_p\text{-Int}_\delta(A)).$$
Hence A is (p,q)-δ-semiopen.
4. It is obvious.
5. It is obvious.
6. Let A be (p,q)-e-open set. Then we have
$$\begin{aligned} A &\subseteq \tau_p\text{-Int}(\tau_q\text{-Cl}_\delta(A)) \cup \tau_q\text{-Cl}(\tau_p\text{-Int}_\delta(A)) \\ &\subseteq \tau_q\text{-Cl}(\tau_p\text{-Int}(\tau_q\text{-Cl}_\delta(A))) \cup \tau_q\text{-Cl}(A) \\ &= \tau_q\text{-Cl}(\tau_p\text{-Int}(\tau_q\text{-Cl}_\delta(A)) \cup A) \\ &= \tau_q\text{-Cl}(\tau_p\text{-Int}(\tau_q\text{-Cl}_\delta(A))). \end{aligned}$$
Hence A is (p,q)-e^*-open.

□

The inverse inclusions in Theorem 3.2 may not hold as shown in the following examples.

Example 3.3. *Let* $X = \{u,v,w\}$ *with topologies* $\tau_1 = \{X,\emptyset,\{u\},\{v\},\{u,v\}\}$ *and* $\tau_2 = \{X,\emptyset,\{u\},\{u,v\},\{u,w\}\}$. *Then the set* $\{u,w\}$ *is* $(1,2)$-*a-open but it is not* τ_1-δ-*open.*

Example 3.4. *Let* $X = \{u,v,w\}$ *with topologies* $\tau_1 = \{X,\emptyset,\{u\},\{v\},\{u,v\}\}$ *and* $\tau_2 = \{X,\emptyset,\{v\},\{u,v\}\}$. *Then the set* $\{u,w\}$ *is* $(1,2)$-δ-*preopen and* $(1,2)$-δ-*semiopen but it is not* $(1,2)$-*a-open.*

Example 3.5. *Let* $X = \{u,v,w\}$ *with topologies* $\tau_1 = \{X,\emptyset,\{w\},\{u,v\}\}$ *and* $\tau_2 = \{X,\emptyset,\{u\},\{w\},\{u,w\}\}$. *Then the set* $\{v,w\}$ *is* $(1,2)$-*e-open but it is not* $(1,2)$-δ-*preopen.*

Example 3.6. *Let* $X = \{u,v,w\}$ *with topologies* $\tau_1 = \{X,\emptyset,\{u\},\{v\},\{u,v\}\},\{v,w\}\}$ *and* $\tau_2 = \{X,\emptyset,\{w\},\{v,w\}\}$. *Then the set* $\{u,v\}$ *is* $(1,2)$-*e-open but it is not* $(1,2)$-δ-*semiopen.*

Example 3.7. *Let $X=\{u,v,w\}$ with topologies $\tau_1=\{X,\emptyset,\{u\}\}$ and $\tau_2=\{X,\emptyset,\{w\},\{u,v\}\}$. Then the set $\{u,v\}$ is $(1,2)$-e^*-open but it is not $(1,2)$-e-open.*

Theorem 3.8. *Let (X,τ_1,τ_2) be a bitopological space. Then the following properties hold:*

1. *Every (p,q)-preopen set is (p,q)-δ-preopen.*
2. *Every (p,q)-δ-semiopen set is (p,q)-semiopen.*
3. *Every (p,q)-a-open set is (p,q)-α-open.*
4. *Every (p,q)-β-open set is (p,q)-e^*-open.*

Proof. The proof is obvious. □

The inverse inclusions in Theorem 3.8 may not hold as shown in the following examples.

Example 3.9. *Let $X=\{u,v,w\}$ with topologies $\tau_1=\{X,\emptyset,\{u\}\}$ and $\tau_2=\{X,\emptyset,\{w\},\{v,w\}\}$. Then the set $\{u,v\}$ is $(1,2)$-δ-preopen and so $(1,2)$-e^*-open. But it is not $(1,2)$-β-open and so not $(1,2)$-preopen.*

Example 3.10. *Let $X=\{u,v,w\}$ with topologies $\tau_1=\{X,\emptyset,\{u\},\{u,v\},\{u,w\}\}$ and $\tau_2=\{X,\emptyset,\{v\},\{u,v\}\}$. Then the set $\{u,w\}$ is $(1,2)$-α-open and so $(1,2)$-semiopen. But it is not $(1,2)$-δ-semiopen and so not $(1,2)$-a-open.*

Remark 3.11. *The notions of (p,q)-b-open set and (p,q)-e-open set are independent as shown in the following example.*

Example 3.12. *Let $X=\{u,v,w,z\}$ with topologies $\tau_1=\{X,\emptyset,\{u\},\{v\},\{u,v\},\{u,w\},\{u,v,w\}\}$ and $\tau_2=\{X,\emptyset,\{v\},\{u,z\},\{u,v,z\},\{u,w,z\}\}$. Then the set $\{v,w\}$ is $(1,2)$-e-open but it is not $(1,2)$-b-open and the set $\{u,z\}$ is $(1,2)$-b-open but it is not $(1,2)$-e-open.*

Theorem 3.13. *The union of any collection of (p,q)-e^*-open (resp. (p,q)-δ-preopen, (p,q)-δ-semiopen, (p,q)-a-open, (p,q)-e-open) sets is (p,q)-e^*-open (resp. (p,q)-δ-preopen, (p,q)-δ-semiopen, (p,q)-a-open, (p,q)-e-open).*

Proof. We prove only for a collection of (p,q)-e^*-open sets. The others can be proved in a similar way. Let $\{A_k : k\in\Delta\}$ be a collection of (p,q)-e^*-open sets. Then $A_k\subseteq\tau_q\text{-Cl}(\tau_p\text{-Int}(\tau_q\text{-Cl}_\delta(A_k)))$ for every $k\in\Delta$. Hence we have

$$\begin{aligned}\cup_{k\in\Delta}A_k &\subseteq \cup_{k\in\Delta}\tau_q\text{-Cl}(\tau_p\text{-Int}(\tau_q\text{-Cl}_\delta(A_k)))\\ &\subseteq \tau_q\text{-Cl}(\cup_{k\in\Delta}\tau_p\text{-Int}(\tau_q\text{-Cl}_\delta(A_k)))\\ &\subseteq \tau_q\text{-Cl}(\tau_p\text{-Int}(\cup_{k\in\Delta}\tau_q\text{-Cl}_\delta(A_k)))\\ &\subseteq \tau_q\text{-Cl}(\tau_p\text{-Int}(\tau_q\text{-Cl}_\delta(\cup_{k\in\Delta}A_k)))\end{aligned}$$

which means $\cup_{k\in\Delta}A_k$ is (p,q)-e^*-open. □

Remark 3.14. *The intersection of two (p,q)-e^*-open (resp. (p,q)-δ-preopen, (p,q)-δ-semiopen, (p,q)-a-open, (p,q)-e-open) sets need not be (p,q)-e^*-open (resp. (p,q)-δ-preopen, (p,q)-δ-semiopen, (p,q)-a-open, (p,q)-e-open) as shown from the following example.*

Example 3.15. *Let $X=\{u,v,w,z\}$, $\tau_1=\{X,\emptyset,\{v\},\{z\},\{v,z\},\{u,v,w\}\}$ and $\tau_2=\{X,\emptyset,\{u\},\{z\},\{u,v\},\{u,z\},\{u,v,z\}\}$. Then the sets $\{u,w\}$ and $\{w,z\}$ are $(1,2)$-e^*-open (also $(1,2)$-e-open) but their intersection $\{w\}$ is not (p,q)-e^*-open (so it is not $(1,2)$-e-open) in (X,τ_1,τ_2).*

Definition 3.16. *Let A be a subset of a bitopological space (X,τ_1,τ_2). A point x of X is called an (p,q)-e^*-interior point (resp. (p,q)-δ-pre-interior point, (p,q)-δ-semi-interior point, (p,q)-a-interior point, (p,q)-e-interior point) of A if there exists an (p,q)-e^*-open (resp. (p,q)-δ-preopen, (p,q)-δ-semiopen, (p,q)-a-open, (p,q)-e-open) set U such that $x\in U\subseteq A$. The set of all (p,q)-e^*-interior (resp. (p,q)-δ-pre-interior, (p,q)-δ-semi-interior, (p,q)-a-interior, (p,q)-e-interior) points of A is called (p,q)-e^*-interior (resp. (p,q)-δ-pre-interior, (p,q)-δ-semi-interior, (p,q)-a-interior, (p,q)-e-interior) of A and is denoted by (p,q)-e^*-Int(A) (resp. (p,q)-δ-p-Int(A), (p,q)-δ-s-Int(A), (p,q)-a-Int(A), (p,q)-e-Int(A)).*

Theorem 3.17. *Let A and B be subsets of (X,τ_1,τ_2). Then the following properties hold:*

1. *(p,q)-e^*-Int(A) (resp. (p,q)-δ-p-Int(A), (p,q)-δ-s-Int(A), (p,q)-a-Int(A), (p,q)-e-Int(A)) is the union of all (p,q)-e^*-open (resp. (p,q)-δ-preopen, (p,q)-δ-semiopen, (p,q)-a-open, (p,q)-e-open) sets contained in A.*
2. *(p,q)-e^*-Int(A) (resp. (p,q)-δ-p-Int(A), (p,q)-δ-s-Int(A), (p,q)-a-Int(A), (p,q)-e-Int(A)) is the largest (p,q)-e^*-open (resp. (p,q)-δ-preopen, (p,q)-δ-semiopen, (p,q)-a-open, (p,q)-e-open) set contained in A.*
3. *(p,q)-e^*-Int(A) (resp. (p,q)-δ-p-Int(A), (p,q)-δ-s-Int(A), (p,q)-a-Int(A), (p,q)-e-Int(A)) is an (p,q)-e^*-open (resp. (p,q)-δ-preopen, (p,q)-δ-semiopen, (p,q)-a-open, (p,q)-e-open) set.*
4. *A is (p,q)-e^*-open (resp. (p,q)-δ-preopen, (p,q)-δ-semiopen, (p,q)-a-open, (p,q)-e-open) if and only if $A=(p,q)$-e^*-Int(A) (resp. $A=(p,q)$-δ-p-Int(A), $A=(p,q)$-δ-s-Int(A), $A=(p,q)$-a-Int(A), $A=(p,q)$-e-Int(A)).*
5. *If $A\subseteq B$, then (p,q)-e^*-Int$(A)\subseteq(p,q)$-e^*-Int(B) (resp. (p,q)-δ-p-Int$(A)\subseteq(p,q)$-δ-p-Int(B), (p,q)-δ-s-Int$(A)\subseteq(p,q)$-δ-s-Int(B), (p,q)-a-Int$(A)\subseteq(p,q)$-a-Int(B), (p,q)-e-Int$(A)\subseteq(p,q)$-e-Int(B).*
6. *(p,q)-e^*-Int$(A)\cup(p,q)$-e^*-Int$(B)\subseteq(p,q)$-e^*-Int$(A\cup B)$ (resp. (p,q)-δ-p-Int$(A)\cup(p,q)$-δ-p-Int$(B)\subseteq(p,q)$-δ-p-Int$(A\cup B)$, (p,q)-δ-s-Int$(A)\cup(p,q)$-δ-s-Int$(B)\subseteq(p,q)$-δ-s-Int$(A\cup B)$, (p,q)-a-Int$(A)\cup(p,q)$-a-Int$(B)\subseteq(p,q)$-a-Int$(A\cup B)$, (p,q)-e-Int$(A)\cup(p,q)$-e-Int$(B)\subseteq(p,q)$-e-Int$(A\cup B)$.*
7. *(p,q)-e^*-Int$(A\cap B)\subseteq(p,q)$-e^*-Int$(A)\cap(p,q)$-e^*-Int(B) (resp. (p,q)-δ-p-Int$(A\cap B)\subseteq(p,q)$-δ-p-Int$(A)\cap(p,q)$-δ-p-Int(B), (p,q)-δ-s-Int$(A\cap B)\subseteq(p,q)$-δ-s-Int$(A)\cap(p,q)$-δ-s-Int(B), (p,q)-a-Int$(A\cap B)\subseteq(p,q)$-a-Int$(A)\cap(p,q)$-a-Int(B), (p,q)-e-Int$(A\cap B)\subseteq(p,q)$-e-Int$(A)\cap(p,q)$-e-Int(B).*

The inverse inclusions in 6 and 7 of Theorem 3.17 do not hold.

Example 3.18. *Let X, τ_1 and τ_2 be as in Example 3.15. Since $\{u,w\}$, $\{u\}$ are (p,q)-e^*-open and $\{w\}$ is not (p,q)-e^*-open, we have*

$$\{u,w\} = (p,q)\text{-}e^*\text{-}Int\{u,w\} \nsubseteq (p,q)\text{-}e^*\text{-}Int\{u\} \cup (p,q)\text{-}e^*\text{-}Int\{w\} = \{u\}.$$

Example 3.19. *Let X, τ_1 and τ_2 be as in Example 3.15. Since $\{u,w\}$, $\{w,z\}$ are (p,q)-e^*-open and $\{w\}$ is not (p,q)-e^*-open, we have*

$$\{w\} = (p,q)\text{-}e^*\text{-}Int\{u,w\} \cap (p,q)\text{-}e^*\text{-}Int\{w,z\} \nsubseteq (p,q)\text{-}e^*\text{-}Int\{w\} = \emptyset.$$

Definition 3.20. *The complement of an (p,q)-e^*-open (resp. (p,q)-δ-preopen, (p,q)-δ-semiopen, (p,q)-a-open, (p,q)-e-open) set is called (p,q)-e^*-closed (resp. (p,q)-δ-preclosed, (p,q)-δ-semiclosed, (p,q)-a-closed, (p,q)-e-closed).*

Lemma 3.21. *Let A be a subset of (X,τ_1,τ_2). Then the following hold:*

1. *A is (p,q)-δ-preclosed set if and only if τ_p-$Cl(\tau_q$-$Int_\delta(A)) \subseteq A$,*
2. *A is (p,q)-δ-semiclosed set if and only if τ_q-$Int(\tau_p$-$Cl_\delta(A)) \subseteq A$,*
3. *A is (p,q)-a-closed set if and only if τ_p-$Cl(\tau_q$-$Int(\tau_p$-$Cl_\delta(A))) \subseteq A$,*
4. *A is (p,q)-e-closed set if and only if τ_p-$Cl(\tau_q$-$Int_\delta(A)) \cap \tau_q$-$Int(\tau_p$-$Cl_\delta(A)) \subseteq A$,*
5. *A is (p,q)-e^*-closed set if and only if τ_q-$Int(\tau_p$-$Cl(\tau_q$-$Int_\delta(A))) \subseteq A$,*

where $p,q = 1,2$ and $p \neq q$.

Proof. The proof follows from the definitions. □

Theorem 3.22. *The intersection of any collection of (p,q)-e^*-closed (resp. (p,q)-δ-preclosed, (p,q)-δ-semiclosed, (p,q)-a-closed, (p,q)-e-closed) sets is (p,q)-e^*-closed (resp. (p,q)-δ-preclosed, (p,q)-δ-semiclosed, (p,q)-a-closed, (p,q)-e-closed).*

Proof. The proof follows from definitions and Theorem 3.13. □

Proposition 3.23. *For any subset A of (X,τ_1,τ_2), $A \cap \tau_q$-$Cl(\tau_p$-$Int_\delta(A))$ is (p,q)-δ-semiopen.*

Proof. The following inclusion holds

$$\begin{aligned} A \cap \tau_q\text{-Cl}(\tau_p\text{-Int}_\delta(A)) &\subseteq \tau_q\text{-Cl}(\tau_p\text{-Int}_\delta(A)) \\ &= \tau_q\text{-Cl}(\tau_p\text{-Int}_\delta(A) \cap \tau_p\text{-Int}_\delta(\tau_q\text{-Cl}(\tau_p\text{-Int}_\delta(A)))) \\ &= \tau_q\text{-Cl}(\tau_p\text{-Int}_\delta(A \cap \tau_q\text{-Cl}(\tau_p\text{-Int}_\delta(A)))) \end{aligned}$$

and so $A \cap \tau_q$-Cl$(\tau_p$-Int$_\delta(A))$ is (p,q)-δ-semiopen. □

Proposition 3.24. *Let A be a subset of (X,τ_1,τ_2). Then (p,q)-δ-s-$Int(A) = A \cap \tau_q$-$Cl(\tau_p$-$Int_\delta(A))$.*

Proof. Let $B = (p,q)$-δ-s-Int(A). Since B is (p,q)-δ-semiopen and $B \subseteq A$, we have $B \subseteq \tau_q$-Cl$(\tau_p$-Int$_\delta(B)) \subseteq \tau_q$-Cl$(\tau_p$-Int$_\delta(A))$. Hence the inclusion

$$B \subseteq A \cap \tau_q\text{-Cl}(\tau_p\text{-Int}_\delta(A))$$

holds. By Proposition 3.23, $A \cap \tau_q$-Cl$(\tau_p$-Int$_\delta(A))$ is (p,q)-δ-semiopen and contained in A. Thus we obtain

$$A \cap \tau_q\text{-Cl}(\tau_p\text{-Int}_\delta(A)) \subseteq B$$

since B is the largest (p,q)-δ-semiopen set contained in A. We conclude that the equality (p,q)-δ-s-Int$(A) = A \cap \tau_q$-Cl$(\tau_p$-Int$_\delta(A))$ holds. □

Proposition 3.25. *For any subset A of (X,τ_1,τ_2), $A \cap \tau_p$-$Int(\tau_q$-$Cl(\tau_p$-$Int_\delta(A)))$ is (p,q)-a-open.*

Proof. It is similar to the proof of Proposition 3.23. □

Proposition 3.26. *Let A be a subset of (X,τ_1,τ_2). Then (p,q)-a-$Int(A) = A \cap \tau_p$-$Int(\tau_q$-$Cl(\tau_p$-$Int_\delta(A)))$.*

Proof. It is similar to the proof of Proposition 3.24. □

Proposition 3.27. *Let A be a subset of (X,τ_1,τ_2). If A is (p,q)-δ-semiopen, then $A \cap \tau_p$-$Int(\tau_q$-$Cl_\delta(A))$ is (p,q)-δ-preopen.*

Proof. Let A be (p,q)-δ-semiopen. Then we have τ_q-Cl$_\delta(A) = \tau_q$-Cl$_\delta(\tau_p$-Int$(A))$. Hence the following relation holds:

$$\begin{aligned} A \cap \tau_p\text{-Int}(\tau_q\text{-Cl}_\delta(A)) &= A \cap \tau_p\text{-Int}(\tau_q\text{-Cl}_\delta(\tau_p\text{-Int}(A))) \\ &\subseteq \tau_p\text{-Int}(\tau_q\text{-Cl}_\delta(\tau_p\text{-Int}(A))) \\ &= \tau_p\text{-Int}(\tau_q\text{-Cl}_\delta(\tau_p\text{-Int}(A) \cap \tau_p\text{-Int}(\tau_q\text{-Cl}_\delta(A)))) \\ &\subseteq \tau_p\text{-Int}(\tau_q\text{-Cl}_\delta(A \cap \tau_p\text{-Int}(\tau_q\text{-Cl}_\delta(A)))) \end{aligned}$$

which means $A \cap \tau_p$-Int$(\tau_q$-Cl$_\delta(A))$ is (p,q)-δ-preopen. □

Proposition 3.28. *Let A be a subset of (X,τ_1,τ_2). Then (p,q)-δ-p-$Int(A) \subseteq A \cap \tau_p$-$Int(\tau_q$-$Cl_\delta(A))$.*

Proof. It is similar to the proof of Proposition 3.24. □

Proposition 3.29. *Let A be a subset of (X,τ_1,τ_2). If A is (q,p)-δ-semiclosed, then $A \cap \tau_q$-$Cl(\tau_p$-$Int(\tau_q$-$Cl_\delta(A)))$ is (p,q)-e^*-open.*

Proof. It is similar to the proof of Proposition 3.27. □

Proposition 3.30. *Let A be a subset of (X,τ_1,τ_2). Then (p,q)-e^*-$Int(A) \subseteq A \cap \tau_q$-$Cl(\tau_p$-$Int(\tau_q$-$Cl_\delta(A)))$.*

Proof. It is similar to the proof of Proposition 3.24. □

Definition 3.31. *Let A be a subset of (X,τ_1,τ_2). A point x of X is called an (p,q)-e^*-cluster point (resp. (p,q)-δ-pre-cluster point, (p,q)-δ-semi-cluster point, (p,q)-a-cluster point, (p,q)-e-cluster point) of A if $U \cap A \neq \emptyset$ for every (p,q)-e^*-open (resp. (p,q)-δ-preopen, (p,q)-δ-semiopen, (p,q)-a-open, (p,q)-e-open) set U in X. The set of all (p,q)-e^*-cluster (resp. (p,q)-δ-pre-cluster, (p,q)-δ-semi-cluster, (p,q)-a-cluster, (p,q)-e-cluster) points of A is called (p,q)-e^*-closure (resp. (p,q)-δ-pre-closure, (p,q)-δ-semi-closure, (p,q)-a-closure, (p,q)-e-closure) of A and is denoted by (p,q)-e^*-$Cl(A)$ (resp. (p,q)-δ-p-$Cl(A)$, (p,q)-δ-s-$Cl(A)$, (p,q)-a-$Cl(A)$, (p,q)-e-$Cl(A)$).*

Theorem 3.32. *Let A and B be subsets of (X,τ_1,τ_2). Then the following properties hold:*

1. *(p,q)-e^*-$Cl(A)$ (resp. (p,q)-δ-p-$Cl(A)$, (p,q)-δ-s-$Cl(A)$, (p,q)-a-$Cl(A)$, (p,q)-e-$Cl(A)$) is the intersection of all (p,q)-e^*-closed (resp. (p,q)-δ-preclosed, (p,q)-δ-semiclosed, (p,q)-a-closed, (p,q)-e-closed) sets containing A.*
2. *(p,q)-e^*-$Cl(A)$ (resp. (p,q)-δ-p-$Cl(A)$, (p,q)-δ-s-$Cl(A)$, (p,q)-a-$Cl(A)$, (p,q)-e-$Cl(A)$) is the smallest (p,q)-e^*-closed (resp. (p,q)-δ-preclosed, (p,q)-δ-semiclosed, (p,q)-a-closed, (p,q)-e-closed) set containing A.*
3. *(p,q)-e^*-$Cl(A)$ (resp. (p,q)-δ-p-$Cl(A)$, (p,q)-δ-s-$Cl(A)$, (p,q)-a-$Cl(A)$, (p,q)-e-$Cl(A)$) is an (p,q)-e^*-closed (resp. (p,q)-δ-preclosed, (p,q)-δ-semiclosed, (p,q)-a-closed, (p,q)-e-closed) set.*
4. *A is (p,q)-e^*-closed (resp. (p,q)-δ-preclosed, (p,q)-δ-semiclosed, (p,q)-a-closed, (p,q)-e-closed) if and only if $A = (p,q)$-e^*-$Cl(A)$ (resp. $A = (p,q)$-δ-p-$Cl(A)$, $A = (p,q)$-δ-s-$Cl(A)$, $A = (p,q)$-a-$Cl(A)$, $A = (p,q)$-e-$Cl(A)$).*
5. *If $A \subseteq B$, then (p,q)-e^*-$Cl(A) \subseteq (p,q)$-e^*-$Cl(B)$ (resp. (p,q)-δ-p-$Cl(A) \subseteq (p,q)$-δ-p-$Cl(B)$, (p,q)-δ-s-$Cl(A) \subseteq (p,q)$-δ-s-$Cl(B)$, (p,q)-a-$Cl(A) \subseteq (p,q)$-a-$Cl(B)$, (p,q)-e-$Cl(A) \subseteq (p,q)$-e-$Cl(B)$).*
6. *(p,q)-e^*-$Cl(A) \cup (p,q)$-e^*-$Cl(B) \subseteq (p,q)$-e^*-$Cl(A \cup B)$ (resp. (p,q)-δ-p-$Cl(A) \cup (p,q)$-δ-p-$Cl(B) \subseteq (p,q)$-δ-p-$Cl(A \cup B)$, (p,q)-δ-s-$Cl(A) \cup (p,q)$-δ-s-$Cl(B) \subseteq (p,q)$-δ-s-$Cl(A \cup B)$, (p,q)-a-$Cl(A) \cup (p,q)$-a-$Cl(B) \subseteq (p,q)$-a-$Cl(A \cup B)$, (p,q)-e-$Cl(A) \cup (p,q)$-e-$Cl(B) \subseteq (p,q)$-e-$Cl(A \cup B)$.*
7. *(p,q)-e^*-$Cl(A \cap B) \subseteq (p,q)$-$e^*$-$Cl(A) \cap (p,q)$-$e^*$-$Cl(B)$ (resp. (p,q)-δ-p-$Cl(A \cap B) \subseteq (p,q)$-$\delta$-p-$Cl(A) \cap (p,q)$-$\delta$-p-$Cl(B)$, (p,q)-δ-s-$Cl(A \cap B) \subseteq (p,q)$-$\delta$-s-$Cl(A) \cap (p,q)$-$\delta$-s-$Cl(B)$, (p,q)-a-$Cl(A \cap B) \subseteq (p,q)$-a-$Cl(A) \cap (p,q)$-a-$Cl(B)$, (p,q)-e-$Cl(A \cap B) \subseteq (p,q)$-e-$Cl(A) \cap (p,q)$-e-$Cl(B)$.*

The inverse inclusions in 6 and 7 of Theorem 3.32 do not hold.

Example 3.33. *Let X, τ_1 and τ_2 be as in Example 3.15. Since $\{v,z\}$, $\{u,v\}$ are (p,q)-e^*-closed and $\{u,v,z\}$ is not (p,q)-e^*-closed, we have*

$$X = (p,q)\text{-}e^*\text{-}Cl\{u,v,z\} \nsubseteq (p,q)\text{-}e^*\text{-}Cl\{u,v\} \cup (p,q)\text{-}e^*\text{-}Cl\{v,z\} = \{u,v,z\}.$$

Example 3.34. *Let X, τ_1 and τ_2 be as in Example 3.15. Since $\{v,w,z\}$, $\{v,z\}$ are (p,q)-e^*-closed and $\{u,v,z\}$ is not (p,q)-e^*-closed, we have*

$$\{v,w,z\} = (p,q)\text{-}e^*\text{-}Cl\{v,w,z\} \cap (p,q)\text{-}e^*\text{-}Cl\{u,v,z\} \nsubseteq (p,q)\text{-}e^*\text{-}Cl\{v,z\} = \{v,z\}.$$

Proposition 3.35. *For any subset A of (X,τ_1,τ_2), $A \cup \tau_q$-$Int(\tau_p$-$Cl_\delta(A))$ is (p,q)-δ-semiclosed.*

Proof. The following inclusion holds

$$\begin{aligned} X\backslash(A \cup \tau_q\text{-Int}(\tau_p\text{-Cl}_\delta(A))) &= X \cap [(X\backslash A) \cap (X\backslash \tau_q\text{-Int}(\tau_p\text{-Cl}_\delta(A)))] \\ &= (X\backslash A) \cap (\tau_q\text{-Cl}(\tau_p\text{-Int}_\delta(X\backslash A))). \end{aligned}$$

Since $(X\backslash A) \cap (\tau_q\text{-Cl}(\tau_p\text{-Int}_\delta(X\backslash A)))$ is (p,q)-δ-semiopen, we have $A \cup \tau_q\text{-Int}(\tau_p\text{-Cl}_\delta(A))$ is (p,q)-δ-semiclosed. □

Proposition 3.36. *Let A be a subset of (X,τ_1,τ_2). Then (p,q)-δ-s-$Cl(A) = A \cup \tau_q$-$Int(\tau_p$-$Cl_\delta(A))$.*

Proof. Let $C = (p,q)$-δ-s-Cl(A). Since C is (p,q)-δ-semiclosed and $A \subseteq C$, we have $\tau_q\text{-Int}(\tau_p\text{-Cl}_\delta(A)) \subseteq \tau_q\text{-Int}(\tau_p\text{-Cl}_\delta(C)) \subseteq C$. Hence the inclusion

$$A \cup \tau_q\text{-Int}(\tau_p\text{-Cl}_\delta(A)) \subseteq C$$

holds. By Proposition 3.35, $A \cup \tau_q\text{-Int}(\tau_p\text{-Cl}_\delta(A))$ is (p,q)-δ-semiclosed and it contains A. Thus we obtain

$$C \subseteq A \cup \tau_q\text{-Int}(\tau_p\text{-Cl}_\delta(A))$$

since C is the smallest (p,q)-δ-semiclosed set containing A. We conclude that the equality (p,q)-δ-s-Cl$(A) = A \cup \tau_q\text{-Int}(\tau_p\text{-Cl}_\delta(A))$ holds. □

Proposition 3.37. *For any subset A of (X,τ_1,τ_2), $A \cup \tau_p$-$Cl(\tau_q$-$Int(\tau_p$-$Cl_\delta(A)))$ is (p,q)-a-closed.*

Proof. It is similar to the proof of Proposition 3.35. □

Proposition 3.38. *Let A be a subset of (X,τ_1,τ_2). Then (p,q)-a-$Cl(A) = A \cup \tau_p$-$Cl(\tau_q$-$Int(\tau_p$-$Cl_\delta(A)))$.*

Proof. It is similar to the proof of Proposition 3.36. □

Proposition 3.39. *Let A be a subset of (X,τ_1,τ_2). If A is (p,q)-δ-semiclosed, then $A \cup \tau_p$-$Cl(\tau_q$-$Int_\delta(A))$ is (p,q)-δ-preclosed.*

Proof. Let A be (p,q)-δ-semiclosed. Then $X\backslash A$ is (p,q)-δ-semiopen. From Proposition 3.27, we have $(X\backslash A) \cap \tau_p\text{-Int}(\tau_q\text{-Cl}_\delta(X\backslash A))$ is (p,q)-δ-preopen and therefore its complement $A \cup \tau_p\text{-Cl}(\tau_q\text{-Int}_\delta(A))$ is (p,q)-δ-preclosed. □

Proposition 3.40. *Let A be a subset of (X,τ_1,τ_2). Then (p,q)-δ-p-$Cl(A) \supseteq A \cup \tau_p$-$Cl(\tau_q$-$Int_\delta(A))$.*

Proof. It is similar to the proof of Proposition 3.36. □

Proposition 3.41. *Let A be a subset of (X,τ_1,τ_2). If A is (q,p)-δ-semiopen, then $A \cup \tau_q$-$Int(\tau_p$-$Cl(\tau_q$-$Int_\delta(A)))$ is (p,q)-e^*-closed.*

Proof. It is similar to the proof of Proposition 3.39. □

Proposition 3.42. *Let A be a subset of (X,τ_1,τ_2). Then (p,q)-e^*-$Cl(A) \supseteq A \cup \tau_q$-$Int(\tau_p$-$Cl(\tau_q$-$Int_\delta(A)))$.*

Proof. It is similar to the proof of Proposition 3.36. □

Theorem 3.43. *Let A be a subset of (X,τ_1,τ_2). Then the following statements hold:*

1. $X\backslash(p,q)$-δ-p-$Int(A) = (p,q)$-δ-p-$Cl(X\backslash A)$
   $X\backslash(p,q)$-δ-s-$Int(A) = (p,q)$-δ-s-$Cl(X\backslash A)$
   $X\backslash(p,q)$-a-$Int(A) = (p,q)$-a-$Cl(X\backslash A)$
   $X\backslash(p,q)$-e-$Int(A) = (p,q)$-e-$Cl(X\backslash A)$
   $X\backslash(p,q)$-e^*-$Int(A) = (p,q)$-e^*-$Cl(X\backslash A)$

2. $X\backslash(p,q)$-δ-p-$Cl(A) = (p,q)$-δ-p-$Int(X\backslash A)$
   $X\backslash(p,q)$-δ-s-$Cl(A) = (p,q)$-δ-s-$Int(X\backslash A)$
   $X\backslash(p,q)$-a-$Cl(A) = (p,q)$-a-$Int(X\backslash A)$
   $X\backslash(p,q)$-e-$Cl(A) = (p,q)$-e-$Int(X\backslash A)$
   $X\backslash(p,q)$-e^*-$Cl(A) = (p,q)$-e^*-$Int(X\backslash A)$

Proof. We only prove for the last equalities of (1) and (2). The other cases can be showed similarly.

$$\begin{aligned} x \notin (p,q)\text{-}e^*\text{-Int}(A) \quad &\Leftrightarrow \quad U \cap (X\backslash A) \neq \emptyset \text{ for all } (p,q)\text{-}e^*\text{-open set } U \text{ containing } x \\ &\Leftrightarrow \quad x \in (p,q)\text{-}e^*\text{-Cl}(X\backslash A). \end{aligned}$$

$$\begin{aligned} x \notin (p,q)\text{-}e^*\text{-Cl}(A) \quad &\Leftrightarrow \quad \text{there exists an } (p,q)\text{-}e^*\text{-open set } U \text{ containing } x \\ &\quad\quad \text{such that } U \cap A = \emptyset, \text{ that is } U \subseteq X\backslash A \\ &\Leftrightarrow \quad x \in (p,q)\text{-}e^*\text{-Int}(X\backslash A). \end{aligned}$$

□

Definition 3.44. *Let A be a subset of (X,τ_1,τ_2). A point x of X is called an (p,q)-e^*-limit point (resp. (p,q)-δ-pre-limit point, (p,q)-δ-semi-limit point, (p,q)-a-limit point, (p,q)-e-limit point) of A if $U \cap (A\backslash\{x\}) \neq \emptyset$ for every (p,q)-e^*-open (resp. (p,q)-δ-preopen, (p,q)-δ-semiopen, (p,q)-a-open, (p,q)-e-open) set U containinig x.*

Theorem 3.45. *A subset A of a bitopological space X is (p,q)-e^*-closed (resp. (p,q)-δ-preclosed, (p,q)-δ-semiclosed, (p,q)-a-closed, (p,q)-e-closed) if and only if it contains the set of its (p,q)-e^*-limit points (resp. (p,q)-δ-pre-limit points, (p,q)-δ-semi-limit points, (p,q)-a-limit points, (p,q)-e-limit points).*

Proof. Let A be (p,q)-e^*-closed. Assume that x is an (p,q)-e^*-limit point of A such that x belongs to $X\backslash A$. Since $X\backslash A$ is (p,q)-e^*-open containing x, we have $A \cap X\backslash A \neq \emptyset$ which is a contradiction. Hence A contains all of its (p,q)-e^*-limit points.
Conversely, assume that A contains the set of its (p,q)-e^*-limit points. Then for each $x \in X\backslash A$ there exists an (p,q)-e^*-open set U containing x such that $U \cap A = \emptyset$. Therefore we have $x \in U \subseteq X\backslash A$ which means $x \in (p,q)$-e^*-Int$(X\backslash A)$. We conclude that $X\backslash A = (p,q)$-e^*-Int$(X\backslash A)$, that is $X\backslash A$ is an (p,q)-e^*-open set and so A is (p,q)-e^*-closed.

□

4. (p,q)-e^* continuity

Definition 4.1. *A function $f : (X,\tau_1,\tau_2) \to (Y,\zeta_1,\zeta_2)$ is called (p,q)-e^*-continuous (resp. (p,q)-δ-pre-continuous , (p,q)-δ-semi-continuous , (p,q)-a-continuous , (p,q)-e-continuous) at a point x of X if for every ζ_p open set V of Y containing $f(x)$ there exists an (p,q)-e^*-open (resp. (p,q)-δ-preopen , (p,q)-δ-semiopen , (p,q)-a-open, (p,q)-e-open) set U in X such that $x \in U$ and $f(U) \subseteq V$.*

Theorem 4.2. *Let f be a function from a bitopological space (X,τ_1,τ_2) to another bitopological space (Y,ζ_1,ζ_2). Then the following statements are equivalent:*

1. *f is (p,q)-e^*-continuous (resp. (p,q)-δ-pre-continuous, (p,q)-δ-semi-continuous, (p,q)-a-continuous, (p,q)-e-continuous).*
2. *For every ζ_p open subset G of Y, $f^{-1}(G)$ is an (p,q)-e^*-open (resp. (p,q)-δ-preopen, (p,q)-δ-semiopen, (p,q)-a-open, (p,q)-e-open) set in X.*
3. *For every ζ_p closed subset F of Y, $f^{-1}(F)$ is an (p,q)-e^*-closed (resp. (p,q)-δ-preclosed , (p,q)-δ-semiclosed, (p,q)-a-closed, (p,q)-e-closed) set in X.*
4. *For every subset A of X, $f((p,q)$-e^*-$Cl(A)) \subseteq \zeta_p$-$Cl(f(A))$ (resp. $f((p,q)$-δ-p-$Cl(A)) \subseteq \zeta_p$-$Cl(f(A))$, $f((p,q)$-δ-s-$Cl(A)) \subseteq \zeta_p$-$Cl(f(A))$, $f((p,q)$-a-$Cl(A)) \subseteq \zeta_p$-$Cl(f(A))$, $f((p,q)$-e-$Cl(A)) \subseteq \zeta_p$-$Cl(f(A))$).*

5. *For every subset B of Y, (p,q)-e^*-Cl$(f^{-1}(B)) \subseteq f^{-1}(\zeta_p$-Cl$(B))$ (resp. (p,q)-δ-p-Cl$(f^{-1}(B)) \subseteq f^{-1}(\zeta_p$-Cl$(B))$, (p,q)-δ-s-Cl$(f^{-1}(B)) \subseteq f^{-1}(\zeta_p$-Cl$(B))$, (p,q)-a-Cl$(f^{-1}(B)) \subseteq f^{-1}(\zeta_p$-Cl$(B))$, (p,q)-e-Cl$(f^{-1}(B)) \subseteq f^{-1}(\zeta_p$-Cl$(B))$).*

Proof. By using the fact that $X \backslash f^{-1}(B) = f^{-1}(Y \backslash B)$ for every subset B of Y, the equivalence of (2) and (3) can be easily shown. We prove the other cases.
(1)⇒(2) Assume that f is (p,q)-e^*-continuous. Let G be ζ_p open subset of Y and take any $x \in f^{-1}(G)$. Then $f(x)$ is contained in G. Since f is (p,q)-e^*-continuous at every point of X, there exists an (p,q)-e^*-open set U in X such that $x \in U$ and $f(U) \subseteq G$. It follows that $x \in U \subseteq f^{-1}(G)$ which means $f^{-1}(G)$ is an (p,q)-e^*-open set in X.
(2)⇒(1) Suppose that the preimage of all ζ_p open subset of Y is an (p,q)-e^*-open set in X. Let x be any point of X and V be a ζ_p open set containing $f(x)$. Put $U = f^{-1}(V)$ which is (p,q)-e^*-open by hypotesis and contains x. Then we have $f(U) = f(f^{-1}(V)) \subseteq V$. This concludes the proof.
(1)⇒(4) Let A be a subset of X. Take a point y from $f((p,q)$-e^*-Cl$(A))$. Then there is $x \in (p,q)$-e^*-Cl(A) such that $f(x) = y$. By (p,q)-e^*-continuity of f, for every ζ_p open set of Y containing $y = f(x)$ we have $f(U) \subseteq V$, where U is (p,q)-e^*-open containing x. Also U and A have non-empty intersection since x is an (p,q)-e^*-cluster point of A. Hence we obtain that $V \cap f(A) \neq \emptyset$ which means $y \in \zeta_p$-Cl$(f(A))$. Thus the inclusion $f((p,q)$-e^*-Cl$(A)) \subseteq \zeta_p$-Cl$(f(A))$ holds for any subset A of X.
(4)⇒(5) Let B be a subset of Y. Put $A = f^{-1}(B)$. Then we have ζ_p-Cl$(f(A)) \subseteq \zeta_p$-Cl(B). This last inclusion with (4) implies $f((p,q)$-e^*-Cl$(A)) \subseteq \zeta_p$-Cl(B). By taking the pre-image of both sides, we obtain (p,q)-e^*-Cl$(A) \subseteq f^{-1}(\zeta_p$-Cl$(B))$.
(5)⇒(3) Assume that the inclusion in (5) holds for every subset of Y. Let F be ζ_p-closed in Y. Then by using the fact that $F = \zeta_p$-Cl(F), we have (p,q)-e^*-Cl$(f^{-1}(F)) \subseteq f^{-1}(F)$. Hence it is obvious that $f^{-1}(F)$ is (p,q)-e^*-closed in X.

□

Corollary 4.3. *Let f be a function from a bitopological space (X,τ_1,τ_2) to another bitopological space (Y,ζ_1,ζ_2). Then the following properties hold:*

1. *Every (p,q)-a-continuous function is (p,q)-δ-pre-continuous.*
2. *Every (p,q)-a-continuous function is (p,q)-δ-semi-continuous.*
3. *Every (p,q)-δ-pre-continuous function is (p,q)-e-continuous.*
4. *Every (p,q)-δ-semi-continuous function is (p,q)-e-continuous.*
5. *Every (p,q)-e-continuous function is (p,q)-e^*-continuous.*

Definition 4.4. *A function $f : (X,\tau_1,\tau_2) \to (Y,\zeta_1,\zeta_2)$ is called (p,q)-weakly-e^*-continuous (resp. (p,q)-weakly-δ-pre-continuous, (p,q)-weakly-δ-semi-continuous, (p,q)-weakly-a-continuous, (p,q)-weakly-e-continuous) if for each point x of X and each ζ_p open set V of Y containing $f(x)$, there exists an (p,q)-e^*-open (resp. (p,q)-δ-preopen , (p,q)-δ-semiopen , (p,q)-a-open, (p,q)-e-open) set U of X containing x such that $f(U) \subseteq \zeta_q$-Cl(V).*

Obviously, (p,q)-e^*-continuity (resp. (p,q)-δ-pre-continuity, (p,q)-δ-semi-continuity, (p,q)-a-continuity, (p,q)-e-continuity) of a function implies (p,q)-weakly-e^*-continuity (resp. (p,q)-weakly-δ-pre-continuity, (p,q)-weakly-δ-semi-continuity, (p,q)-weakly-a-continuity, (p,q)-weakly-e-continuity).

Theorem 4.5. *A function $f : (X,\tau_1,\tau_2) \to (Y,\zeta_1,\zeta_2)$ is (p,q)-weakly-e^*-continuous (resp. (p,q)-weakly-δ-pre-continuous, (p,q)-weakly-δ-semi-continuous, (p,q)-weakly-a-continuous, (p,q)-weakly-e-continuous) if and only if $f^{-1}(V) \subseteq (p,q)$-$e^*$-Int$(f^{-1}(\zeta_q$-Cl$(V)))$ (resp. $f^{-1}(V) \subseteq (p,q)$-$\delta$-$p$-Int$(f^{-1}(\zeta_q$-Cl$(V)))$, $f^{-1}(V) \subseteq (p,q)$-$\delta$-$s$-Int$(f^{-1}(\zeta_q$-Cl$(V)))$, $f^{-1}(V) \subseteq (p,q)$-$a$-Int$(f^{-1}(\zeta_q$-Cl$(V)))$, $f^{-1}(V) \subseteq (p,q)$-$e$-Int$(f^{-1}(\zeta_q$-Cl$(V)))$) for every ζ_p open set V of Y.*

Proof. Let $x \in f^{-1}(V)$ for any ζ_p open set V of Y. Suppose that f is (p,q)-weakly-e^*-continuous. Then there is an (p,q)-e^*-open set U such that $x \in U$ and $f(U) \subseteq \zeta_q$-Cl(V). From the last inclusion, we obtain $U \subseteq f^{-1}(\zeta_q$-Cl$(V))$ which means x is an (p,q)-e^*-interior point of $f^{-1}(\zeta_q$-Cl$(V))$.
Conversely, suppose that the inclusion $f^{-1}(V) \subseteq (p,q)$-$e^*$-Int$(f^{-1}(\zeta_q$-Cl$(V)))$ holds for any x in X and any ζ_p open set V containing $f(x)$. Put $U = (p,q)$-e^*-Int$(f^{-1}(\zeta_q$-Cl$(V)))$. Then we have U is an (p,q)-e^*-open set containing x such that $f(U) \subseteq \zeta_q$-Cl(V). Hence f is (p,q)-weakly-e^*-continuous. □

Definition 4.6. *A function $f : (X,\tau_1,\tau_2) \to (Y,\zeta_1,\zeta_2)$ is said to satisfy (p,q)-e^*-interiority condition (resp. (p,q)-δ-pre-interiority condition, (p,q)-δ-semi-interiority condition, (p,q)-a-interiority condition, (p,q)-e-interiority condition) if (p,q)-e^*-Int$(f^{-1}(\zeta_q$-Cl$(V))) \subseteq f^{-1}(V)$ (resp. (p,q)-δ-p-Int$(f^{-1}(\zeta_q$-Cl$(V))) \subseteq f^{-1}(V)$, (p,q)-δ-s-Int$(f^{-1}(\zeta_q$-Cl$(V))) \subseteq f^{-1}(V)$, (p,q)-a-Int$(f^{-1}(\zeta_q$-Cl$(V))) \subseteq f^{-1}(V)$, (p,q)-e-Int$(f^{-1}(\zeta_q$-Cl$(V))) \subseteq f^{-1}(V)$) for every ζ_q open set V of Y.*

Theorem 4.7. *If $f : (X,\tau_1,\tau_2) \to (Y,\zeta_1,\zeta_2)$ is an (p,q)-weakly-e^*-continuous (resp. (p,q)-weakly-δ-pre-continuous, (p,q)-weakly-δ-semi-continuous, (p,q)-weakly-a-continuous, (p,q)-weakly-e-continuous) function satisfying (p,q)-e^*-interiority condition (resp. (p,q)-δ-pre-interiority condition, (p,q)-δ-semi-interiority condition, (p,q)-a-interiority condition, (p,q)-e-interiority condition), then f is (p,q)-e^*-continuous (resp. (p,q)-δ-pre-continuous, (p,q)-δ-semi-continuous, (p,q)-a-continuous, (p,q)-e-continuous).*

Proof. Let G be ζ_p open in Y. By Theorem 4.5, we have $f^{-1}(G) \subseteq (p,q)$-$e^*$-Int$(f^{-1}(\zeta_q$-Cl$(G)))$. The reverse inclusion also holds since f satisfies (p,q)-e^*-interiority condition. Hence we have $f^{-1}(G)$ is an (p,q)-e^*-open set in X. So we conclude that f is (p,q)-e^*-continuous. □

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