Journal of Engineering and Technology
DergiPark AKADEMİK

Journal of Engineering and Tecnology 3;2 (2019) 29-34

Exponential growth of solutions for a parabolic system

[a]Erhan Pişkin, *[b]Fatma Ekinci

[a]Dicle University, Department of Mathematics, 21280 Diyarbakır/TURKEY.

[b]Dicle University, Department of Mathematics, Institute of Natural and Applied Sciences, 21280 Diyarbakır/TURKEY.

ARTICLE INFO

Article history:
Received 17 Oct 2019
Received in revised form 14 Nov 2019

Accepted 14 Nov 2019
Available online 25 Dec 2019

Key words:
Exponential growth,
parabolic equation,
multiple nonlinearities.

* Corresponding author.
E-mail address:
ekincifatma2017@gmail.com

ABSTRACT

In this paper, we investigated the initial boundary problem of a class of doubly nonlinear parabolic systems. We prove exponential growth of solution with negative initial energy.

1. Introduction

In this paper, we study the following parabolic system

$$\begin{cases} u_t - \Delta u + |u|^{q-2}u_t = f_1(u,v), & x \in \Omega,\ t > 0, \\ v_t - \Delta v + |v|^{q-2}v_t = f_2(u,v), & x \in \Omega,\ t > 0, \\ u(x,t) = v(x,t) = 0, & x \in \partial\Omega,\ t \geq 0, \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x) & x \in \Omega, \end{cases} \tag{1}$$

where $q > 2$ are real numbers and Ω is a bounded domain in $R^n\ (n \geq 1)$ with smooth boundary $\partial\Omega$. $f_i(u,v)\ (i = 1,2)$ will be given later.

Pang and Qiao [1] studied the blow up properties of the problem (1) with negative and positive initial energy.

In the absence of $|u|^{q-2}u_t$ and $|v|^{q-2}v_t$ term become the following problem

$$\begin{cases} u_t - \Delta u = f_1(u,v), \\ v_t - \Delta v = f_2(u,v). \end{cases} \tag{2}$$

This type of equation aries from a variety of mathematical models in engineering and physical sciences, it appears naturally in the models of physics, chemistry, biology, ecology and so on (see [2-12]). In [13], the authors got the global existence solution, blow-up in finite time solution, and asymptotic behavior of solution for (2).

Currently, in [14] the author also discussed the problem (2). He got global existence of the solutions , the asymptotic stability of solution and the blow up of solution.

In detail, this paper is organized as follows: In the next section, we present some notations and statement of assumptions. In section 3, the growth of solution is given.

2. Preliminaries

In this section, we shall give some assumptions for the proof of our results. Let $\|.\|$, $\|.\|_p$ and $(u,v)=\int_\Omega u(x)v(x)\,dx$ denote the usual $L^2(\Omega)$ norm, $L^p(\Omega)$ norm and inner product of $L^2(\Omega)$, respectively. Throughout this paper, C is used to point out general positive constants.

For the numbers m and q, we suppose that

$$\begin{cases} 2 < q < m \le \frac{n+2}{n-2} \;\; if\ n > 2, \\ 2 < q < m \le +\infty \;\; if\ n = 1,2. \end{cases} \tag{3}$$

Regarding the functions $f_1(u,v),\ f_2(u,v) \in C^1$ such that

$$f_1(u,v) = \frac{\partial F(u,v)}{\partial u}, f_2(u,v) = \frac{\partial F(u,v)}{\partial v}$$

and

$$\begin{cases} k_0(|u|^m + |v|^m) \le F(u,v) \le k_1(|u|^m + |v|^m), \\ \quad uf_1(u,v) + vf_2(u,v) = (m+1)F(u,v) \end{cases} \tag{4}$$

where k_0, k_1 are positive constants.

Combining arguments of [15,12,16], $u(x,t), v(x,t)$ are called a solution of problem (1) on $\Omega \times [0,T)$ if

$$\begin{cases} u,v \in C(0,T;H_0^1(\Omega)) \cap C^1\big(0,T;L^2(\Omega)\big), \\ \quad |u|^{q-2}u_t,\ \ |v|^{q-2}v_t \in L^2(\Omega \times [0,T)) \end{cases} \tag{5}$$

satisfying the initial condition $u(x,0)=u_0(x), v(x,0)=v_0(x)$ and

$$\int_0^t \int_\Omega [\nabla u(s)\nabla w(s) + u_t(s)w(s) + |u|^{q-2}u_t w - f_1(u,v)w]\,dx\,ds = 0, \tag{6}$$

$$\int_0^t \int_\Omega [\nabla v(s)\nabla w(s) + v_t(s)w(s) + |v|^{q-2}v_t w - f_2(u,v)w]\,dx\,ds = 0 \tag{7}$$

for all $w \in\ C(0,T;H_0^1(\Omega))$.

The energy functional associated with problem (1) is

$$E(t) = \frac{1}{2}\|\nabla u\|^2 \ + \frac{1}{p}\|\nabla v\|^2 \ - \int_\Omega F(u,v)dx, \tag{8}$$

where $u,v \in H_0^1(\Omega)$.

Lemma 2.1 Suppose that (3) and (4) hold. $E'(t)$ is noncreasing function $t > 0$ and

$$E'(t) = -\|u_t\|^2 - \|v_t\|^2 - \int_\Omega |u|^{q-2}{u_t}^2 dx - \int_\Omega |v|^{q-2}{v_t}^2\, dx < 0. \tag{9}$$

Proof. Multiplying $Eq.\,(1)_1$ by u_t and $Eq.\,(1)_2$ by v_t and integrating over Ω, we obtain

$$\int_0^t E'(\tau)d\tau = -\left[\int_0^t (\|u_t\|^2 + \|v_t\|^2)d\tau + \int_0^t \int_\Omega |u|^{q-2}{u_t}^2 dx d\tau + \int_0^t \int_\Omega |v|^{q-2}{v_t}^2 dx d\tau\right],$$

$$E(t) - E(0) = -\left[\int_0^t (\|u_t\|^2 + \|v_t\|^2)d\tau + \int_0^t \int_\Omega |u|^{q-2}{u_t}^2 dx d\tau + \int_0^t \int_\Omega |v|^{q-2}{v_t}^2 dx d\tau\right]$$

for $t > 0$.

3. Exponential Growth of Solution

In this section, we state and prove exponential growth result.

Theorem 3.1 Suppose that (3) holds, $u_0, v_0 \in H_0^1(\Omega)$ and $E(0) < 0$. Then, the solution of the system (1) grows exponentially.

Proof. We set

$$H(t) = -E(t). \tag{10}$$

From (10) and (9), we have

$$H'(t) = -E'(t) \geq 0. \tag{11}$$

Since $E(0) < 0$, we get

$$H(0) = -E(0) > 0. \tag{12}$$

By the integrate (11), we get

$$0 < H(0) \leq H(t). \tag{13}$$

By using (10) and (8)

$$H(t) - \int_\Omega F(u,v)dx = -\frac{1}{2}\left(\|\nabla u\|^2 + \|\nabla v\|^2\right) < 0. \tag{14}$$

Then, by using (4), we have

$$0 < H(0) \leq H(t) \leq \int_\Omega F(u,v)dx \leq k_1(\|u\|_m^m + \|v\|_m^m). \tag{15}$$

We define the functional

$$\Phi(t) = H(t) + \frac{\varepsilon}{2}\|u\|^2 + \frac{\varepsilon}{2}\|v\|^2. \tag{16}$$

By differentiating (16) and using Eq.(1), we get

$$\begin{aligned}
\Phi'(t) &= H'(t) + \varepsilon\left(\int_\Omega u u_t dx + \int_\Omega v v_t dx\right) \\
&= \|u_t\|^2 + \|v_t\|^2 - \varepsilon\|\nabla u\|^2 - \varepsilon\|\nabla v\|^2 + \varepsilon\int_\Omega [u f_1(u,v) + v f_2(u,v)]dx \\
&\quad + \int_\Omega |u|^{q-2}{u_t}^2 dx + \int_\Omega |v|^{q-2}{v_t}^2 dx - \varepsilon\int_\Omega |u|^{q-2} u u_t dx - \varepsilon\int_\Omega |v|^{q-2} v v_t dx \\
&= \|u_t\|^2 + \|v_t\|^2 - \varepsilon\|\nabla u\|^2 - \varepsilon\|\nabla v\|^2 + \varepsilon(m+1)\int_\Omega F(u,v)dx \\
&\quad + \int_\Omega |u|^{q-2}{u_t}^2 dx + \int_\Omega |v|^{q-2}{v_t}^2 dx - \varepsilon\int_\Omega |u|^{q-2} u u_t dx \\
&\quad - \varepsilon\int_\Omega |v|^{q-2} v v_t dx.
\end{aligned} \tag{17}$$

In order to estimate the last two terms in the right-hand side of (17), we use the following Young's inequality,

$$ab \le \delta^{-1}a^2 + \delta b^2,$$

so we have

$$\int_\Omega |u|^{q-2}uu_t dx \le \int_\Omega |u|^{\frac{q-2}{2}}u_t|u|^{\frac{q-2}{2}}dx$$
$$\le \delta^{-1}\int_\Omega |u|^{q-2}{u_t}^2 dx + \delta\int_\Omega |u|^q\, dx.$$

Similarly,

$$\int_\Omega |v|^{q-2}vv_t dx \le \delta^{-1}\int_\Omega |v|^{q-2}{v_t}^2 dx + \delta\int_\Omega |v|^q\, dx.$$

Then, (17) becomes

$$\Phi'(t) \ge \|u_t\|^2 + \|v_t\|^2 - \varepsilon\|\nabla u\|^2 - \varepsilon\|\nabla v\|^2 + \varepsilon(m+1)(\|u\|_m^m + \|v\|_m^m)$$
$$-\varepsilon\delta\left(\|u\|_q^q + \|v\|_q^q\right) + (1-\varepsilon\delta^{-1})\int_\Omega |u|^{q-2}{u_t}^2 dx$$
$$+(1-\varepsilon\delta^{-1})\int_\Omega |v|^{q-2}{v_t}^2 dx. \tag{18}$$

By using follows equality that

$$-\|\nabla u\|^2 - \|\nabla v\|^2 = 2H(t) - 2\int_\Omega F(u,v)dx.$$

Hence, (18) becomes

$$\Phi'(t) \ge 2\varepsilon H(t) + \|u_t\|^2 + \|v_t\|^2 + \varepsilon(m-1)(\|u\|_m^m + \|v\|_m^m)$$
$$-\varepsilon\delta\left(\|u\|_q^q + \|v\|_q^q\right) + (1-\varepsilon\delta^{-1})\int_\Omega |u|^{q-2}{u_t}^2 dx$$
$$+(1-\varepsilon\delta^{-1})\int_\Omega |v|^{q-2}{v_t}^2 dx. \tag{19}$$

As the embedding $L^m \hookrightarrow L^q \hookrightarrow L^2, m > q > 2$, we have

$$\begin{cases} \|u\|_q^q \le C\|u\|_m^q \le C(\|u\|_m^m)^{\frac{q}{m}}, \\ \|v\|_q^q \le C\|v\|_m^q \le C(\|v\|_m^m)^{\frac{q}{m}}. \end{cases} \tag{20}$$

Since $0 < \frac{q}{m} < 1$, now applying the following inequality

$$x^l \le (x+1) \le \left(1+\tfrac{1}{z}\right)(x+z),$$

which holds for all $x \ge 0,\ 0 \le l \le 1,\ z > 0$, especially, taking $x = \|u\|_m^m,\ l = \frac{q}{m}$, $z = H(0)$, we get

$$C(\|u\|_m^m)^{\frac{q}{m}} \le \left(1+\frac{1}{H(0)}\right)\left(\|u\|_m^m + H(0)\right),$$

similarly

$$C(\|v\|_m^m)^{\frac{q}{m}} \le \left(1+\tfrac{1}{H(0)}\right)\left(\|v\|_m^m + H(0)\right).$$

Then, from (15) and (20), we get

$$\|u\|_q^q + \|v\|_q^q \le C\left(\|u\|_m^q + \|v\|_m^q\right)$$
$$\le C_1(\|u\|_m^m + \|u\|_m^m). \tag{21}$$

Then, from (21) we obtain

$$\Phi'(t) \ge 2\varepsilon H(t) + \|u_t\|^2 + \|v_t\|^2 + \varepsilon a_1(\|u\|_m^m + \|v\|_m^m)$$
$$+(1-\varepsilon\delta^{-1})\int_\Omega |u|^{q-2}{u_t}^2 dx + (1-\varepsilon\delta^{-1})\int_\Omega |v|^{q-2}{v_t}^2 dx,$$

where δ small enough such that $a_1 = m - 1 - \delta C_1 > 0$ and taking ε and δ small enough such that $1 - \varepsilon\delta^{-1} > 0$, then

$$\Phi'(t) \geq C(H(t) + \|u_t\|^2 + \|v_t\|^2 + \|u\|_m^m + \|v\|_m^m). \quad (22)$$

On the other hand, by definition of $\Phi(t)$ and Poincare's inequality, we get

$$\Phi(t) = H(t) + \frac{\varepsilon}{2}\|u\|^2 + \frac{\varepsilon}{2}\|v\|^2$$
$$\leq C(H(t) + \|\nabla u\|^2 + \|\nabla v\|^2).$$

From definition of $H(t)$, we get

$$\Phi(t) \leq C(H(t) + \|u\|_m^m + \|v\|_m^m) \quad (23)$$
$$\leq C(H(t) + \|u\|_m^m + \|v\|_m^m + \|u_t\|^2 + \|v_t\|^2). \quad (24)$$

From (22) and (24), we arrive at

$$\Phi'(t) \geq r\Phi(t), \quad (25)$$

where r is a positive constant.

Integration of (25) over $(0, t)$ gives us

$$\Phi(t) \geq \Phi(0)\exp(rt).$$

From (23) and (15), we get

$$\Phi(t) \leq H(t) \leq \|u\|_m^m + \|v\|_m^m.$$

Consequently, we show that the solution in the L_m-norm growths exponentially.

4. References

[1] Pang J. and Qiao B. Blow-up of solution for initial boundary value problem of reaction diffusion equations. Journal of Advances in Mathematics 2015;10(1):3138-3144.

[2] Bebernes J. and Eberly D. Mathematical Problems from Combustion Theory. Applied Mathematical Science, Springer-Verlag: Berlin; 1989.

[3] Pao C. V. Nonlinear Parabolic and Elliptic Equations. Plenum, New York; 1992.

[4] Escerh J. and Yin Z. Stable equilibra to parabolic systems in unbounded domains. Journal of Nonlinear Mathematical Physics 2004;11(2):243-255.

[5] Zhou H. Blow-up rates for semilinear reaction-diffusion systems. Journal of Differential Equations 2014;257:843-867.

[6] Escher J. and Yin Z. On the stability of equilibria to weakly coupled parabolic systems in unbounded domains. Nonlinear Analysis 2005;60:1065-1084.

[7] Escobedo M. and Herrero MA. A semilinear reaction diffusion system in a bounded domain. Annali di Matematica Pura ed Applicata 1993;165:315-336.

[8] Escobedo M. and Levine HA. Critical blowup and global existence numbers for a weakly coupled system of reaction-diffusion equations. Archive for Rational Mechanics and Analysis 1995;129:47-100.

[9] Alaa N. Global existence for reaction-diffusion systems with mass control and critical growth with respect to the gradient. Journal of Mathematical Analysis and Applications 2001;253:532-557.

[10] Wang RN. and Tang ZW. Global existence and asymptotic stability of equilibria to reaction-diffusion systems. Journal of Physics A: Mathematical and Theoretical 2009;42, Article ID 235205.

[11] Yadav OP. and Jiwari R. A finite element approach for analysis and computational modelling of coupled reaction diffusion models. Numerical Methods for Partial Differential Equations, 2018;1-21.

[12] Chen HW. Global existence and blow-up for a nonlinear reaction-diffusion system. Journal of Mathematical Analysis and Applications 1997;212:481-492.

[13] Niu Y. Global existence and nonexistence of generalized coupled reaction-diffusion systems. Mathematical Methods in the Applied Sciences 2019; Early view, 1-31.

[14] Ferhat M. Well posedness and asymptotic behavior for coupled quasilinear parabolic system with source term. Electronic Journal of Mathematical Analysis and Applications 2019;7(1):266-282.

[15] Korpusov MO. and Sveshnikov AG. Sufficent close-to-necessary conditions for the blowup of solutions to a strongly nonlinear generalized Boussinesq equation. Computational Mathematics and Mathematical Physics 2008;48(9):1591-1599.

[16] Junning Z. Existence and nonexistence of solutions for $u_t = div(|\nabla u|^{p-2}\nabla u) + f(\nabla u, u, x, t)$. *Journal of Mathematical Analysis and Applications* 1993;172:130--146.