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On Non-Newtonian Power Series and Its Applications

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Abstract

The purpose of this study is to examine *-Abel and *-Dirichlet tests, *-power series, $\beta-$summability in the Cesàro and Abel sense. *-Calculus can be used instead of non-Newtonian calculus.

Keywords: **-Abel and *-Dirichlet tests; *-Power Series; $\beta-$Summability In The Cesàro and Abel Sense.*
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1. Introduction and Preliminaries

Non-Newtonian calculus was firstly introduced and worked by Michael Grossman and Robert Katz between the years 1967 and 1970 [10]. Grossman worked some properties of derivatives and integrals in non-Newtonian calculus [12]. Kadak studied on some sequence spaces in the non-Newtonian complex field [13]. Duyar et al. [6] obtained some basic topologic properties on non-Newtonian real line. Later, Duyar and Erdogan examined the non-Newtonian real series and their convergence conditions [7]. Duyar and Oğur studied some topologic properties on non-Newtonian real numbers [8]. In [5], Çakmak and Başar have introduced the classical sequence spaces with respect to non-Newtonian calculus over the non-Newtonian real field. Later, Tekin and Başar have studied the spaces ω^{*}, l_{∞}^{*}, c^{*}, c_{0}^{*} and l_{p}^{*} over the non-Newtonian complex field $\mathbb{C}^{*}$ and obtained the corresponding results for these spaces in [15], where $\ddot{p}\ddot{\geq}\ddot{1}$. As a continuation of Çakmak and Başar [5], Türkmen and Başar [16] and Tekin and Başar [15]; Çakır has studied the space $\mathbb{C}_{*}(\Omega)$ of *-continuous functions and state that it forms a vector space with respect to the non-Newtonian addition and scalar multiplication, and proved that $\mathbb{C}_{*}(\Omega)$ is a Banach space with some characteristic features of complex numbers and functions in terms of non-Newtonian calculus together with the basic properties of *-boundedness and *-continuity, in [4]. Erdogan and Duyar examined the non-Newtonian improper integrals [9]. Ünlüyol and Salaş studied convexity respect to the non-Newtonian calculus [18]. In this article, Dirichlet's and Abel's Tests are obtained in the sense of non-Newtonian (*-Dirichlet's and *-Abel's Tests), *-power series were introduced, $\beta-$summability and its properties were obtained in the sense of Cesàro and Abel being an application of this approach.
It can be easily observed for the basic principles of Multiplicative Calculus (in short MC) is given in [11]. Plenty of advance studies in (MC) could be found in several articles such as [1, 2, 3, 13, 16] and [17].
A *generator* is defined as an injective function with domain $\mathbb{R}$ and the range of generator is a subset of $\mathbb{R}$. Let us take any generator α with range $A=\mathbb{R}(N)_{\alpha}$ and define $\alpha-$addition, $\alpha-$subtraction, $\alpha-$multiplication, $\alpha-$division and $\alpha-$order as follows;

$$\begin{array}{ll}
\alpha-\text{addition} & x\dot{+}y=\alpha\left(\alpha^{-1}(x)+\alpha^{-1}(y)\right)\\
\alpha-\text{subtraction} & x\dot{-}y=\alpha\left(\alpha^{-1}(x)-\alpha^{-1}(y)\right)\\
\alpha-\text{multiplication} & x\dot{\times}y=\alpha\left(\alpha^{-1}(x)\times\alpha^{-1}(y)\right)\\
\alpha-\text{division} & x\dot{/}y=\alpha\left(\alpha^{-1}(x)/\alpha^{-1}(y)\right)\\
\alpha-\text{order} & x\dot{<}y\ \left(x\dot{\leq}y\right)\Leftrightarrow\alpha^{-1}(x)<\alpha^{-1}(y)\ \left(\alpha^{-1}(x)\leq\alpha^{-1}(y)\right)
\end{array}$$

for $x,y\in\mathbb{R}(N)_{\alpha}$ [10].
$(\mathbb{R}(N)_{\alpha},\dot{+},\dot{\times},\dot{\leq})$ is totally ordered field.
The numbers $x\dot{>}\dot{0}$ are $\alpha-$positive numbers and the numbers $x\dot{<}\dot{0}$ are $\alpha-$negative numbers in $\mathbb{R}(N)_{\alpha}$. $\alpha-$integers are obtained by successive $\alpha-$addition of $\dot{1}$ to $\dot{0}$ and successive $\alpha-$subtraction of $\dot{1}$ from $\dot{0}$. Hence $\alpha-$integers are as follows:

$$\ldots,\alpha(-2),\alpha(-1),\alpha(0),\alpha(1),\alpha(2),\ldots.$$

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For each integer n, we set $\dot{n} = \alpha(n)$. If $\dot{n}$ is an $\alpha-$positive integer, then it is n times sum of $\dot{1}$ [7, 10].
$\alpha-$absolute value of a number $x \in \mathbb{R}(N)_\alpha$ is defined by

$$|x|_\alpha = \alpha\left(\left|\alpha^{-1}(x)\right|\right) = \begin{cases} x & if \quad x\dot{>}\dot{0} \\ \dot{0} & if \quad x=\dot{0} \\ \dot{0}\dot{-}x & if \quad x\dot{<}\dot{0} \end{cases}.$$

For $x \in \mathbb{R}(N)_\alpha$, $\sqrt[p]{x}^\alpha = \alpha\left(\sqrt[p]{\alpha^{-1}(x)}\right)$ and $x^{p_\alpha} = \alpha\left\{\left[\alpha^{-1}(x)\right]^p\right\}$ [10].
Grossman and Katz described the *-calculus with the help of two arbitrary selected generators. In this study, we employ *-calculus. Let α and β be any generators, and let * ("star") denotes the ordered pair of arithmetics ($\alpha-$arithmetic, $\beta-$arithmetic). The following notations will be used [10].

	$\alpha-$arithmetic	$\beta-$arithmetic
Realm	$A\,(=\mathbb{R}(N)_\alpha)$	$B\,(=\mathbb{R}(N)_\beta)$
Addition	$\dot{+}$	$\ddot{+}$
Subtraction	$\dot{-}$	$\ddot{-}$
Multiplication	$\dot{\times}$	$\ddot{\times}$
Division	$\dot{/}\ (or\ \text{-}\alpha)$	$\ddot{/}\ (or\ \text{-}\beta)$
Ordering	$\dot{<}$	$\ddot{<}$

In the $*-$calculus, $a-$arithmetic is used for arguments and $\beta-$arithmetic is used for values.
The isomorphism from $a-$arithmetic to $\beta-$arithmetic is the unique function ι (iota) that possesses the following three properties.
1. ι is one-to-one.
2. ι is on A and onto B.
3. For any numbers u and v in A,

$$\begin{aligned} \iota(u\dot{+}v) &= \iota(u)\ddot{+}\iota(v), \\ \iota(u\dot{-}v) &= \iota(u)\ddot{-}\iota(v), \\ \iota(u\dot{\times}v) &= \iota(u)\ddot{\times}\iota(v), \\ \iota\left(u\dot{/}v\right) &= \iota(u)\ddot{/}\iota(v), v\neq\dot{0}\ \left(or\ \frac{\iota(u)}{\iota(v)}\beta\right) \end{aligned}$$

$$u\dot{<}v \Longleftrightarrow \iota(u)\ddot{<}\iota(v).$$

It turns out that $\iota(x) = \beta\left\{\alpha^{-1}(x)\right\}$ for every number x in A, and that $\iota(\dot{n}) = \ddot{n}$ for every integer n [10].
Let $X \subset \mathbb{R}(N)_\alpha$, $a \in X'^\alpha$, $b \in \mathbb{R}(N)_\beta$ and let $f : X \to \mathbb{R}(N)_\beta$ be a function. If for every $\varepsilon\ddot{>}\ddot{0}$ there exists a number $\delta = \delta(\varepsilon)\dot{>}\dot{0}$ such that $|f(x)\ddot{-}b|_\beta\ddot{<}\varepsilon$ for all $x \in X$ which holds condition $\dot{0}\dot{<}|x\dot{-}a|_\alpha\dot{<}\delta$, then it is said that the *-limit of the function f (in the sense of Cauchy) at the point a is b and this is denoted by

$$*-\lim_{x\to a} f(x) = b.$$

If a sequence $(f(x_n))$ is $\beta-$convergent to the number b for all sequences $(x_n) \subset X-\{a\}$ which are $\alpha-$convergent to a point a, then it is said that the *-limit of the function f (*-sequential limit of the function f) at the point a is b and this is denoted by

$$*-\lim_{x\to a} f(x) = b.$$

The equivalence of *-sequential limit and *-limit in the sense of Cauchy of a function at a point is given in [14].
If the following *-limit exists, we denote it by $\left[\overset{*}{D}f\right](a)$ and call it the *-derivative of f at a, and say that f is *-differentiable at a:

$$*-\lim_{x\to a}\left\{[f(x)\ddot{-}f(a)]\,\ddot{/}\,[\iota(x)\ddot{-}\iota(a)]\right\}.$$

If it exists, $\left[\overset{*}{D}f\right](a)$ is necessarily in B [10].

The derivative of f, denoted by $\overset{*}{D}f$, is the function that assigns each number t in A to the number $\left[\overset{*}{D}f\right](t)$, if it exists.

The *-average of a *-continuous function f on $[r,s]$ is denoted by $\overset{*}{M}{}_r^{*s} f$ and defined to be $\beta-$limit of the $\beta-$convergent sequence whose n th term is $\beta-$average of $f(a_1),...,f(a_n)$, where $a_1,...,a_n$ is the $n-$fold $\alpha-$partition of $[r,s]$.

The *-integral of a *-continuous function f on $[r,s]$, denoted by ${}^*\int_r^s f(x)\,d^*x$, is the number $[\iota(s)\ddot{-}\iota(r)]\,\ddot{\times}\,\overset{*}{M}{}_r^{*s} f$ in B [10].

Let S be a nonempty subset of $\mathbb{R}(N)_\alpha$ and let $k \in \mathbb{N}$. The sequence $(f_k) = (f_1, f_2, ..., f_k, ...)$ is called sequence (f_k) of *-functions (or non-Newtonian function sequence) for functions $f_k : S \subseteq \mathbb{R}(N)_\alpha \to \mathbb{R}(N)_\beta$. Let a sequence (f_k) of *-functions with $f_k : S \subseteq \mathbb{R}(N)_\alpha \to \mathbb{R}(N)_\beta$ be given. If the sequence $(f_k(x_0))$ is β-convergent for $x_0 \in S$, then the sequence (f_k) of *-functions is called *-convergent (or non-Newtonian convergent). The sequence (f_k) of *-functions is said *-pointwise convergent or *-convergent to a function f, if the sequence $(f_k(x))$ is

β-convergent for each $x \in S$ and ${}^{\beta}\lim\limits_{k\to\infty} f_k(x) = f(x)$. In this case, the function f is called *-limit of the sequence (f_k) of *-functions and it is written as follows;

$$*-\lim_{k\to\infty} f_k = f\ (*-pointwise) \text{ or } f_k \xrightarrow{*} f\ (*-pointwise).$$

Let take the sequence (f_k) of *-functions, where $f_k : S \subseteq \mathbb{R}(N)_\alpha \to \mathbb{R}(N)_\beta$. The sequence (f_k) of *-functions *-uniform convergent to the function f on set S, if for any given $\varepsilon \ddot{>} \ddot{0}$, there exists at least a natural number k_0 depends on number ε but not depend on variable x such that $|f_k(x) \ddot{-} f(x)|_\beta \ddot{<} \varepsilon$ for all $k > k_0$ and each $x \in S$. It is denoted by $*-\lim\limits_{k\to\infty} f_k = f\ (*-uniform)$ or $f_k \xrightarrow{*} f\ (*-uniform)$.
Let take sequence (f_k) of *-functions with $f_k : A \subseteq \mathbb{R}(N)_\alpha \to \mathbb{R}(N)_\beta$. The infinite $\beta-$sum

$$_\beta\sum_{k=1}^{\infty} f_k = f_1 \ddot{+} f_2 \ddot{+} \ldots \ddot{+} f_k \ddot{+} \ldots$$

is called series of *-functions (or non-Newtonian function series). The $\beta-$sum $S_k = {}_\beta\sum_{k=1}^{n} f_k$ is called n th partial $\beta-$sum of the series ${}_\beta\sum_{k=1}^{\infty} f_k$ for $n \in \mathbb{N}$. Let the series of *-functions ${}_\beta\sum_{k=1}^{\infty} f_k$ with $f_k : A \subseteq \mathbb{R}(N)_\alpha \to \mathbb{R}(N)_\beta$ and the function $f : A \subseteq \mathbb{R}(N)_\alpha \to \mathbb{R}(N)_\beta$ be given. If the $\beta-$partial sums sequence (S_n), where $S_n = {}_\beta\sum_{k=1}^{n} f_k$ is *-pointwise converges to the function f, then series of *-functions ${}_\beta\sum_{k=1}^{n} f_k$ *-pointwise converges to the function f on the set A and is written as

$$_\beta\sum_{k=1}^{\infty} f_k = f(*-pointwise).$$

In this situation, the function f is called $\beta-$sum (or non-Newtonian sum) of *-series ${}_\beta\sum_{k=1}^{\infty} f_k$ [14].

Definition 1.1. *If $S_k \xrightarrow{*} f\ (*-uniform)$, then the series of *-functions ${}_\beta\sum_{k=1}^{\infty} f_k$ is called *-uniform convergent to the function f on the set A and is written as ${}_\beta\sum_{k=1}^{\infty} f_k = f\ (*-uniform)$ [14].*

2. Results and Discussion

2.1. *-Dirichlet's and *-Abel's Tests

The sequence of *-functions and series of *-functions are examined in [14]. As an application of these *-Abel and *-Dirichlet tests will be given. There are some tests to determine *-uniform convergence in the case of *-Weierstrass M-criterion is not sufficient to determine *-uniform convergence. These statements in *-calculus of two theorems which known as Abel's and Dirichlet's tests in classic calculus are given below. These tests are useful for applications and especially *-power series.
Now, the formula which is named "$\alpha-$Abel's partial sum" is obtained.

Proposition 2.1. *If $\{a_1, a_2, ..., a_n, ...\}$ and $\{b_1, b_2, ..., b_n, ...\}$ are two non-Newtonian real number sequences ($\alpha-$real number sequences) and if $s_n = a_1 \dot{+} a_2 \dot{+} ... \dot{+} a_n$, then the equalities*

$$\begin{aligned} {}_\alpha\sum_{k=1}^{n} (a_k \dot{\times} b_k) &= s_n \dot{\times} b_{n+1} \dot{-} {}_\alpha\sum_{k=1}^{n} [s_k \dot{\times} (b_{k+1} \dot{-} b_k)] && (2.1)\\ &= s_n \dot{\times} b_1 \dot{-} {}_\alpha\sum_{k=1}^{n} [(s_n \dot{-} s_k) \dot{\times} (b_{k+1} \dot{-} b_k)] && (2.2) \end{aligned}$$

hold.

Proof. Since $a_n = s_n \dot{-} s_{n-1}$ with $s_0 = \dot{0}$, we have

$$_\alpha\sum_{k=1}^{n} (a_k \dot{\times} b_k) = {}_\alpha\sum_{k=1}^{n} [(s_k \dot{-} s_{k-1}) \dot{\times} b_k] = {}_\alpha\sum_{k=1}^{n} (s_k \dot{\times} b_k) \dot{-} {}_\alpha\sum_{k=1}^{n} (s_{k-1} \dot{\times} b_k). \quad (2.3)$$

On the other hand, since

$$_\alpha\sum_{k=1}^{n} (s_{k-1} \dot{\times} b_k) = {}_\alpha\sum_{k=1}^{n} (s_k \dot{\times} b_{k+1}) \dot{-} (s_n \dot{\times} b_{n+1}),$$

if we consider this sum in the equality (2.3), then

$$_\alpha\sum_{k=1}^{n} (a_k \dot{\times} b_k) = (s_n \dot{\times} b_{n+1}) \dot{-} {}_\alpha\sum_{k=1}^{n} [s_k \dot{\times} (b_{k+1} \dot{-} b_k)]$$

is obtained. Again, if we write

$$b_{n+1} = {}_\alpha\sum_{k=1}^{n} (b_{k+1} \dot{-} b_k) \dot{+} b_1$$

instead of b_{n+1} in the equality (2.1), then we obtain the equality (2.2). □

Definition 2.2. *If the sequence (φ_n) of *-functions with $\varphi_n : A \to \mathbb{R}(N)_\beta$, where $A \subset \mathbb{R}(N)_\alpha$, satisfies the condition $\varphi_n(x)\ddot{\leq}\varphi_{n+1}(x)$ $(\varphi_{n+1}(x)\ddot{\leq}\varphi_n(x))$ for all $n \in \mathbb{N}$ and all $x \in A$, then the sequence (φ_n) is called *-monotone increasing (*-monotone decreasing). If the sequence (φ_n) of *-functions is *-monotone increasing or *-monotone decreasing, then the sequence (φ_n) is called *-monotone.*

Theorem 2.3. *Let sequence (φ_n) of *-functions with $\varphi_n : A \to \mathbb{R}(N)_\beta$ where $A \subset \mathbb{R}(N)_\alpha$ and let (φ_n) possess the following properties.*
*1. (φ_n) is *-monotone decreasing and*
2. There exists a number $M\ddot{\geq}\ddot{0}$ such that $|\varphi_n(x)|_\beta \ddot{\leq} M$ for all $x \in A$ and all $n \in \mathbb{N}$.

*In this case, if the *-series ${}_\beta\sum_{n=1}^{\infty} f_n$ is *-uniform convergent on the set A, then the *-series ${}_\beta\sum_{n=1}^{\infty} (f_n \ddot{\times} \varphi_n)$ is also *-uniform convergent on the set A.*

Proof. Let $s_n(x) = {}_\beta\sum_{k=1}^{n} f_k(x)$ and $h_n(x) = {}_\beta\sum_{k=1}^{n} (\varphi_k(x)\ddot{\times} f_k(x))$. By virtue of the equality (2.2) in Proposition 2.1,

$$h_n(x)\ddot{-}h_m(x) = [s_n(x)\ddot{-}s_m(x)]\ddot{\times}\varphi_1(x)\dot{+}\,{}_\beta\sum_{k=m+1}^{n}[s_n(x)\ddot{-}s_k(x)]\ddot{\times}[\varphi_{k+1}(x)\ddot{-}\varphi_k(x)]$$

for $n > m$. From here,

$$|h_n(x)\ddot{-}h_m(x)|_\beta \ddot{\leq} |s_n(x)\ddot{-}s_m(x)|_\beta \ddot{\times} |\varphi_1(x)|_\beta \dot{+}\,{}_\beta\sum_{k=m+1}^{n} |s_n(x)\ddot{-}s_k(x)|_\beta \ddot{\times} |\varphi_{k+1}(x)\ddot{-}\varphi_k(x)|_\beta$$

is written. $|\varphi_{k+1}(x)\ddot{-}\varphi_k(x)|_\beta = \varphi_k(x)\ddot{-}\varphi_{k+1}(x)$ since $\varphi_{k+1}\ddot{\leq}\varphi_k$. Since ${}_\beta\sum_{n=1}^{\infty} f_n$ is *-uniform convergent, there exists a natural number N corresponds to an arbitrary number $\varepsilon\ddot{>}\ddot{0}$ such that $|s_n(x)\ddot{-}s_m(x)|_\beta \ddot{<} \frac{\varepsilon}{\ddot{3}\ddot{\times}M}\beta$ for all $n,m > N$. Thus,

$$\begin{aligned}
|h_n(x)\ddot{-}h_m(x)|_\beta &\ddot{<} \frac{\varepsilon}{\ddot{3}}\beta\dot{+}\left(\frac{\varepsilon}{\ddot{3}\ddot{\times}M}\beta\right)\ddot{\times}\,{}_\beta\sum_{k=m+1}^{n}[\varphi_k(x)\ddot{-}\varphi_{k+1}(x)] \\
&= \frac{\varepsilon}{\ddot{3}}\beta\dot{+}\left(\frac{\varepsilon}{\ddot{3}\ddot{\times}M}\beta\right)\ddot{\times}[\varphi_{m+1}(x)\ddot{-}\varphi_{n+1}(x)] \\
&\ddot{\leq} \frac{\varepsilon}{\ddot{3}}\beta\dot{+}\left(\frac{\varepsilon}{\ddot{3}\ddot{\times}M}\beta\right)\ddot{\times}\left[|\varphi_{m+1}(x)|_\beta\dot{+}|\varphi_{n+1}(x)|_\beta\right] \\
&\ddot{\leq} \frac{\varepsilon}{\ddot{3}}\beta\dot{+}\frac{\varepsilon}{\ddot{3}}\beta\dot{+}\frac{\varepsilon}{\ddot{3}}\beta = \varepsilon
\end{aligned}$$

is found for all $x \in A$. Then, in accordance with *-Cauchy criterion [14], ${}_\beta\sum_{n=1}^{\infty}(f_n(x)\ddot{\times}\varphi_n(x))$ is *-uniform convergent. □

Remark 2.4. *Particularly, by choosing the functions φ_n and f_n as constant, useful tests are also obtained for non-Newtonian real number series.*

Remark 2.5. *In the case of the sequence (φ_n) of *-functions is *-monotone increasing, similar test can be obtained. For this, if $(\ddot{0}\ddot{-}\varphi_n)$ is taken instead of φ_n above, then the proof is completed.*

Theorem 2.6. *(*-Dirichlet's Test) Let $s_n(x) = {}_\beta\sum_{m=1}^{n} f_m(x)$ for a sequence of *-functions $f_n : A \subset \mathbb{R}(N)_\alpha \to \mathbb{R}(N)_\beta$. Suppose that there is a constant M such that $|s_n(x)|_\beta \ddot{\leq} M$ for all $x \in A$ and all $n \in \mathbb{N}$. If a sequence (g_n) of *-functions satisfies the conditions $g_n : A \subset \mathbb{R}(N)_\alpha \to \mathbb{R}(N)_\beta$, $g_n \xrightarrow{*} \ddot{0}(*-uniform)$, $g_n\ddot{\geq}\ddot{0}$ and $g_{n+1}(x)\ddot{\leq}g_n(x)$, then the *-series ${}_\beta\sum_{n=1}^{\infty}(f_n\ddot{\times}g_n)$ is *-uniform convergent on the set A.*

Proof. In the light of Theorem 2.3, by taking $\varphi_n = g_n$ if the equality (2.1) is considered in Proposition 2.1 for $n > m$, then

$$h_n(x)\ddot{-}h_m(x) = s_n(x)\ddot{\times}\varphi_{n+1}(x)\ddot{-}s_m(x)\ddot{\times}\varphi_{m+1}(x)\ddot{-}\,{}_\beta\sum_{k=m+1}^{n} s_k(x)\ddot{\times}(\varphi_{k+1}(x)\ddot{-}\varphi_k(x))$$

is written. Since $\varphi_k\ddot{\geq}\ddot{0}$ and $\varphi_{k+1}\ddot{\leq}\varphi_k$, we have

$$\begin{aligned}
|h_n(x)\ddot{-}h_m(x)|_\beta &\ddot{\leq} M\ddot{\times}(\varphi_{n+1}(x)\dot{+}\varphi_{m+1}(x))\dot{+}M\ddot{\times}\,{}_\beta\sum_{k=m+1}^{n}(\varphi_k(x)\ddot{-}\varphi_{k+1}(x)) \\
&= M\ddot{\times}(\varphi_{n+1}(x)\dot{+}\varphi_{m+1}(x)\dot{+}\varphi_{m+1}(x)\ddot{-}\varphi_{n+1}(x)) \\
&= \ddot{2}\ddot{\times}M\ddot{\times}\varphi_{m+1}(x).
\end{aligned}$$

Since $\varphi_n = g_n$ is *-uniform convergent to the number $\ddot{0}$, a natural number N which corresponds an arbitrary $\varepsilon\ddot{>}\ddot{0}$ can be chosen such that the inequality $\varphi_m(x)\ddot{<}\frac{\varepsilon}{\ddot{2}\ddot{\times}M}\beta$ holds for all $x \in A$ when $m > N$. Therefore, $|h_n(x)\ddot{-}h_m(x)|_\beta \ddot{<} \varepsilon$ is obtained for all $x \in A$ since $n,m > N$. This completes the proof. □

Remark 2.7. *Theorems 2.3 and 2.6 are similar to each other, but they are not same. The conditions which are wanted for (φ_n) in Theorem 2.3 does not require that the sequence of *-functions is *-uniform convergent. Furthermore, the condition $\varphi_n\ddot{\geq}\ddot{0}$ is not wanted for the sequence (φ_n) of *-functions in Theorem 2.3.*

For example, consider the *-alternating series ${}_{\beta}\sum\limits_{n=0}^{\infty}\left(\ddot{0}\ddot{-}\ddot{1}\right)^{n_\beta}\ddot{\times}g_n(x)$. Let $g_n(x)\ddot{\geq}\ddot{0}$, $g_n\overset{*}{\longrightarrow}\ddot{0}\ (*-uniform)$ and $g_{n+1}\ddot{\leq}g_n$ here. If $f_n(x)=\left(\ddot{0}\ddot{-}\ddot{1}\right)^{n_\beta}$ is taken, by virtue of the *-Dirichlet's test, the series ${}_{\beta}\sum\limits_{n=0}^{\infty}\left(\ddot{0}\ddot{-}\ddot{1}\right)^{n_\beta}\ddot{\times}g_n(x)$ is *-uniform convergent. According to this, *-alternating series is *-convergent whose general term $\beta-$absolutely converges to the point $\ddot{0}$.

Example 2.8. *The *-series* ${}_{\beta}\sum\limits_{n=1}^{\infty}\frac{\left(\ddot{0}\ddot{-}\ddot{1}\right)^{n_\beta}}{\ddot{n}}\beta\,\ddot{\times}\,\ddot{e}^{\left(-n.\alpha^{-1}(x)\right)_\beta}$ *is *-uniform convergent on* $\dot{[}\dot{0},\dot{+}\infty\dot{]}$*:*

By virtue of Theorem 2.6, if $\varphi_n(x)=\ddot{e}^{\left(-n.\alpha^{-1}(x)\right)_\beta}$ is taken, then $\varphi_{n+1}(x)\ddot{\leq}\varphi_n(x)$ and $|\varphi_n(x)|_\beta=\left|\ddot{e}^{\left(-n.\alpha^{-1}(x)\right)_\beta}\right|_\beta\ddot{\leq}\ddot{1}$ for $x\dot{>}\dot{0}$. Otherwise, since we know that the *-series ${}_{\beta}\sum\limits_{n=1}^{\infty}\left(\ddot{0}\ddot{-}\ddot{1}\right)^{n_\beta}\ddot{\times}\frac{\ddot{1}}{\ddot{n}}\beta$ is *-convergent, in view of *-Dirichlet's test, given series is *-uniform convergent.

Example 2.9. *The function f defined by* $f(x)={}_{\beta}\sum\limits_{n=1}^{\infty}\frac{\left(\ddot{0}\ddot{-}\ddot{1}\right)^{n_\beta}}{\ddot{n}}\beta\,\ddot{\times}\,\ddot{e}^{\left(-n.\alpha^{-1}(x)\right)_\beta}$ *is *- continuous:*

The functions $f_n(x)=\frac{\left(\ddot{0}\ddot{-}\ddot{1}\right)^{n_\beta}}{\ddot{n}}\beta\,\ddot{\times}\,\ddot{e}^{\left(-n.\alpha^{-1}(x)\right)_\beta}$ for $n=1,2,3,\ldots$ are *-continuous. By Example 2.8, the *-series ${}_{\beta}\sum\limits_{n=1}^{\infty}f_n$ is *-uniform convergent. By virtue of Corollary 3 in [14], *-limit function f is *-continuous.

2.2. *-Power Series

Definition 2.10. *The general statement of *-power series of* $\iota(x\dot{-}x_0)=\iota(x)\ddot{-}\iota(x_0)$ *is formed*

$$
{}_{\beta}\sum_{k=0}^{\infty}a_k\ddot{\times}\left(\iota(x)\ddot{-}\iota(x_0)\right)^{k_\beta}=a_0\ddot{+}\left[a_1\ddot{\times}\left(\iota(x)\ddot{-}\iota(x_0)\right)\right]\ddot{+}\left[a_2\ddot{\times}\left(\iota(x)\ddot{-}\iota(x_0)\right)^{2_\beta}\right]\ddot{+}\ldots \tag{2.4}
$$

where $a_0,a_1,\ldots,a_k,\ldots\in\mathbb{R}(N)_\beta$ *are constants and* $x,x_0\in\mathbb{R}(N)_\alpha$. *If* $x_0=\dot{0}$, *then the *-power series is*

$$
{}_{\beta}\sum_{k=0}^{\infty}a_k\ddot{\times}\iota(x)^{k_\beta}=a_0\ddot{+}\left[a_1\ddot{\times}\iota(x)\right]\ddot{+}\left[a_2\ddot{\times}\iota(x)^{2_\beta}\right]\ddot{+}\ldots\ .
$$

Theorem 2.11. *Let the *-power series* ${}_{\beta}\sum\limits_{k=0}^{\infty}a_k\ddot{\times}\left(\iota(x)\ddot{-}\iota(x_0)\right)^{k_\beta}$ *be given and let* $\frac{\ddot{1}}{r}\beta={}^{\beta}\limsup\limits_{k\to\infty}\sqrt[k]{|a_k|_\beta}^{\,\beta}$ *(or* $\frac{\ddot{1}}{r}\beta={}^{\beta}\lim\limits_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\beta\right|_\beta$ *if* ${}^{\beta}\lim\limits_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\beta\right|_\beta$ *exist). Then,*

1. *the series is *-absolute convergent if* $|\iota(x)\ddot{-}\iota(x_0)|_\beta\ddot{<}r$,
2. *the series is *-divergent if* $|\iota(x)\ddot{-}\iota(x_0)|_\beta\ddot{>}r$,
3. *the series is *-uniform convergent for* $|\iota(x)\ddot{-}\iota(x_0)|_\beta\ddot{\leq}p$ *if* $\ddot{0}\ddot{<}p\ddot{<}r$.

Proof. If *-Cauchy's root test [7] is applied to the series ${}_{\beta}\sum\limits_{k=0}^{\infty}a_k\ddot{\times}\left(\iota(x)\ddot{-}\iota(x_0)\right)^{k_\beta}$, then we have

$$
\begin{aligned}
{}^{\beta}\limsup\sqrt[k]{\left|a_k\ddot{\times}\left(\iota(x)\ddot{-}\iota(x_0)\right)^{k_\beta}\right|_\beta}^{\,\beta} &= {}^{\beta}\lim_{n\to\infty}\left({}^{\beta}\sup_{k\geq n}\sqrt[k]{\left|a_k\ddot{\times}\left(\iota(x)\ddot{-}\iota(x_0)\right)^{k_\beta}\right|_\beta}^{\,\beta}\right)\\
&= \left[{}^{\beta}\lim_{n\to\infty}\left({}^{\beta}\sup_{k\geq n}\sqrt[k]{|a_k|_\beta}^{\,\beta}\right)\right]\ddot{\times}|\iota(x)\ddot{-}\iota(x_0)|_\beta\\
&= \frac{|\iota(x)\ddot{-}\iota(x_0)|_\beta}{r}\beta\ .
\end{aligned}
$$

Therefore, the series is *-convergent if $\frac{|\iota(x)\ddot{-}\iota(x_0)|_\beta}{r}\beta\ddot{<}\ddot{1}$, namely if $|\iota(x)\ddot{-}\iota(x_0)|_\beta\ddot{<}r$, and the series is *-divergent if $\frac{|\iota(x)\ddot{-}\iota(x_0)|_\beta}{r}\beta\ddot{>}\ddot{1}$ namely if $|\iota(x)\ddot{-}\iota(x_0)|_\beta\ddot{>}r$. Here, if ${}^{\beta}\limsup\sqrt[k]{|a_k|_\beta}^{\,\beta}=\ddot{0}$, then the series is *-convergent for all x. Hence $r=\ddot{+}\infty$. If $|\iota(x)\ddot{-}\iota(x_0)|_\beta\ddot{\leq}p$, then $\left|a_k\ddot{\times}\left(\iota(x)\ddot{-}\iota(x_0)\right)^{k_\beta}\right|_\beta\ddot{\leq}|a_k|_\beta\ddot{\times}p^{k_\beta}$. If *-Cauchy's root test [7] is applied to the series ${}_{\beta}\sum\limits_{k=1}^{\infty}a_k\ddot{\times}p^{k_\beta}$, since

$$
{}^{\beta}\limsup_{k\to\infty}\sqrt[k]{|a_k\ddot{\times}p^{k_\beta}|_\beta}^{\,\beta}=\left[{}^{\beta}\lim_{n\to\infty}\left({}^{\beta}\sup_{k\geq n}\sqrt[k]{|a_k|_\beta}^{\,\beta}\right)\right]\ddot{\times}p=\frac{p}{r}\beta\ddot{<}\ddot{1},
$$

it is seen that the series ${}_{\beta}\sum\limits_{k=1}^{\infty}a_k\ddot{\times}p^{k_\beta}$ is convergent. Thus, in accordance with *-Weierstrass M-Criterion [14], the series ${}_{\beta}\sum\limits_{k=0}^{\infty}a_k\ddot{\times}\left(\iota(x)\ddot{-}\iota(x_0)\right)^{k_\beta}$ is *-uniform convergent.

If the $\beta-$limit

$$
{}^{\beta}\lim_{k\to\infty}\left|\frac{a_{k+1}\ddot{\times}\left(\iota(x)\ddot{-}\iota(x_0)\right)^{(k+1)_\beta}}{a_k\ddot{\times}\left(\iota(x)\ddot{-}\iota(x_0)\right)^{k_\beta}}\beta\right|_\beta={}^{\beta}\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\beta\right|_\beta\ddot{\times}|\iota(x)\ddot{-}\iota(x_0)|_\beta
$$

exists which is obtained by applying $\beta-$rate test [7] to the series ${}_\beta\sum_{k=0}^{\infty} a_k \ddot{\times} (\iota(x) \ddot{-} \iota(x_0))^{k_\beta}$, then

$$ {}^\beta\lim_{k\to\infty} \left| \frac{a_{k+1}}{a_k}\beta \right|_\beta \ddot{\times} |\iota(x) \ddot{-} \iota(x_0)|_\beta \ddot{<} \ddot{1}, $$

namely

$$ |\iota(x) \ddot{-} \iota(x_0)|_\beta \ddot{<} \frac{\ddot{1}}{{}^\beta\lim_{k\to\infty} \left| \frac{a_{k+1}}{a_k}\beta \right|_\beta}\beta = {}^\beta\lim_{k\to\infty} \left| \frac{a_k}{a_{k+1}}\beta \right|_\beta = r $$

is written. The *-power series is *-convergent if $|\iota(x) \ddot{-} \iota(x_0)|_\beta \ddot{<} r$, and *-divergent if $|\iota(x) \ddot{-} \iota(x_0)|_\beta \ddot{>} r$. If $|\iota(x) \ddot{-} \iota(x_0)|_\beta \ddot{\leq} p$, then $\left| a_k \ddot{\times} (\iota(x) \ddot{-} \iota(x_0))^{k_\beta} \right|_\beta \ddot{\leq} |a_k|_\beta \ddot{\times} p^{k_\beta}$. If the $\beta-$rate test [7] is applied to the series ${}_\beta\sum_{k=1}^{\infty} a_k \ddot{\times} p^{k_\beta}$, then

$$ {}^\beta\lim_{k\to\infty} \left| \frac{a_{k+1} \ddot{\times} p^{(k+1)_\beta}}{a_k \ddot{\times} p^{k_\beta}}\beta \right|_\beta = {}^\beta\lim_{k\to\infty} \left| \frac{a_{k+1}}{a_k}\beta \right|_\beta \ddot{\times} p = \frac{p}{r}\beta \ddot{<} \ddot{1}. $$

Therefore the series is *-convergent. Hence, by *-Weierstrass M-Criterion [14], the *-power series is *-uniform convergent. □

Definition 2.12. *The number $\iota^{-1}(r)$ is called radius of non-Newtonian convergence (radius of *-convergence) of the *-power series in the equality (2.4) for the number r in Theorem 2.11. $\dot{(}x_0 \dot{-} \iota^{-1}(r), x_0 \dot{+} \iota^{-1}(r)\dot{)}$ is called the interval of non-Newtonian convergence (or interval of *-convergence). By checking the end points, we decide whether or not to include them to interval of *-convergence.*

Example 2.13. *The radius of *-convergence of the series ${}_\beta\sum_{k=1}^{\infty} \frac{\iota(x)^{k_\beta}}{\ddot{k}}\beta$ is $\ddot{1}$ and the interval of *-convergence of the series ${}_\beta\sum_{k=1}^{\infty} \frac{\iota(x)^{k_\beta}}{\ddot{k}}\beta$ is $\dot{[}\dot{0} \dot{-} \dot{1}, \dot{1}\dot{)}$:*

Since

$$ r = {}^\beta\lim_{k\to\infty} \left| \frac{a_k}{a_{k+1}}\beta \right|_\beta = {}^\beta\lim_{k\to\infty} \frac{\frac{\ddot{1}}{\ddot{k}}\beta}{\frac{\ddot{1}}{\ddot{k} \ddot{+} \ddot{1}}\beta}\beta = {}^\beta\lim_{k\to\infty} \frac{\ddot{k} \ddot{+} \ddot{1}}{\ddot{k}}\beta = \ddot{1}, $$

radius of *-convergence is $\iota^{-1}(r) = \iota^{-1}(\ddot{1}) = \dot{1}$. The interval of *-convergence is found as $\dot{(}\dot{0} \dot{-} \dot{1}, \dot{1}\dot{)}$ since $x_0 = \dot{0}$. For the end point $x = \dot{1}$, $\beta-$harmonic series ${}_\beta\sum_{k=1}^{\infty} \frac{\ddot{1}}{\ddot{k}}\beta$ is *-divergent, for the initial point $x = \dot{0} \dot{-} \dot{1}$, $\beta-$alternating series [7] ${}_\beta\sum_{k=1}^{\infty} \frac{(\ddot{0} \ddot{-} \ddot{1})^{k_\beta}}{\ddot{k}}\beta$ is convergent. Therefore, the set of *-convergence of the series ${}_\beta\sum_{k=1}^{\infty} \frac{\iota(x)^{k_\beta}}{\ddot{k}}\beta$ is the interval $\dot{[}\dot{0} \dot{-} \dot{1}, \dot{1}\dot{)}$.

Let the interval of *-convergence of the series ${}_\beta\sum_{k=0}^{\infty} a_k \ddot{\times} (\iota(x) \ddot{-} \iota(x_0))^{k_\beta}$ be $\dot{(}x_0 \dot{-} \iota^{-1}(r), x_0 \dot{+} \iota^{-1}(r)\dot{)}$. Let define $f(x) = b_{x_1}$ if $x_1 \in \dot{(}x_0 \dot{-} \iota^{-1}(r), x_0 \dot{+} \iota^{-1}(r)\dot{)}$ and if $b_{x_1} = {}_\beta\sum_{k=0}^{\infty} a_k \ddot{\times} (\iota(x) \ddot{-} \iota(x_0))^{k_\beta}$. According to this, the function $f : \dot{(}x_0 \dot{-} \iota^{-1}(r), x_0 \dot{+} \iota^{-1}(r)\dot{)} \to \mathbb{R}(N)_\beta$ is defined by

$$ f(x) = {}_\beta\sum_{k=0}^{\infty} a_k \ddot{\times} (\iota(x) \ddot{-} \iota(x_0))^{k_\beta}. $$

Remark 2.14. *Domain of the function f can be bigger than the interval of *-convergence of the *-power series. For example, the interval of *-convergence of ${}_\beta\sum_{k=0}^{\infty} \iota(x)^{k_\beta}$ is $\dot{(}\dot{0} \dot{-} \dot{1}, \dot{1}\dot{)}$ and $f(x) = \frac{\ddot{1}}{\ddot{1} \ddot{-} \iota(x)}\beta = {}_\beta\sum_{k=0}^{\infty} \iota(x)^{k_\beta}$ on this interval. This function f is defined and *-continuous for all $x \in \mathbb{R}(N)_\alpha$ except for $x = \dot{1}$.*

By Theorem 2.11, the following corollaries are obtained.

Corollary 2.15. *(1) The function f is *-continuous on $\dot{(}x_0 \dot{-} \iota^{-1}(r), x_0 \dot{+} \iota^{-1}(r)\dot{)}$. Because there exist a number $p \in \mathbb{R}(N)_\alpha$ such that*

$$ x_0 \dot{-} \iota^{-1}(r) \dot{<} x_0 \dot{-} p \dot{<} x_1 \dot{<} x_0 \dot{+} p \dot{<} x_0 \dot{+} \iota^{-1}(r) $$

*for each $x_1 \in \dot{(}x_0 \dot{-} \iota^{-1}(r), x_0 \dot{+} \iota^{-1}(r)\dot{)}$. In this case, by third item of Theorem 2.11, *-power series is *-uniform convergent on $\dot{[}x_0 \dot{-} p, x_0 \dot{+} p\dot{]} \subset \dot{(}x_0 \dot{-} \iota^{-1}(r), x_0 \dot{+} \iota^{-1}(r)\dot{)}$, and by virtue of Corollary 3 in [14], the function f is *-continuous.*
*(2) By *-uniform convergence and Corollary 4 in [14], *-power series is term by term *-integrable.*
*(3) By third item of Theorem 2.11 again and Corollary 5 in [14], *-derivation*

$$ \left[\overset{*}{D} f \right](x) = {}_\beta\sum_{k=1}^{\infty} \ddot{k} \ddot{\times} a_k \ddot{\times} (\iota(x) \ddot{-} \iota(x_0))^{(k-1)_\beta} $$

*is found through term by term *-differentiating.*

This new series (-derivative series) has same radius of *-convergence with the original series, it is *-uniform convergent for* $\ddot{0}\ddot{<}p\ddot{<}r$ *and* $\left[\overset{*}{D}f\right](x)$ *is *-continuous. If this process keeps on, then every order *-derivative of the function* f *exist and nth order *-derivative*

$$\left[\overset{*}{D^n}f\right](x) = {}_\beta\sum_{k=n}^{\infty}\ddot{k}\ddot{\times}\left(\ddot{k}\ddot{-}\ddot{1}\right)\ddot{\times}\ldots\ddot{\times}\left(\ddot{k}\ddot{-}\ddot{n}\ddot{+}\ddot{1}\right)\ddot{\times}a_k\ddot{\times}\left(\iota(x)\ddot{-}\iota(x_0)\right)^{(k-n)_\beta}$$

is found for $x\in\dot{(}x_0\dot{-}\iota^{-1}(r),x_0\dot{+}\iota^{-1}(r)\dot{)}$. $\left[\overset{*}{D^n}f\right](x_0) = a_n\ddot{\times}\ddot{n}!_\beta$ *or* $a_n = \dfrac{\left[\overset{*}{D^n}f\right](x_0)}{\ddot{n}!_\beta}\beta$ *is obtained for* $x=x_0$.

Therefore, we can give the following theorem without proof.

Theorem 2.16. *Let the radius of *-convergence of *-power series* ${}_\beta\sum\limits_{k=0}^{\infty}a_k\ddot{\times}\left(\iota(x)\ddot{-}\iota(x_0)\right)^{k_\beta}$ *be* $\iota^{-1}(r)\dot{>}\dot{0}$ *and let* $f(x) = {}_\beta\sum\limits_{k=1}^{\infty}a_k\ddot{\times}\left(\iota(x)\ddot{-}\iota(x_0)\right)^{k_\beta}$. *Then the function* f *is every order *-differentiable and* $a_k = \dfrac{\left[\overset{*}{D^n}f\right](x_0)}{\ddot{k}!_\beta}\beta$.

Corollary 2.17. *Sum of a *-power series is a *-continuous function and its every order *-derivative exists. Such a series is term by term *-differentiable, and the radius of *-convergence of *-derivative series with first series are equal.*

Proof. The proof is trivial for $x_0=\dot{0}$. For the *-power series ${}_\beta\sum\limits_{k=0}^{\infty}a_k\ddot{\times}\iota(x)^{k_\beta}$, terms $g_k(x)=a_k\ddot{\times}\iota(x)^{k_\beta}$ are *-differentiable. The *-derivative functions $\left[\overset{*}{D}g_k\right](x)=\ddot{k}\ddot{\times}a_k\ddot{\times}\iota(x)^{(k-1)_\beta}$ is *-continuous and

$$\frac{\ddot{1}}{R}\beta = {}^\beta\limsup_{k\to\infty}\sqrt[k]{\left|\ddot{k}\ddot{\times}a_k\right|_\beta}^{\beta}$$

where R is the radius of *-convergence of *-derivative power series ${}_\beta\sum\limits_{k=1}^{\infty}\left[\ddot{k}\ddot{\times}a_k\ddot{\times}\iota(x)^{(k-1)_\beta}\right]$. But, since $\sqrt[k]{\ddot{k}}^{\beta}\xrightarrow{\beta}\ddot{1}$,

$$\frac{\ddot{1}}{R}\beta = {}^\beta\limsup_{k\to\infty}\sqrt[k]{\left|\ddot{k}\ddot{\times}a_k\right|_\beta}^{\beta} = \frac{\ddot{1}}{r}\beta.$$

Namely $R=r$ is obtained. Hence, $\iota^{-1}(R)=\iota^{-1}(r)$. Then, this *-series of *-derivatives is *-uniform convergent on a closed $\alpha-$interval whose radius smaller than $\iota^{-1}(r)$. Therefore, by Corollary 5 in [14], the $\beta-$sum of *-derivative series is *-derivative of $\beta-$sum of the *-series in beginning. □

Now, the concept $\beta-$summability in the sense of Cesàro will be given which is an application for *-power series.

Definition 2.18. *Let*

$$S_n = {}_\beta\sum_{k=1}^{n}a_k \text{ and } \sigma_n = \frac{\ddot{1}}{\ddot{n}}\beta\ddot{\times}{}_\beta\sum_{k=1}^{n}S_k$$

*be defined for *-series* ${}_\beta\sum a_k$. *From the definition of* σ_n, σ_n *is* $\beta-$*average of first n partial* $\beta-$*sums. Additionally,* σ_n *can be written in the form*

$$\sigma_n = {}_\beta\sum_{k=1}^{n}\left(\ddot{1}\ddot{-}\frac{\ddot{k}\ddot{-}\ddot{1}}{\ddot{n}}\beta\right)\ddot{\times}a_k\,.$$

If ${}^\beta\lim\limits_{n\to\infty}\sigma_n=A$, *then it is said that the series* ${}_\beta\sum a_k$ *is* $\beta-$*Cesàro summable of order one (or* $(C,1)$ $\beta-$*summable) and* $(C,1)$ $\beta-$*sum of this series is A. If* $(C,1)$ $\beta-$*sum of *-series* ${}_\beta\sum a_k$ *is A, then*

$${}_\beta\sum_{k=1}^{\infty}a_k = A \quad (C,1)$$

is written.

The idea here has arisen from need of give a meaning to *-divergent series. For example,

$$\frac{\ddot{1}}{\ddot{2}}\beta = \ddot{1}\ddot{-}\ddot{1}\ddot{+}\ddot{1}\ddot{-}\ddot{1}\ddot{+}\ddot{1}\ddot{-}\ldots \quad (C,1)\,.$$

Indeed, here $(S_n)=(\ddot{1},\ddot{0},\ddot{1},\ddot{0},\ldots)$ and

$$T_n = {}_\beta\sum_{k=1}^{n}S_k = \ddot{1},\ddot{1},\ddot{2},\ddot{2},\ddot{3},\ddot{3},\ldots\ .$$

Then $\sigma_{2n}=\dfrac{\ddot{n}}{\ddot{2}\ddot{\times}\ddot{n}}\beta$ and $\sigma_{2n+1}=\dfrac{\ddot{n}\ddot{+}\ddot{1}}{\left(\ddot{2}\ddot{\times}\ddot{n}\right)\ddot{+}\ddot{1}}\beta$, hence ${}^\beta\lim\limits_{n\to\infty}\sigma_n=\dfrac{\ddot{1}}{\ddot{2}}\beta$.

New summability method can be defined by taking $\beta-$average of σ_n's which is stronger than $(C,1)$ $\beta-$summability. The value

$$^{\beta}\lim_{n\to\infty}\frac{\ddot{1}}{\ddot{n}}\beta\ddot{\times}(\sigma_1\ddot{+}\sigma_2\ddot{+}...\ddot{+}\sigma_n)$$

is called $(C,2)$ $\beta-$sum of the *-series $_{\beta}\sum_{k=1}^{\infty}a_k$.

(C,r) can be defined in a similar way for $r=1,2,3,\ldots$. Some properties of $(C,1)$ $\beta-$summability are given below.

Proposition 2.19. *(1) If* $_{\beta}\sum_{k=1}^{\infty}a_k=A\quad(C,1)$ *and* $_{\beta}\sum_{k=1}^{\infty}b_k=B\quad(C,1)$*, then* $_{\beta}\sum_{k=1}^{\infty}[(y\ddot{\times}a_k)\ddot{+}(z\ddot{\times}b_k)]=(y\ddot{\times}A)\ddot{+}(z\ddot{\times}B)\quad(C,1)$.

(2) If $_{\beta}\sum_{k=1}^{\infty}a_k=A\quad(C,1)$*, then* $_{\beta}\sum_{k=1}^{\infty}a_{k+1}=A\ddot{-}a_1\quad(C,1)$.

(3) (-Regularity) If* $\beta-$*sum of the series* $_{\beta}\sum_{k=1}^{\infty}a_k$ *is A, then* $_{\beta}\sum_{k=1}^{\infty}a_k=A\quad(C,1)$.

Proof. (3) $_{\beta}\sum_{k=1}^{\infty}a_k=A$, then $S_n\xrightarrow{*}A$. Let $B\ddot{<}A$. Then, there exists a number $n_0\in\mathbb{N}$ such that $S_n\ddot{\geq}B$ for all $n\geq n_0$.

$$\sigma_n=\frac{\ddot{1}}{\ddot{n}}\beta\ddot{\times}(S_1\ddot{+}...\ddot{+}S_{n_0}\ddot{+}S_{n_0+1}\ddot{+}...\ddot{+}S_n)\ddot{\geq}\frac{\ddot{1}}{\ddot{n}}\beta\ddot{\times}(S_1\ddot{+}...\ddot{+}S_{n_0})\ddot{+}\frac{\ddot{n}\ddot{-}\ddot{n}_0}{\ddot{n}}\beta\ddot{\times}B.$$

Therefore $^{\beta}\liminf\sigma_n\ddot{\geq}B$. Since $B\ddot{<}A$ is taken arbitrary, $^{\beta}\liminf\sigma_n\ddot{\geq}A$. Similarly, it is seen that $^{\beta}\limsup\sigma_n\ddot{\leq}A$. Hence, $^{\beta}\lim_{n\to\infty}\sigma_n=A$ is obtained. □

Another $\beta-$summability method is named "Abel $\beta-$sum".

Definition 2.20. *If* $*-\lim_{x\to\dot{1}^-}\left({_{\beta}\sum_{k=0}^{\infty}}a_k\ddot{\times}\imath(x)^{k_\beta}\right)=A$ *for the series* $_{\beta}\sum_{k=0}^{\infty}a_k$*, then it is said that this series is* $\beta-$*summable in the sense of Abel and its Abel* $\beta-$*sum is A. In this case,* $_{\beta}\sum_{k=0}^{\infty}a_k=A\ (Abel)$ *is written.*

For example,

$$\frac{\ddot{1}}{\ddot{2}}\beta=\ddot{1}\ddot{-}\ddot{1}\ddot{+}\ddot{1}\ddot{-}\ddot{1}\ddot{+}\ddot{1}\ddot{-}...(Abel).$$

Indeed, for $|x|_{\alpha}\dot{<}\dot{1}$, it is seen that

$$f(x)=\ddot{1}\ddot{-}\imath(x)\ddot{+}\imath(x)^{2_\beta}\ddot{-}...=\ {_{\beta}\sum_{k=0}^{\infty}}(\ddot{0}\ddot{-}\ddot{1})^{k_\beta}\ddot{\times}\left(\imath(x)^{k_\beta}\right)=\frac{\ddot{1}}{\ddot{1}\ddot{+}\imath(x)}\beta$$

and

$$*-\lim_{x\to\dot{1}^-}f(x)=\frac{\ddot{1}}{\ddot{2}}\beta.$$

Abel $\beta-$sum and $(C,1)$ $\beta-$sum gives the same result which always holds. Firstly, *-regularity of $\beta-$summability in the sense of Abel will be shown.

Theorem 2.21. *If* $_{\beta}\sum_{k=0}^{\infty}a_k=A$*, then *-series* $_{\beta}\sum_{k=0}^{\infty}a_k\ddot{\times}(\imath(x))^{k_\beta}$ **-convergent for* $|x|_{\alpha}\dot{<}\dot{1}$ *and* $*-\lim_{x\to\dot{1}^-}\left({_{\beta}\sum_{k=0}^{\infty}}a_k\ddot{\times}(\imath(x))^{k_\beta}\right)=A$.

Proof. Suppose that $A=\ddot{0}$ by changing the number a_0 properly. Since a_0 is bounded, by virtue of Theorem 2.11, *-series $_{\beta}\sum_{k=0}^{\infty}a_k\ddot{\times}(\imath(x))^{k_\beta}$ is *-convergent for $|x|_{\alpha}\dot{<}\dot{1}$.

If $S_n=\ {_{\beta}\sum_{k=0}^{n}}a_k$ is taken, since S_n is $\beta-$bounded as $n\to\infty$, series $_{\beta}\sum S_k\ddot{\times}(\imath(x))^{k_\beta}$ is *-convergent for $|x|_{\alpha}\dot{<}\dot{1}$ similarly. On the other hand, when $A=\ddot{0}$, $S_n\xrightarrow{*}\ddot{0}$ as $n\to\infty$. Therefore,

$$\begin{aligned}f(x)&=S_0\ddot{+}\ {_{\beta}\sum_{k=1}^{\infty}}(S_k\ddot{-}S_{k-1})\ddot{\times}(\imath(x))^{k_\beta}\\&=(\ddot{1}\ddot{-}\imath(x))\ddot{\times}\ {_{\beta}\sum_{k=0}^{\infty}}S_k\ddot{\times}(\imath(x))^{k_\beta}\end{aligned}$$

is written. Since $S_n\xrightarrow{*}\ddot{0}$, for given $\varepsilon\ddot{>}\ddot{0}$, a $n_0\in\mathbb{N}$ can be found such that $|S_n|_{\beta}\ddot{\leq}\varepsilon$ when $n\geq n_0$. Then

$$\begin{aligned}|f(x)|_{\beta}&\ddot{\leq}(\ddot{1}\ddot{-}\imath(x))\ddot{\times}\left|{_{\beta}\sum_{k=0}^{n_0}}S_k\ddot{\times}(\imath(x))^{k_\beta}\right|_{\beta}\ddot{+}(\ddot{1}\ddot{-}\imath(x))\ddot{\times}\left|{_{\beta}\sum_{k=n_0+1}^{\infty}}S_k\ddot{\times}(\imath(x))^{k_\beta}\right|_{\beta}\\&\ddot{\leq}(\ddot{1}\ddot{-}\imath(x))\ddot{\times}\left|{_{\beta}\sum_{k=0}^{n_0}}S_k\ddot{\times}(\imath(x))^{k_\beta}\right|_{\beta}\ddot{+}(\ddot{1}\ddot{-}\imath(x))\ddot{\times}\varepsilon\ddot{\times}\imath(x)^{(n_0+1)_\beta}\ddot{\times}\frac{\ddot{1}}{\ddot{1}\ddot{-}\imath(x)}\beta\\&\ddot{\leq}(\ddot{1}\ddot{-}\imath(x))\ddot{\times}\left|{_{\beta}\sum_{k=0}^{n_0}}S_k\ddot{\times}(\imath(x))^{k_\beta}\right|_{\beta}\ddot{+}\varepsilon\end{aligned}$$

is obtained. Hence, $^{*}-\limsup\limits_{x\to\dot{1}^{-}}|f(x)|_{\beta}\ddot{\leq}\varepsilon$ and $^{*}-\lim\limits_{x\to\dot{1}^{-}}f(x)=\ddot{0}$ is found since $\varepsilon\ddot{>}\ddot{0}$ arbitrary. □

According to this, if a *-power series is *-convergent on an $\alpha-$closed interval, then $\beta-$sum of that series is also *-continuous on end points.

Indeed, Abel $\beta-$sum method is stronger than $(C,1)$ $\beta-$sum method.

Remark 2.22. *If sequence $\frac{a_n}{\ddot{n}}\beta$ is $\beta-$bounded for n is sufficiently large, then it is written that $a_n=O(n)$ and if $\frac{a_n}{\ddot{n}}\beta\overset{*}{\longrightarrow}\ddot{0}$, then it is written that $a_n=o(n)$.*

Theorem 2.23. *If ${}_{\beta}\sum\limits_{k=0}^{\infty}a_k=A\ (C,1)$, then ${}_{\beta}\sum\limits_{k=0}^{\infty}a_k=A\ (Abel)$.*

Proof. Suppose that $A=\ddot{0}$ as in the proof of Theorem 2.21. Let define $S_n={}_{\beta}\sum\limits_{k=0}^{n}a_k$ and $T_n={}_{\beta}\sum\limits_{k=0}^{n}S_k$. By hypothesis, $T_n=o(n)$. From here, $S_n=T_n\ddot{-}T_{n-1}=O(n)$ and $a_n=S_n\ddot{-}S_{n-1}=O(n)$ can be written. Then all of three *-series ${}_{\beta}\sum a_k\ddot{\times}(\iota(x))^{k_\beta}$, ${}_{\beta}\sum S_k\ddot{\times}(\iota(x))^{k_\beta}$ and ${}_{\beta}\sum T_k\ddot{\times}(\iota(x))^{k_\beta}$ are *-convergent for $|x|_{\alpha}\dot{<}\dot{1}$. Therefore

$$\begin{aligned} f(x) &= {}_{\beta}\sum a_k\ddot{\times}(\iota(x))^{k_\beta}=(\ddot{1}\ddot{-}\iota(x))\ddot{\times}{}_{\beta}\sum S_k\ddot{\times}(\iota(x))^{k_\beta}\\ &= (\ddot{1}\ddot{-}\iota(x))^{2_\beta}\ddot{\times}{}_{\beta}\sum T_k\ddot{\times}(\iota(x))^{k_\beta}\end{aligned}$$

is written. Since $T_n=O(n)$, there exists $n_0\in\mathbb{N}$ such that $|T_n|_{\beta}\ddot{\leq}\varepsilon\ddot{\times}\ddot{n}$ for $n>n_0$ which corresponds every given $\varepsilon\ddot{>}\ddot{0}$. According to this,

$$\begin{aligned} |f(x)|_{\beta} &\ddot{\leq} (\ddot{1}\ddot{-}\iota(x))^{2_\beta}\ddot{\times}\left|{}_{\beta}\sum_{k\leq n_0}T_k\ddot{\times}(\iota(x))^{k_\beta}\right|_{\beta}\ddot{+}(\ddot{1}\ddot{-}\iota(x))^{2_\beta}\ddot{\times}\left({}_{\beta}\sum_{k>n_0}T_k\ddot{\times}(\iota(x))^{k_\beta}\right)\\ &\ddot{\leq} (\ddot{1}\ddot{-}\iota(x))^{2_\beta}\ddot{\times}\left|{}_{\beta}\sum_{k\leq n_0}T_k\ddot{\times}(\iota(x))^{k_\beta}\right|_{\beta}\ddot{+}(\ddot{1}\ddot{-}\iota(x))^{2_\beta}\ddot{\times}\varepsilon\ddot{\times}\iota(x)\ddot{\times}(\ddot{1}\ddot{-}\iota(x))^{(-2)_\beta}\end{aligned}$$

is found. Whence,

$$*-\lim_{x\to\dot{1}^{-}}\sup|f(x)|_{\beta}\ddot{\leq}\varepsilon$$

is obtained. Then, $*-\lim\limits_{x\to\dot{1}^{-}}f(x)=\ddot{0}$ as in Theorem 2.21. □

Example 2.24. *The *-series ${}_{\beta}\sum\limits_{k=1}^{\infty}(\ddot{0}\ddot{-}\ddot{1})^{k_\beta}\ddot{\times}\ddot{k}$ is not $(C,1)$ $\beta-$summable:*

Since

$$\begin{aligned} a_n &: (\ddot{0}\ddot{-}\ddot{1}),\ddot{2},(\ddot{0}\ddot{-}\ddot{3}),\ddot{4},(\ddot{0}\ddot{-}\ddot{5}),\ddot{6},...\\ S_n &: (\ddot{0}\ddot{-}\ddot{1}),\ddot{1},(\ddot{0}\ddot{-}\ddot{2}),\ddot{2},(\ddot{0}\ddot{-}\ddot{3}),\ddot{3},...\\ T_n &: (\ddot{0}\ddot{-}\ddot{1}),\ddot{0},(\ddot{0}\ddot{-}\ddot{2}),\ddot{0},(\ddot{0}\ddot{-}\ddot{3}),\ddot{0},...\\ T_{2n} &= \ddot{0},T_{2n-1}=(\ddot{0}\ddot{-}\ddot{n}),\sigma_{2n}=\ddot{0}\ \textit{and}\ \sigma_{2n-1}=\frac{(\ddot{0}\ddot{-}\ddot{n})}{(\ddot{2}\ddot{\times}\ddot{n}\ddot{-}\ddot{1})}\beta\overset{\beta}{\to}\frac{\ddot{0}\ddot{-}\ddot{1}}{\ddot{2}}\beta\end{aligned}$$

are obtained. Then, ${}^{\beta}\lim\limits_{n\to\infty}\sigma_n$ does not exist.

Example 2.25. *Show that ${}_{\beta}\sum\limits_{k=1}^{\infty}(\ddot{0}\ddot{-}\ddot{1})^{k_\beta}\ddot{\times}\ddot{k}\ddot{\times}(\iota(x))^{k_\beta}=\frac{\ddot{0}\ddot{-}\ddot{1}}{\ddot{4}}\beta$ (Abel).*

$$\begin{aligned} {}_{\beta}\sum_{k=1}^{\infty}(\ddot{0}\ddot{-}\ddot{1})^{k_\beta}\ddot{\times}\ddot{k}\ddot{\times}(\iota(x))^{k_\beta} &= \iota(x)\ddot{\times}\overset{*}{D}\left({}_{\beta}\sum_{k=1}^{\infty}(\ddot{0}\ddot{-}\ddot{1})^{k_\beta}\ddot{\times}(\iota(x))^{k_\beta}\right)\\ &= \iota(x)\ddot{\times}\overset{*}{D}\left(\frac{\ddot{1}}{\ddot{1}\ddot{+}\iota(x)}\beta\ddot{-}\ddot{1}\right)\quad(|x|_{\alpha}\dot{<}\dot{1})\\ &= \frac{\ddot{0}\ddot{-}\iota(x)}{(\ddot{1}\ddot{+}\iota(x))^{2_\beta}}\beta.\end{aligned}$$

Since $f(x)=\frac{\ddot{0}\ddot{-}\iota(x)}{(\ddot{1}\ddot{+}\iota(x))^{2_\beta}}\beta$ and $*-\lim\limits_{x\to\dot{1}^{-}}f(x)=\frac{\ddot{0}\ddot{-}\ddot{1}}{\ddot{4}}\beta$, ${}_{\beta}\sum\limits_{k=1}^{\infty}(\ddot{0}\ddot{-}\ddot{1})^{k_\beta}\ddot{\times}\ddot{k}\ddot{\times}(\iota(x))^{k_\beta}=\frac{\ddot{0}\ddot{-}\ddot{1}}{\ddot{4}}\beta$ (*Abel*) is obtained.

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