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On Developable Ruled Surfaces in Pseudo-Galilean Space

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Abstract: In this paper, we investigated the ruled surfaces in the three-dimensional pseudo-Galilean space. We obtained the distribution parameter of the ruled surface by using the Frenet frame of directrix curve. Moreover, we derived the necessary conditions to construct a developable ruled surface in the pseudo-Galilean space.

Keyword: Developable, Distribution parameter, Pseudo-Galilean space, Ruled surface.

1 Introduction

A surface formed by a one-parameter family of straight lines is called a ruled surface. The ruled surfaces are one of the most important topics of the differential geometry since, the ruled surfaces appear in many different areas such as geometric design and manufacturing [6, 7, 9, 11], tool path planning and robot motion planning [13]. So, many geometers studied on the ruled surfaces in different spaces such as Minkowski and Galilean. Turgut and Hacısalihoglu [14, 15] defined timelike ruled surfaces and obtained some properties in the Minkowski space. Yaylı [17] obtained the distribution parameter of a spacelike ruled surface with respect to Frenet frame. O. Röschel [12] introduced the ruled surfaces in the Galilean space. He classified ruled surfaces into three types in the Galilean space. After that, Kamenarovic [8] obtained the defining equations for the ruled surfaces of types *A, B* and *C* in the Galilean space. Divjak and Milin-Sipus [4, 5] calculated the special curves on the ruled surfaces in the Galilean and pseudo-Galilean spaces. Recently, Milin-Sipus [10] investigated the ruled Weingarten surfaces in the Galilean space.

The pseudo-Galilean space is a Cayley-Klein space equipped with the projective metric of signature $(0, 0, +, -)$. The absolute figure of the pseudo-Galilean geometry consists of an ordered triple $\{\omega, f, I\}$, where ω is the real (absolute) plane, f is the real line (absolute line) in ω. I is the fixed hyperbolic involution of points of f. More information can be found in [1–3, 6, 16].

Definition 1.1. Let $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{b} = (b_1, b_2, b_3)$ be vectors in the pseudo-Galilean space. The scalar product is given by

$$<\mathbf{a}, \mathbf{b}>= a_1 b_1. \tag{1}$$

The scalar product of two isotropic vectors $\mathbf{p} = (0, p_2, p_3)$ and $\mathbf{q} = (0, q_2, q_3)$ is given by

$$<\mathbf{p}, \mathbf{q}>_1= p_2 q_2 - p_3 q_3. \tag{2}$$

Definition 1.2. The isotropic angle measure ϑ between two vectors $\mathbf{u} = (1, u_2, u_3)$ and $\mathbf{v} = (1, v_2, v_3)$ is defined by

$$\vartheta = \|\mathbf{u} - \mathbf{v}\|_1 = \sqrt{|(u_2 - v_2)^2 - (u_3 - v_3)^2|}. \tag{3}$$

On the otherhand, the angle measure between two isotropic vectors $\mathbf{p} = (0, p_2, p_3)$ and $\mathbf{q} = (0, q_2, q_3)$ is given by

$$\cosh\theta = \frac{<\mathbf{p}, \mathbf{q}>_1}{\|\mathbf{p}\|_1 \|\mathbf{q}\|_1}. \tag{4}$$

Definition 1.3. An admissible curve c is given by the parametrization

$$r(u) = (u, y(u), z(u)). \tag{5}$$

The Frenet frame is given by

$$\begin{aligned} \mathbf{t} &= (1, y'(u), z'(u)) \\ \mathbf{n} &= \frac{1}{\kappa}(0, y''(u), z''(u)) \\ \mathbf{b} &= \frac{1}{\kappa}(0, z''(u), y''(u)) \end{aligned}$$

where $\kappa = \sqrt{|y''^2 - z''^2|}$ is the curvature.

Frenet formulas are given by

$$\frac{d}{dx}\begin{bmatrix} \mathbf{t} \\ \mathbf{n} \\ \mathbf{b} \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ 0 & 0 & \tau \\ 0 & \tau & 0 \end{bmatrix}\begin{bmatrix} \mathbf{t} \\ \mathbf{n} \\ \mathbf{b} \end{bmatrix} \tag{6}$$

where $\tau = \frac{1}{\kappa^2}\det[r', r'', r''']$ is the torsion.

There are there types of the ruled surfaces in the pseudo-Galilean space.

Definition 1.4. A ruled surface of type *A* is parametrized by

$$\Phi_A(x,u) = m(x) + u\mathbf{a}(x) \tag{7}$$

where the directrix curve $m(x) = (x, y(x), z(x))$ does not lie in an pseudo-Euclidean plane and the generator $\mathbf{a}(x) = (1, a_2(x), a_3(x))$ is non-isotropic.

Definition 1.5. A ruled surface of type *B* can be parametrized by

$$\Phi_B(x,u) = r(x) + u\mathbf{a}(x) \tag{8}$$

where the directrix curve $r(x) = (0, y(x), z(x))$ lies in an pseudo-Euclidean plane and $\mathbf{a}(x) = (1, a_2(x), a_3(x))$ is the generator.

Definition 1.6. A ruled surface of type *C* is given by the parametrization

$$\Phi_C(x,u) = n(x) + u\mathbf{k}(x) \tag{9}$$

where the directrix $n(x) = (x, y(x), 0)$ lies in an isotropic plane and the generator $\mathbf{k}(x) = (0, a_2(x), a_3(x))$ is isotropic.

The ruled surfaces of type B and C are the special case of the ruled surfaces of type A. Moreover, the distribution parameters of the ruled surfaces of type B and C never vanish since, there are not developable ruled surfaces of type B and C. Hence, we omit them in this study.

2 Ruled surface of type *A*

The natural frame of the ruled surface of type *A* is defined by

$$\begin{aligned} \mathbf{e}_1(x) &= (1, a_2(x), a_3(x)) \\ \mathbf{e}_2(x) &= \frac{1}{\kappa}(0, a_2'(x), a_3'(x)) \\ \mathbf{e}_3(x) &= \frac{1}{\kappa}(0, a_3'(x), a_2'(x)) \end{aligned} \tag{10}$$

where $\kappa = \sqrt{\left|{a_2'}^2 - {a_3'}^2\right|}$.

Frenet formulas are given as follows

$$\frac{d}{dx}\begin{bmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ 0 & 0 & \tau \\ 0 & \tau & 0 \end{bmatrix}\begin{bmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{bmatrix} \tag{11}$$

where $\tau = \frac{\det(\mathbf{a}, \mathbf{a}', \mathbf{a}'')}{\kappa^2}$ is called the torsion.

The parameter of distribution d is given by

$$d = \frac{\det(m', \mathbf{a}, \mathbf{a}')}{\langle \mathbf{a}', \mathbf{a}' \rangle_1}. \tag{12}$$

The striction curve of the ruled surface of type *A* can be given by

$$s(x) = m(x) + u(x)\mathbf{a}(x)$$

where

$$u(x) = \frac{\langle \mathbf{a} - m', \mathbf{a}' \rangle_1}{\langle \mathbf{a}', \mathbf{a}' \rangle_1}. \tag{13}$$

Using the directrix curve $m(x)$ of the ruled surface of type *A*, we can define an another moving frame which is called the Frenet frame $\{\mathbf{t}, \mathbf{n}, \mathbf{b}\}$. The Frenet frame can be given in the following form

$$\begin{aligned} \mathbf{t} &= (1, y'(x), z'(x)) \\ \mathbf{n} &= \frac{1}{\kappa^*}(0, y''(x), z''(x)) \\ \mathbf{b} &= \frac{1}{\kappa^*}(0, z''(x), y''(x)) \end{aligned} \tag{14}$$

where $\kappa^* = \sqrt{|y''^2 - z''^2|}$.

Frenet formulas are obtained by

$$\frac{d}{dx}\begin{bmatrix} \mathbf{t} \\ \mathbf{n} \\ \mathbf{b} \end{bmatrix} = \begin{bmatrix} 0 & \kappa^* & 0 \\ 0 & 0 & \tau^* \\ 0 & \tau^* & 0 \end{bmatrix} \begin{bmatrix} \mathbf{t} \\ \mathbf{n} \\ \mathbf{b} \end{bmatrix} \tag{15}$$

where $\tau^* = \dfrac{1}{\kappa^{*2}} \det[m'(x), m''(x), m'''(x)]$.

The two moving coordinate systems $\{\mathbf{t}, \mathbf{n}, \mathbf{b}\}$ and $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ are orthogonal coordinate systems in the pseudo-Galilean space which represent the moving space H and the fixed space H', respectively.

CASE 1 Assume that spine curve $\gamma(u)$ is an admissible spacelike curve with a timelike normal vector $\mathbf{n}$. Therefore, the binormal $\mathbf{b}$ is a spacelike vector.

Theorem 2.1. Let $\{\mathbf{t}, \mathbf{n}, \mathbf{b}\}$ and $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be two orthogonal coordinate systems along the ruled surface of type *A*. The distribution parameter can be given by

$$d = \frac{\tau^*(x_3^2 - x_2^2) + \kappa^* x_3}{x_2^2 \tau^{*2} - (\kappa^* - x_3\tau^*)^2}. \tag{16}$$

Proof: The generator vector $\mathbf{a}(x)$ of the ruled surface of type *A* can be written in terms of Frenet frame base vectors $\{\mathbf{t}, \mathbf{n}, \mathbf{b}\}$ as follows

$$\mathbf{a} = x_1\mathbf{t} + x_2\mathbf{n} + x_3\mathbf{b} \tag{17}$$

where $x_1, x_2, x_3 \in \mathbb{R}$.

Using (1) and (17) it is easy to see that

$$\langle \mathbf{a}, \mathbf{t} \rangle = x_1 = 1. \tag{18}$$

Since $\mathbf{n}$ and $\mathbf{b}$ are isotropic vectors, it is convenient to rewrite (17) in the following form

$$\mathbf{a} - \mathbf{t} = x_2\mathbf{n} + x_3\mathbf{b}.$$

Then, using (2) gives

$$x_2 = -\langle \mathbf{a} - \mathbf{t}, \mathbf{n} \rangle_1 \tag{19}$$

and

$$x_3 = \langle \mathbf{a} - \mathbf{t}, \mathbf{b} \rangle_1. \tag{20}$$

In addition, differentiating (17) then, substituting (15) into the result gives

$$\mathbf{a}' = (\kappa^* + x_3\tau^*)\mathbf{n} + x_2\tau^*\mathbf{b}. \tag{21}$$

From (12), (17) and (21), we have the distribution parameter of the ruled surface of type *A* in the following form

$$d = \frac{\tau^*(x_3^2 - x_2^2) + \kappa^* x_3}{x_2^2 \tau^{*2} - (\kappa^* - x_3\tau^*)^2}. \tag{22}$$

A ruled surface is said to be developable if and only if the parameter of distribution d is zero. Hence, we state the following corollary:

Corollary 2.1. The ruled surface of type *A* is developable, if and only if the directrix curve is a helix with the following Frenet curvatures

$$\frac{\kappa^*}{\tau^*} = \frac{x_2^2 - x_3^2}{x_3}. \tag{23}$$

Theorem 2.2. Let $\{\mathbf{t}, \mathbf{n}, \mathbf{b}\}$ and $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be two orthogonal coordinate systems along the ruled surface of type *A*. The relation matrix between the two moving coordinate systems $\{\mathbf{t}, \mathbf{n}, \mathbf{b}\}$ and $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ are given by

$$\begin{bmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{bmatrix} = \begin{bmatrix} 1 & \dfrac{\kappa\sin\phi}{\tau^*} & \dfrac{\kappa^* - \kappa\cos\phi}{\tau^*} \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} \mathbf{t} \\ \mathbf{n} \\ \mathbf{b} \end{bmatrix} \tag{24}$$

where ϕ is the pseudo-Euclidean angle between the vectors $\mathbf{e}_2$ and $\mathbf{n}$.

Proof: It is easy to see that combining (21) and $\mathbf{a}' = \kappa\mathbf{e}_2$ gives

$$\mathbf{e}_2 = (\frac{\kappa^* + x_3\tau^*}{\kappa})\mathbf{n} + \frac{x_2\tau^*}{\kappa}\mathbf{b}. \tag{25}$$

This last equation implies that $\mathbf{e}_2$ is on the normal plane spanned by $\{\mathbf{n}, \mathbf{b}\}$. Since the isotropic vectors are on the pseudo-Euclidean planes in the pseudo-Galilean space. we can define the hyperbolic angle between the isotropic vectors. In order to obtain the relationship between the natural and Frenet frames. Let ϕ be the hyperbolic angle between the timelike isotropic vectors $\mathbf{e}_2$ and $\mathbf{n}$, we have

$$\mathbf{e}_2 = \cosh\phi\mathbf{n} + \sinh\phi\mathbf{b}. \tag{26}$$

It is easy to see that

$$\mathbf{e}_3 = \sinh\phi\mathbf{n} + \cosh\phi\mathbf{b}. \tag{27}$$

From (25), (26) and (27), we have

$$x_2 = \frac{\kappa\sinh\phi}{\tau^*} \tag{28}$$

and

$$x_3 = \frac{-\kappa^* + \kappa\cosh\phi}{\tau^*}. \tag{29}$$

Substituting (28) and (29) into (17) gives

$$\mathbf{a} = \mathbf{t} + \frac{\kappa\sinh\phi}{\tau^*}\mathbf{n} + \frac{-\kappa^* + \kappa\cosh\phi}{\tau^*}\mathbf{b}. \tag{30}$$

We may express the results in the following matrix form as

$$\begin{bmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{bmatrix} = \begin{bmatrix} 1 & \frac{\kappa\sinh\phi}{\tau^*} & \frac{-\kappa^* + \kappa\cosh\phi}{\tau^*} \\ 0 & \cosh\phi & \sinh\phi \\ 0 & \sinh\phi & \cosh\phi \end{bmatrix} \begin{bmatrix} \mathbf{t} \\ \mathbf{n} \\ \mathbf{b} \end{bmatrix}. \tag{31}$$

Combining (26) and $\mathbf{a}' = \kappa\mathbf{e}_2$, we obtain

$$\mathbf{a}' = \kappa\cosh\phi\mathbf{n} + \kappa\sinh\phi\mathbf{b}. \tag{32}$$

Corollary 2.2. From (12), (30) and (32), the distribution parameter of the ruled surface of type *A* is obtained by

$$d = \frac{\kappa - \kappa^*\cosh\phi}{\tau^*\kappa}. \tag{33}$$

Corollary 2.3. The ruled surface of type *A* is developable, if and only if there is a relation between κ and κ^* in the following form

$$\kappa = \kappa^*\cosh\phi. \tag{34}$$

Theorem 2.3. The striction curve of the ruled surface of type *A* can be given by

$$s(x) = m(x) + \frac{\kappa^*\sin\phi}{\kappa\tau^*}\mathbf{a}(x). \tag{35}$$

Proof: Substituting (28) and (29) into (17) gives

$$u(x) = \frac{\kappa^*\sinh\phi}{\kappa\tau^*} \tag{36}$$

and we obtain the equation (35).

3 Examples

Example 3.1. Let the ruled surface of type *A* be given by the parametrization

$$\Phi_A(x,u) = (x, 2\sinh x, 2\cosh x) + u((1, 3\cosh x + \sqrt{3}\sinh x, 3\sinh x + \sqrt{3}\cosh x)). \tag{37}$$

The natural frame is obtained by

$$\begin{aligned} \mathbf{e}_1 &= (1, 3\cosh x + \sqrt{3}\sinh x, 3\sinh x + \sqrt{3}\cosh x) \\ \mathbf{e}_2 &= (0, 3\sinh x + \sqrt{3}\cosh x, 3\cosh x + \sqrt{3}\sinh x) \\ \mathbf{e}_3 &= (0, 3\cosh x + \sqrt{3}\sinh x, 3\sinh x + \sqrt{3}\cosh x) \end{aligned}. \tag{38}$$

The Frenet frame is

$$\begin{aligned} \mathbf{t} &= (1, 2\cosh x, 2\sinh x) \\ \mathbf{n} &= (0, \sinh x, \cosh x) \\ \mathbf{b} &= (0, \cosh x, \sinh x) \end{aligned} \tag{39}$$

where $\kappa^* = 2, \tau^* = 1$.

From (19) and (20), we have

$$x_1 = 1, x_2 = \sqrt{3}, x_3 = 1$$

It follows that

$$\frac{\kappa^*}{\tau^*} = \frac{x_2^2 - x_3^2}{x_3} = 2.$$

Thus this is a developable ruled surface of type A shown in Fig. 1.

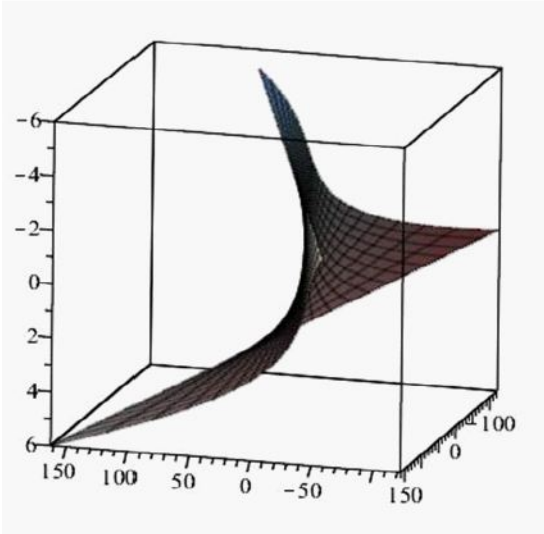

Fig. 1: A developable ruled surface of type A

4 References

[1] M. Dede, *Tubular surfaces in Galilean space*, Math. Commun., **18** (2013), 209-217.
[2] M. Dede, C. Ekici, A. C. Cöken, *On the parallel surfaces in Galilean space*, Hacettepe Journal of Mathematics and Statistics, **42** (2013), 605-615.
[3] B. Divjak, *Curves in pseudo-Galilean geometry*, Annales Universitatis Scientiarum Budapest, **41** (1998), 117-128.
[4] B. Divjak, Z. Milin-Sipus, *Special curves on ruled surfaces in Galilean and pseudo-Galilean space*, Acta Math. Hungar.,**98** (2003), 203-215.
[5] B. Divjak, Z. Milin-Sipus, *Minding isometries of ruled surfaces in pseudo-Galilean space*, J. Geom.,**77** (2003), 35-47.
[6] C. Ekici, M. Dede, *On the Darboux vector of ruled surfaces in pseudo-Galilean space*, Math. and Comp. App., **16** (2011), 830-838.
[7] H. W. Guggenheimer, *Differential geometry*, New York: McGraw-Hill, 1963.
[8] I. Kamenarovic, *Existence theorems for ruled surfaces in the Galilean Space* $\mathbb{G}_3$, Rad HAZU Math., **456** (1991), 183-196.
[9] E. Kasap, S. Yüce, N. Kuruoğlu, *The Involute-Evolute Offsets of Ruled Surfaces*, Iranian Journal of Science & Technology, Transaction A, **33** (2009), 195-201.
[10] Z. Milin-Sipus, *Ruled Weingarten surfaces in Galilean space*, Periodica Mathematica Hungarica, **56** (2008), 213-225.
[11] B. Ravani, T.S. Ku, *Bertrand Offsets of ruled and developable surfaces*, Comp. Aided Geom. Design,**23** (1991), 145-152.
[12] O. M. Röschel, *Die geometrie des Galileischen raumes*, Leoben: Habilitationsschrift, 1984.
[13] B. S. Ryuh, G.R. Pennock, *Accurate motion of Robot End-Effector using the curvature theory of ruled surfaces*, Journal of Mechanisms, Transmissions, and Automation in Design, **10** (1988), 383-387.
[14] A. Turgut, H. H. Hacısalihoğlu, *Timelike ruled surfaces in the Minkowski 3-space*, Far East J. Math. Sci.,**5** (1997), 83-90.
[15] A. Turgut, H. H. Hacısalihoğlu, *Timelike ruled surfaces in the Minkowski 3-space II*, Turkish J. Math. **22** (1998), 33-46.
[16] I. M. A. Yaglom, *Simple non-Euclidean geometry and its physical basis*, New York: Springer-Verlag, 1979.
[17] Y. Yaylı, On the motion of the Frenet vectors and spacelike ruled surfaces in the Minkowski 3-Space, Mathematical & Computational Applications, **5** (2000), 49-55.

