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*Araştırma Makalesi - Research Article*

# Genelleştirilmiş Tetranacci Sayilarini İçeren Circulant Matrislerin Normu Üzerine

Fatma YEŞİL BARAN[1*], Tevfik YETİŞ[2]

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**ÖZ**

Bu çalışmada başlangıç değerleri $\mathcal{T}_0 = a$, $\mathcal{T}_1 = b$, $\mathcal{T}_2 = c$, $\mathcal{T}_3 = d$ olan ve $n \geq 4$ için,

$$\mathcal{T}_n = p\mathcal{T}_{n-1} + q\mathcal{T}_{n-2} + r\mathcal{T}_{n-3} + s\mathcal{T}_{n-4}$$

rekürans bağıntısı ile tanımlanan $(\mathcal{T}_n)_{n\in\mathbb{N}}$ Genelleştirilmiş Tetranacci Dizisi için Binet formülü elde edilerek, bu dizinin ilk $n$ teriminin toplamı formülüze edilmiştir. Genelleştirilmiş Tetranacci sayı dizisi için üreteç fonksiyonuna ulaşılmıştır. Ayrıca elemanları genelleştirilmiş Tetranacci sayı dizisinin elemanlarından oluşan circulant matrisler için bazı matris normları hesaplanmıştır.

***Anahtar Kelimeler- Circulant Matris, Tetranacci Dizisi, Matris Normu.***

1*Sorumlu yazar iletişim: *fatma.yesilbaran@amasya.edu.tr* (https://orcid.org/0000-0001-8613-2706)
*Computer Engineering, Amasya University, Şeyhcui Neighborhood, Kemal Nehrozoğlu Street, 05100 AMASYA.*
2*İletişim: mitimat01@gmail.com* (https://orcid.org/0000-0002-7835-3178)
*Mathematics, Horasan Vocational And Technical Anatolian High School, Horasan/ERZURUM*

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# On The Norm Of Cırculant Matrıces Vıa Generalızed Tetranaccı Numbers

**ABSTRACT**

In this study, the sum of first $\boldsymbol{n}$ terms of this series is formulated by obtaining the Binet formula for the generalized Tetranacci sequence $(\mathcal{T}_n)_{n\in\mathbb{N}}$, whose initial values are $\mathcal{T}_0 = a$, $\mathcal{T}_1 = b$, $\mathcal{T}_2 = c$, $\mathcal{T}_3 = d$ and defined by the

$$\mathcal{T}_n = p\mathcal{T}_{n-1} + q\mathcal{T}_{n-2} + r\mathcal{T}_{n-3} + s\mathcal{T}_{n-4}$$

recurrence relation for $n \geq 4$. The generating function is obtained for generalized Tetranacci number sequence. In addition, some matrix norms are calculated for the circulant matrices consisting of elements of the generalized Tetranacci number sequence.

***Keywords*- Circulant Matrix, Tetranacci Sequence, Matrix Norm.**

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## I. INTRODUCTION

Fibonacci numbers have been one of research topics in mathematics in the past. In the studies on this subject and every new research, it has been seen that this number series have new features. The Tetranacci numbers sequence is a sequence of numbers that continues by summing up the four terms preceding each term. This series is called "Quadranacci" in Latin and "Tetranacci" in Greek. Firstly, Feinberg described the Tetranacci series in 1963 [1]. Also, Waddill studied the Tetranacci series more extensively in his work titled "The Tetranacci Sequence and Generalizations" in 1992 [2].

Nowadays, the relationship between mathematics and other sciences has become a stubborn fact. We see this relationship between engineering, science and mathematics. Circulant matrices, in particular, appear in many scientific and engineering applications to model combinatorial problems. Circulant matrices play a significant role in the solution of some differential equations, digital filters, communication, image processing, signaling and Toeplitz matrices. Lind defined the circulant matrices which its elements consisting of Fibonacci numbers and calculated the determinants of these matrices [3]. Davis, gave the properties of circulant matrices [4]. Öcal, Tuğlu and Altınışık examined the stability and continuity of generalized Fibonacci and Lucas numbers and obtained the Binet formula for these series [5]. Alptekin calculated eigenvalues of circulant matrices which elements of Pell, Pell-Lucas and Modified Pell numbers. Also she obtained the Euclidean norm of semicirculant matrices whose elements are these numbers [6]. Solak obtained the upper and lower bounds of Euclidean and spectral norms for circulant matrices which the elements of Fibonacci and Lucas numbers [7]. Alptekin, Mansour and Tuğlu calculated the spectral norms and eigenvalues of circulant matrices which elements consisting of Horadam numbers and also calculated the Euclidean norm of semicirculant matrices consisting of these numbers [8].

Shen and Cen obtained bounds for spectral norms of circulant matrices whose elements are Fibonacci and Lucas [9]. Bahşi and Solak calculated the spectral norms of circulant matrices with hyper-Fibonacci and hyper-Lucas numbers [10]. Tuğlu and Kızılateş examined the upper and lower boundary problems for the spectral norms of the geometric circulant matrix whose elements include generalized Horadam numbers, they obtained the bounds for the spectral norms and also calculated the norms of the $r-$circulant matrix whose elements are generalized with Horadam numbers [11 − 13]. Also they obtained some norms of special matrices which include harmonic Fibonacci number and Quadra Lucas-Jacobsthal number [14,15]. Polatli calculated bounds for the spectral norms of $r-$circulant matrices with a type of Catalan triangle numbers [16].

Bahşi calculated the matrix norms of circulant matrices consisting of elements of the Tribonacci sequence by changing the initial conditions of the generalized Fibonacci and Lucas numbers [17].

Özkoç and Ardıyok calculated the spectral and Euclidean norms of the circulant and negacyclic matrices consisting of Tetranacci sequence and its complement Tetranacci sequence [18]. Taşçı and Acar defined Gaussian Tetranacci numbers with their initial values being Gaussian integers [19].

The aim of this study is to generalize the recurrence relation for Tetranacci numbers which is defined as

$$M_n = M_{n-1} + M_{n-2} + M_{n-3} + M_{n-4}$$

for initial conditions which are $M_0 = M_1 = M_2 = 0$ , $M_3 = 1$ and to form a new generalized Tetranacci sequence $(\mathcal{T}_n)_{n\in\mathbb{N}}$ and obtain some properties of this sequence with using this generalization.

## II. PRELIMINARIES

**Definition:** Let $(a_k)$ be an array with real terms.

$$g(x) = \sum_{k=0}^{\infty} a_k x^k$$

is called the generating function of the sequence $(a_k)$.

**Definition:** Let $a, b, p, q \in \mathbb{Z}$ and $H_0 = a$ , $H_1 = b$. Horadam sequence is defined as

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$$H_{n+2} = pH_{n+1} + qH_n.$$

The elements of this sequence are called Horadam numbers [20].

**Definition:** The Toeplitz matrix is of the type $nxn$, that satisfies the condition $t_{k,j} = t_{k-j}$ for

$T_n = [t_{k,j}; k, j = 0, 1, \dots, n-1]$. So ,

$$T_n = \begin{bmatrix} t_0 & t_{-1} & t_{-2} & \cdots & t_{-(n-1)} \\ t_1 & t_0 & t_{-1} & \cdots & \vdots \\ t_2 & t_1 & t_0 & \cdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ t_{n-1} & \cdots & \cdots & \cdots & t_0 \end{bmatrix}$$

expresses matrices in the form of Toeplitz matrices are used in the solution of differential and integral oscillation functions, mathematics, physics and statistics equations. The $C(c)$ circulant matrix is a special case of the Toeplitz matrix and its definition is as follows.

**Definition:** Let $c = (c_0, c_1, \dots, c_{n-1})^T$. The matrix $C(c)$ of type $nxn$, which is $j - i \equiv k \ (modn)$, is called the circulant matrix and is shown as

$$C(c) = \begin{bmatrix} c_0 & c_1 & c_2 & \cdots & c_{n-1} \\ c_{n-1} & c_0 & c_1 & \cdots & c_{n-2} \\ c_{n-2} & c_{n-1} & c_0 & \cdots & c_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_1 & c_2 & c_3 & \cdots & c_0 \end{bmatrix}$$

[4]. Karner, Schneid and Ueberhuber have defined the right and left circulant matrices [21]. Pollock obtained the spectral decomposition of circulant matrices and defined symmetric circulant matrices and studied Fourier transforms of this matrix [22]. These matrices have many applications in numerical analysis, optimization, digital image processing, mathematical statistics and modern technology.

**Definition:** Let $c = (c_0, c_1, \dots, c_{n-1})^r$ for $i, j = 1, 2, \dots, n$ and

$$C_{ij} = \begin{cases} c_{j-i} & , j \geq i \\ r.c_{n+j-i}, & j < i. \end{cases}$$

$C_r(c) = [C_{ij}]$ matrix is called $r$ −circulant matrix and shown in the form as

$$C_r(c) = \begin{bmatrix} c_0 & c_1 & c_2 & \cdots & c_{n-1} \\ rc_{n-1} & c_0 & c_1 & \cdots & c_{n-2} \\ rc_{n-2} & rc_{n-1} & c_0 & \cdots & c_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ rc_1 & rc_2 & rc_3 & \cdots & c_0 \end{bmatrix}$$

If $r = 1$ in $r$ −circulant matrices, circulant matrices are obtained.

BŞEÜ Fen Bilimleri Dergisi
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**Definition:** The sequence $(M_n)_{n\in\mathbb{N}}$ is called the Tetranacci sequence as

$$M_n = M_{n-1} + M_{n-2} + M_{n-3} + M_{n-4} \quad (n \geq 4)$$

where initial conditions for $M_0 = M_1 = M_2 = 0$, $M_3 = 1$ .The elements of this sequence are called Tetranacci numbers [2].

**Definition:** Binet formula for the Tetranacci sequence is

$$M_n = \frac{\alpha^n}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)} + \frac{\beta^n}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)} + \frac{\gamma^n}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)} + \frac{\delta^n}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)}$$

[23]. Here $\alpha, \beta, \gamma, \delta$ are the roots of the characteristic equation.

$$\alpha = \frac{1}{4} + \frac{1}{2}R + \frac{1}{2}\sqrt{\frac{11}{4} - R^2 + \frac{13}{4}R^{-1}}$$

$$\beta = \frac{1}{4} + \frac{1}{2}R - \frac{1}{2}\sqrt{\frac{11}{4} - R^2 + \frac{13}{4}R^{-1}}$$

$$\gamma = \frac{1}{4} - \frac{1}{2}R + \frac{1}{2}\sqrt{\frac{11}{4} - R^2 - \frac{13}{4}R^{-1}}$$

$$\delta = \frac{1}{4} - \frac{1}{2}R - \frac{1}{2}\sqrt{\frac{11}{4} - R^2 - \frac{13}{4}R^{-1}}$$

where

$$R = \sqrt{\frac{11}{12} + \left(-\frac{65}{54} + \sqrt{\frac{563}{108}}\right)^{\frac{1}{3}} + \left(-\frac{65}{54} - \sqrt{\frac{563}{108}}\right)^{\frac{1}{3}}}$$

**Definition**: The generating function for $(M_n)_{n\in\mathbb{N}}$ Tetranacci sequence is

$$M(x) = \sum_{n=0}^{\infty} M_n x^n = \frac{x^3}{1-x-x^2-x^3-x^4}$$

[4].

**Definition**: Let $A$ be matrix of type $nxn$. The maximum column and row total norm of matrix $A$ is defined as respectively

i. $\|A\|_1 = \max_{1\leq j\leq n} \sum_{i=1}^{n} |a_{ij}|$

ii. $\|A\|_\infty = \max_{1\leq i\leq n} \sum_{j=1}^{n} |a_{ij}|$ .

## III. MAIN SECTION

**Definition:** The sequence $(\mathcal{T}_n)_{n\in\mathbb{N}}$ is defined as the Generalized Tetranacci sequence with recurrence relation as

$$\mathcal{T}_n = p\mathcal{T}_{n-1} + q\mathcal{T}_{n-2} + r\mathcal{T}_{n-3} + s\mathcal{T}_{n-4} \quad (n \geq 4) \tag{1}$$

where initial conditions for $\mathcal{T}_0 = a$, $\mathcal{T}_1 = b$, $\mathcal{T}_2 = c$, $\mathcal{T}_3 = d$ and $1 - p - q - r - s \neq 0$.

**Theorem:** The generating function for $(\mathcal{T}_n)_{n\in\mathbb{N}}$ generalized Tetranacci numbers is

BİLECİK ŞEYH EDEBALİ ÜNİVERSİTESİ

BŞEÜ Fen Bilimleri Dergisi
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$$\mathcal{T}(x)=\frac{a+x(b-ap)+x^2(c-bp-aq)+x^3(d-cp-bq-ar)}{1-px-qx^2-rx^3-sx^4}$$

***Proof*:** Let the generating function of generalized Tetranacci numbers is

$$\mathcal{T}(x)=\sum_{n=0}^{\infty}\mathcal{T}_n x^n.$$

Then

$$(1-px-qx^2-rx^3-sx^4)\sum_{n=0}^{\infty}\mathcal{T}_n x^n=\sum_{n\geq 0}^{\infty}\mathcal{T}_n x^n-p\sum_{n\geq 1}^{\infty}\mathcal{T}_{n-1}x^n-q\sum_{n\geq 2}^{\infty}\mathcal{T}_{n-2}x^n$$

$$-r\sum_{n\geq 3}^{\infty}\mathcal{T}_{n-3}x^n-s\sum_{n\geq 4}^{\infty}\mathcal{T}_{n-4}x^n$$

$$=\mathcal{T}_0+\mathcal{T}_1x+\mathcal{T}_2x^2+\mathcal{T}_3x^3-p[\mathcal{T}_0x+\mathcal{T}_1x^2+\mathcal{T}_2x^3]$$

$$-q[\mathcal{T}_0x^2+\mathcal{T}_1x^3]-r\mathcal{T}_0x^3$$

$$-\sum_{n\geq 4}^{\infty}\{\mathcal{T}_n-p\mathcal{T}_{n-1}-q\mathcal{T}_{n-2}-r\mathcal{T}_{n-3}-s\mathcal{T}_{n-4}\}x^n$$

is obtained. From (1) and initial conditions,

$$(1-px-qx^2-rx^3-sx^4)\sum_{n=0}^{\infty}\mathcal{T}_n x^n=a+x(b-ap)+x^2(c-bp-aq)-x^3(d-cp-bq-ar\,)$$

is delivered. Then

$$\sum_{n=0}^{\infty}\mathcal{T}_n x^n=\frac{a+x(b-ap)+x^2(c-bp-aq)+x^3(d-cp-bq-ar)}{1-px-qx^2-rx^3-sx^4}.$$

Therefore, the generating function for $(\mathcal{T}_n)_{n\in\mathbb{N}}$ is

$$\mathcal{T}(x)=\frac{a+x(b-ap)+x^2(c-bp-aq)+x^3(d-cp-bq-ar)}{1-px-qx^2-rx^3-sx^4}.$$

Following table is about of the obtained generating functions of series of numbers are given for some special values in for initial conditions and coefficients.

**Table 1.** Generating Functions of Number Sequences

| ***a*** | ***b*** | ***c*** | ***d*** | ***p*** | ***q*** | ***r*** | ***s*** | **Generating Function of Series** |
|---|---|---|---|---|---|---|---|---|
| 0 | 1 | 1 | 2 | 1 | 1 | 0 | 0 | Fibonacci |
| 0 | 1 | 1 | 2 | 1 | 1 | 0 | 0 | Lucas |
| 0 | 1 | 2 | 6 | 2 | 1 | 0 | 0 | Pell |
| 0 | 1 | 1 | 3 | 1 | 2 | 0 | 0 | Jacobsthal |
| 0 | 1 | $k$ | $k^2+1$ | $k$ | 1 | 0 | 0 | $k$ –Fibonacci |
| 0 | 0 | 0 | 1 | 1 | 1 | 1 | 1 | Tetranacci |

**Theorem:** Let $\alpha,\beta,\gamma$ and $\delta$ are roots of characteristic equation of $1-px-qx^2-rx^3-sx^4=0$. Binet formula for $(\mathcal{T}_n)_{n\in\mathbb{N}}$ is obtained as

$$\mathcal{T}_n=\frac{A\alpha^n}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)}+\frac{B\beta^n}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)}+\frac{C\gamma^n}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)}+\frac{D\delta^n}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)}$$

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where

$$D = d - cp - bq - ar$$

$$C = (\gamma - \delta)[c - bp - aq] + D$$

$$B = -\frac{(b-ap)[(\gamma-\beta)(\gamma-\delta)(\delta-\beta)(\delta-\gamma)](\gamma-\delta)}{(\gamma-\delta)^2}$$

$$A = \frac{a(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)(\beta-\gamma)(\beta-\delta)(\gamma-\delta)+B(\alpha-\gamma)(\alpha-\delta)(\gamma-\delta)}{(\beta-\gamma)(\beta-\delta)(\gamma-\delta)} + \frac{-C(\alpha-\beta)(\alpha-\delta)(\beta-\delta)+D(\alpha-\beta)(\alpha-\gamma)(\beta-\gamma)}{(\beta-\gamma)(\beta-\delta)(\gamma-\delta)}$$

***Proof:***

Let $\alpha, \beta, \gamma$ and $\delta$ are roots of equation of $1 - px - qx^2 - rx^3 - sx^4 = 0$. Then, if simple fractionation method is applied

$$\mathcal{T}(x) = \frac{K}{(1-\alpha x)} + \frac{L}{(1-\beta x)} + \frac{M}{(1-\gamma x)} + \frac{N}{(1-\delta x)}$$

can be written.

$$\mathcal{T}(x) = \frac{a+x(b-ap)+x^2(c-bp-aq)+x^3(d-cp-bq-ar)}{1-px-qx^2-rx^3-sx^4}$$

$$= \frac{K(1-\beta x)(1-\gamma x)(1-\delta x)+L(1-\alpha x)(1-\gamma x)(1-\delta x)}{(1-\alpha x)(1-\beta x)(1-\gamma x)(1-\delta x)} + \frac{M(1-\alpha x)(1-\beta x)(1-\delta x)+N(1-\alpha x)(1-\beta x)(1-\gamma x)}{(1-\alpha x)(1-\beta x)(1-\gamma x)(1-\delta x)}$$

From above equality of terms of the same, we reach the system of linear equations as

$$\{K + L + M + N\} = a$$

$$\begin{Bmatrix} -K(\beta+\gamma+\delta) - L(\alpha+\gamma+\delta) \\ -M(\alpha+\beta+\delta) - N(\alpha+\beta+\gamma) \end{Bmatrix} = b - ap$$

$$\begin{Bmatrix} K(\beta\gamma+\gamma\delta+\beta\delta) + L(\alpha\gamma+\gamma\delta+\alpha\delta) \\ +M(\alpha\beta+\alpha\delta+\beta\delta) + N(\alpha\beta+\alpha\gamma+\beta\gamma) \end{Bmatrix} = c - bp - aq$$

$$\{-K\beta\gamma\delta - L\alpha\gamma\delta - M\alpha\beta\delta - N\alpha\beta\gamma\} = d - cp - bq - ar$$

Then we obtain

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ -(\beta+\gamma+\delta) & -(\alpha+\gamma+\delta) & -(\alpha+\beta+\delta) & -(\alpha+\beta+\gamma) \\ \beta\gamma+\gamma\delta+\beta\delta & \alpha\gamma+\chi\delta+\alpha\delta & \alpha\beta+\alpha\delta+\beta\delta & \alpha\beta+\alpha\gamma+\beta\gamma \\ -\beta\gamma\delta & -\alpha\gamma\delta & -\alpha\beta\delta & -\alpha\beta\gamma \end{bmatrix} \begin{bmatrix} K \\ L \\ M \\ N \end{bmatrix} = \begin{bmatrix} a \\ b-ap \\ c-bp-aq \\ d-cp-bq-ar \end{bmatrix}.$$

This system is reduced to following system

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & \beta-\alpha & \gamma-\alpha & \delta-\alpha \\ 0 & 0 & \gamma^2+\alpha\beta-\alpha\gamma-\beta\gamma & \delta^2+\alpha\beta-\alpha\delta-\beta\delta \\ 0 & 0 & 0 & \delta^3-\alpha\delta^2-\beta\delta^2-\alpha\beta\gamma+\alpha\beta\delta+\alpha\gamma\delta+\beta\gamma\delta \end{bmatrix} \begin{bmatrix} K \\ L \\ M \\ N \end{bmatrix} = \begin{bmatrix} a \\ b-ap \\ c-bp-aq \\ d-cp-bq-ar \end{bmatrix}$$

BŞEÜ Fen Bilimleri Dergisi
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and from there

$$N = \frac{d-cp-bq-ar}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)}$$

is delivered. Let $D = d - cp - bq - ar$ and $N$ can be rewritten as

$$N = \frac{D}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)}$$

Similarly, after necessary corrections using $N$ value

$$M = \frac{(\gamma-\delta)[c-bp-aq]+D}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)}$$

is obtained. Using M and after similar regulations, L is obtained as

$$L = \frac{B}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)}$$

where

$$B = -\frac{(b-ap)(\gamma-\beta)(\gamma-\delta)(\delta-\beta)(\delta-\gamma)}{(\gamma-\delta)^2} - \frac{C(\delta-\beta)(\delta-\gamma)+D(\gamma-\beta)(\gamma-\delta)}{(\gamma-\delta)^2}$$

and

$$C = (\gamma-\delta)[c-bp-aq]+D$$

Because of $K = a - L - M - N$ and if $L, M, N$ values are written and necessary corrections are made, $K$ is obtained as

$$K = \frac{a(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)(\beta-\gamma)(\beta-\delta)(\gamma-\delta)+B(\alpha-\gamma)(\alpha-\delta)(\gamma-\delta)}{(\beta-\gamma)(\beta-\delta)(\gamma-\delta)(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)} + \frac{-C(\alpha-\beta)(\alpha-\delta)(\beta-\delta)+D(\alpha-\beta)(\alpha-\gamma)(\beta-\gamma)}{(\beta-\gamma)(\beta-\delta)(\gamma-\delta)(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)}$$

where

$$A = \frac{a(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)(\beta-\gamma)(\beta-\delta)(\gamma-\delta)+B(\alpha-\gamma)(\alpha-\delta)(\gamma-\delta)}{(\beta-\gamma)(\beta-\delta)(\gamma-\delta)} + \frac{-C(\alpha-\beta)(\alpha-\delta)(\beta-\delta)+D(\alpha-\beta)(\alpha-\gamma)(\beta-\gamma)}{(\beta-\gamma)(\beta-\delta)(\gamma-\delta)}.$$

Then $K$ is rewritten as

$$K = \frac{A}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)}$$

Therefore;

$$\mathcal{T}(x) = \frac{A}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)}\frac{1}{(1-\alpha x)} + \frac{B}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)}\frac{1}{(1-\beta x)} + \frac{C}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)}\frac{1}{(1-\gamma x)} + \frac{D}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)}\frac{1}{(1-\delta x)}$$

is delivered. So,

$$\mathcal{T}(x) = \sum_{n=0}^{\infty}\{\frac{A\alpha^n}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)} + \frac{B\beta^n}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)} + \frac{C\gamma^n}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)} + \frac{D\delta^n}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)}\}x^n$$

can be written. From equality of coefficients

$$\mathcal{T}_n = \frac{A\alpha^n}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)} + \frac{B\beta^n}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)} + \frac{C\gamma^n}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)} + \frac{D\delta^n}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)}$$

is obtained. This formula is the Binet Formula of the Generalized Tetranacci Sequence.

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In the function $\mathcal{T}_n$ in equation (1), if some specific values are selected for initial conditions and coefficients, the Binet formulas of the number sequences available in the literature are reached. Following table is about to these special values.

**Table 2.** Binet Formulas of Number Sequences

| $\mathcal{T}_0$ | $\mathcal{T}_1$ | $\mathcal{T}_2$ | $\mathcal{T}_3$ | $p$ | $q$ | $r$ | $s$ | Recurrence Relation | Binet Formula |
|---|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 1 | 1 | 1 | 1 | 1 | $M_n = M_{n-1} + M_{n-2} + M_{n-3} + M_{n-4}$ | $M_n = \frac{\alpha^n}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)} + \frac{\beta^n}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)} + \frac{\gamma^n}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)} + \frac{\delta^n}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)}$ [18] |
| 0 | 0 | 1 | | 1 | 1 | 1 | 0 | $T_n = T_{n-1} + T_{n-2} + T_{n-3}$ | $T_n = \frac{\alpha^{n+1}}{(\alpha-\beta)(\alpha-\gamma)} + \frac{\beta^{n+1}}{(\beta-\alpha)(\beta-\gamma)} + \frac{\gamma^{n+1}}{(\gamma-\alpha)(\gamma-\beta)}$ [24]. |
| 0 | 1 | | | 1 | 1 | 0 | 0 | $F_n = F_{n-1} + F_{n-2}$ | $F_n = \frac{\alpha^n-\beta^n}{\alpha-\beta}$ |

**Theorem:** Let $\theta = p+q+r+s$ and $1-\theta \neq 0$ ,

$$\sum_{i=4}^{n} \mathcal{T}_i = \frac{1}{1-\theta}\{-\theta\mathcal{T}_n - (\theta-p)\mathcal{T}_{n-1} - (r+s)\mathcal{T}_{n-2} - s\mathcal{T}_{n-3} + s\mathcal{T}_0 + (r+s)\mathcal{T}_1 + (\theta-p)\mathcal{T}_2 + \theta\mathcal{T}_3\} \quad (2)$$

***Proof:***

From (1) ,

$$\mathcal{T}_n - p\,\mathcal{T}_{n-1} = q\mathcal{T}_{n-2} + r\mathcal{T}_{n-3} + s\mathcal{T}_{n-4}$$

is obtained. After some operations and necessary regulations, following equation is delivered.

$$\mathcal{T}_4(1-p-q-r-s) + \mathcal{T}_5(1-p-q-r-s) + \cdots + \mathcal{T}_{n-4}(1-p-q-r-s) + \mathcal{T}_{n-3}(1-p-q-r) + \mathcal{T}_{n-2}(1-p-q) + \mathcal{T}_{n-1}(1-p) + \mathcal{T}_n = \mathcal{T}_3(p+q+r+s) + \mathcal{T}_2(q+r+s) + \mathcal{T}_1(r+s) + s\mathcal{T}_0$$

Then, above equation is formulated by

$$\sum_{i=4}^{n}(1-p-q-r-s)\mathcal{T}_i = -s\mathcal{T}_{n-3} - (r+s)\mathcal{T}_{n-2} - (q+r+s)\mathcal{T}_{n-1} - (p+q+r+s)\mathcal{T}_n + s\mathcal{T}_0 + (r+s)\mathcal{T}_1 + (q+r+s)\mathcal{T}_2 + (p+q+r+s)\mathcal{T}_3.$$

Therefore

$$\sum_{i=4}^{n} \mathcal{T}_i = \frac{1}{1-\theta}\{-\theta\mathcal{T}_n - (\theta-p)\mathcal{T}_{n-1} - (r+s)\mathcal{T}_{n-2} - s\mathcal{T}_{n-3} + s\mathcal{T}_0 + (r+s)\mathcal{T}_1 + (\theta-p)\mathcal{T}_2 + \theta\mathcal{T}_3\}$$

is obtained.

**Theorem:** Let $A = circ(\mathcal{T}_n)$ be defined as

$$\begin{bmatrix} \mathcal{T}_0 & \mathcal{T}_1 & \mathcal{T}_2 & \dots & \mathcal{T}_{n-1} \\ \mathcal{T}_{n-1} & \mathcal{T}_0 & \mathcal{T}_1 & \dots & \mathcal{T}_{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathcal{T}_1 & \mathcal{T}_2 & \mathcal{T}_3 & \dots & \mathcal{T}_3 \end{bmatrix}$$

and $1-\theta \neq 0$ The maximum column total norm of matrix $A$ is

$$\|A\|_1 = \frac{1}{1-\theta}\{-\theta\mathcal{T}_{n-1} - (\theta-p)\mathcal{T}_{n-2} - (r+s)\mathcal{T}_{n-3} - s\mathcal{T}_{n-4} + (1-p-q-r)\mathcal{T}_0 + (1-p-q)\mathcal{T}_1 + (1-p)\mathcal{T}_2 + \mathcal{T}_3\} \quad (3)$$

***Proof:*** From definition of maximum total column norm, we write

$$\|A\|_1 = \max_{1\le j\le n} \sum_{i=1}^{n}|a_{ij}| = \sum_{i=0}^{n-1}\mathcal{T}_i$$

and

$$\sum_{i=0}^{n-1}\mathcal{T}_i = \mathcal{T}_0 + \mathcal{T}_1 + \mathcal{T}_2 + \mathcal{T}_3 + \sum_{i=4}^{n-1}\mathcal{T}_i.$$

From (2),

$$\sum_{i=0}^{n-1}\mathcal{T}_i = \frac{1}{1-\theta}\{-\theta\mathcal{T}_{n-1} - (\theta-p)\mathcal{T}_{n-2} - (r+s)\mathcal{T}_{n-3} - s\mathcal{T}_{n-4}\} + \left(\frac{r+s}{1-\theta}+1\right)\mathcal{T}_1 + \left(\frac{\theta-p}{1-\theta}+1\right)\mathcal{T}_2 + \left(\frac{\theta}{1-\theta}+1\right)\mathcal{T}_3.$$

Thus we obtain as

$$\|A\|_1 = \frac{1}{1-\theta}\{-\theta\mathcal{T}_{n-1} - (\theta-p)\mathcal{T}_{n-2} - (r+s)\mathcal{T}_{n-3} - s\mathcal{T}_{n-4} + (1-p-q-r)\mathcal{T}_0 + (1-p-q)\mathcal{T}_1 + (1-p)\mathcal{T}_2 + \mathcal{T}_3\}.$$

**Theorem:** Let $A = circ(\mathcal{T}_n)$ . The maximum row total norm of matrix $A$ is

$$\|A\|_\infty = \frac{1}{1-\theta}\{-\theta\mathcal{T}_{n-1} - (\theta-p)\mathcal{T}_{n-2} - (r+s)\mathcal{T}_{n-3} - s\mathcal{T}_{n-4} + (1-p-q-r)\mathcal{T}_0 + (1-p-q)\mathcal{T}_1 + (1-p)\mathcal{T}_2 + \mathcal{T}_3\}.$$

***Proof:***

From (2) and (3),

$$\|A\|_\infty = \|A\|_1 = \frac{1}{1-\theta}\{-\theta\mathcal{T}_{n-1} - (\theta-p)\mathcal{T}_{n-2} - (r+s)\mathcal{T}_{n-3} - s\mathcal{T}_{n-4} + (1-p-q-r)\mathcal{T}_0 + (1-p-q)\mathcal{T}_1 + (1-p)\mathcal{T}_2 + \mathcal{T}_3\}$$

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