Results in Nonlinear Analysis 2 (2019) No. 4, 193–199
Available online at www.nonlinear-analysis.com

Results in Nonlinear Analysis
ISSN 2636-7556

Peer Reviewed Scientific Journal

Series Form Solution to Two Dimensional Heat Equation of Fractional Order

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Abstract

In this article we develop series type solution to two dimensional wave equation involving external source term of fractional order. For the require result, we use iterative Laplace transform. The solution is computed in series form which is rapidly convergent to exact value. Some examples are given to illustrate the establish results.

Keywords: Two dimensional wave equation, Series solution, Iterative Laplace transform.
2010 MSC: 26A33; 34A08; 35R11.

1. Introduction

In last few decades the area of fractional calculus has been given much attentions by the researchers. This is because of the significant applications and accurate and realistic description of many physical and biological phenomenon of real world. As fractional differential operator is a global operator which provides greater degree of freedom. Therefore the concerned area has been given much attentions and plenty of research articles, monograph, books, etc address

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Received October 30, 2019 Accepted: November 27, 2019, Online: December 25, 2019.

the said area in different aspects [8, 5, 2, 12]. The so far aspects been investigated for arbitrary order differential equations are qualitative theory and numerical analysis. In this regards plenty of research articles, books are available.

To deal the aforesaid differential equations of fractional superluxurious tools and methods were formed in past. These tools including integral transform of Fourier, Laplace, Sumudu, etc. Laplace transform is a powerful transform to solve many linear problems of fractional differential equations, see for detail [4, 6, 7]. Therefore in this paper we are going to compute series type solution by iterative method using Laplace transform for the given two dimensional wave equations

$$\begin{aligned} \mathcal{D}_t^\theta \mathrm{u}(x,y,t) &= \mathcal{D}_x^2 \mathrm{u}(x,y,t) + \mathcal{D}_y^2 \mathrm{u}(x,y,t) + f(x,y,t),\ 0 < \theta \leq 1, \\ \mathrm{u}(x,y,0) = &= g(x,y), \end{aligned} \tag{1.1}$$

where $\mathcal{D}$ stands for Caputo fractional derivative and

$$f \in C([0,\infty) \times [0,\infty) \times [0,\infty), [0,\infty)),\ g \in ([0,\infty) \times [0,\infty), [0,\infty)).$$

The heat equation has many application and due to this it has been studied for different dimensions by using various techniques lik Laplace transform, Fourier transform, etc, see for detail [15, 1, 9, 14, 10, 11]. In [3], the authors studied two dimensional heat equation via double Laplace transform in the absence of external source function $f(x,y,t)$. Here we investigate said equation of two dimension in the presence of external source function via Laplace transform. Various numerical examples and their plots are given to discuss the required analysis.

2. Background Materials

Here we recall some definition from [8, 5, 4].

Definition 2.1. For a function in three variables say $u(x,y,t)$ we define fractional integral corresponding to t as

$$\mathbf{I}_t^\theta \mathrm{u}(x,y,t) = \frac{1}{\Gamma(\theta)} \int_0^t (t-\eta)^{\theta-1} d\eta,\ \theta > 0,$$

such that integral on right exists.

Definition 2.2. For a function in three variables say $u(x,y,t)$ we define fractional partial derivative corresponding to t as

$$\mathcal{D}_t^\theta \mathrm{u}(x,y,t) = \frac{1}{\Gamma(n-\theta)} \int_0^t (t-\eta)^{n-\theta-1} \frac{\partial^n}{\partial \eta^n}[\mathrm{u}(x,y,\eta)] d\eta,\ \theta > 0,$$

such that integral on right exists and $n = [\theta] + 1$. If $\theta \in (0,1]$, then one has

$$\mathcal{D}_t^\theta \mathrm{u}(x,y,t) = \frac{1}{\Gamma(1-\theta)} \int_0^t (t-\eta)^{-\theta} \frac{\partial}{\partial \eta}[\mathrm{u}(x,y,\eta)] d\eta.$$

Lemma 1. *The Laplace transform of* $\mathcal{D}_t^\theta \mathrm{u}(x,y,t)$ *is defined as*

$$\mathcal{L}[\mathcal{D}_t^\theta \mathrm{u}(x,y,t)] = s^\theta \mathcal{L}[\mathrm{u}(x,y,t)] - \sum_{k=0}^{n-1} s^{\theta-k-1} \frac{\partial^k}{\partial t^k}[\mathrm{u}(x,y,0)].$$

3. Main work

Here we apply the Laplace transform to (4.5) as

$$\mathcal{L}[\mathcal{D}_t^{\theta}\mathrm{u}(x,y,t)] = \mathcal{L}[\mathcal{D}_x^2\mathrm{u}(x,y,t)+\mathcal{D}_y^2\mathrm{u}(x,y,t)+f(x,y,t)], \tag{3.1}$$

on using initial condition we have

$$\begin{aligned} s^{\theta}\mathcal{L}[\mathrm{u}(x,y,t)] &= s^{\theta-1}g(x,y)+\mathcal{L}[\mathcal{D}_x^2\mathrm{u}(x,y,t)+\mathcal{D}_y^2\mathrm{u}(x,y,t)+f(x,y,t)], \\ \mathcal{L}[\mathrm{u}(x,y,t)] &= \frac{g(x,y)}{s}+\frac{1}{s^{\theta}}\mathcal{L}\Big[\mathcal{D}_x^2\mathrm{u}(x,y,t)+\mathcal{D}_y^2\mathrm{u}(x,y,t)+f(x,y,t)\Big]. \end{aligned} \tag{3.2}$$

Let we assume the solution as $\mathrm{u}(x,y,t)=\sum_{n=0}^{\infty}\mathrm{u}_n(x,y,t)$, then (3.2) gives

$$\mathcal{L}[\sum_{n=0}^{\infty}\mathrm{u}_n(x,y,t)] = \frac{g(x,y)}{s}+\frac{1}{s^{\theta}}\mathcal{L}\Big[\mathcal{D}_x^2\sum_{n=0}^{\infty}\mathrm{u}_n(x,y,t)+\sum_{n=0}^{\infty}\mathrm{u}_n(x,y,t)+f(x,y,t)\Big]. \tag{3.3}$$

Comparing terms on both sides, we have

$$\begin{aligned} \mathcal{L}[\mathrm{u}_0(x,y,t)] &= \frac{g(x,y)}{s}+\frac{1}{s^{\theta}}\mathcal{L}\Big[f(x,y,t)\Big], \\ \mathcal{L}[\mathrm{u}_1(x,y,t)] &= \frac{1}{s^{\theta}}\mathcal{L}\Big[\mathcal{D}_x^2\mathrm{u}_0(x,y,t)+\mathcal{D}_y^2\mathrm{u}_0(x,y,t)\Big], \\ \mathcal{L}[\mathrm{u}_2(x,y,t)] &= \frac{1}{s^{\theta}}\mathcal{L}\Big[\mathcal{D}_x^2\mathrm{u}_1(x,y,t)+\mathcal{D}_y^2\mathrm{u}_(x,y,t)\Big], \\ &\vdots \\ \mathcal{L}[\mathrm{u}_{n+1}(x,y,t)] &= \frac{1}{s^{\theta}}\mathcal{L}\Big[\mathcal{D}_x^2\mathrm{u}_n(x,y,t)+\mathcal{D}_y^2\mathrm{u}_n(x,y,t)\Big],\ n\geq 0. \end{aligned} \tag{3.4}$$

Evaluating inverse Laplace transform, we have

$$\begin{aligned} \mathrm{u}_0(x,y,t) &= g(x,y)+\mathcal{L}^{-1}\Big[\frac{1}{s^{\theta}}\mathcal{L}\Big[f(x,y,t)\Big]\Big], \\ \mathrm{u}_1(x,y,t) &= \mathcal{L}^{-1}\Big[\frac{1}{s^{\theta}}\mathcal{L}\Big[\mathcal{D}_x^2\mathrm{u}_0(x,y,t)+\mathcal{D}_y^2\mathrm{u}_0(x,y,t)\Big]\Big], \\ &\vdots \\ \mathrm{u}_{n+1}(x,y,t) &= \mathcal{L}^{-1}\Big[\frac{1}{s^{\theta}}\mathcal{L}\Big[\mathcal{D}_x^2\mathrm{u}_n(x,y,t)+\mathcal{D}_y^2\mathrm{u}_n(x,y,t)\Big]\Big],\ n\geq 0. \end{aligned} \tag{3.5}$$

Hence the required series solution is given by

$$\mathrm{u}(x,y,t)=\mathrm{u}_0(x,y,t)+\mathrm{u}_1(x,y,t)+\mathrm{u}_2(x,y,t)+\cdots. \tag{3.6}$$

This is an infinite series which is convergent as already proved in [13].

4. Examples

In this section, we present three examples of two dimensional wave fractional order differential equations, then the proposed method are applied to obtain the approximate results.

Example 4.1. Consider the given heat equation under the initial condition as

$$\begin{aligned} \mathcal{D}_t^{\theta}\mathrm{u}(x,y,t) &= \mathcal{D}_x^2\mathrm{u}(x,y,t)+\mathcal{D}_y^2\mathrm{u}(x,y,t)+x+y+1,\ 0<\theta\leq 1,\\ \mathrm{u}(x,y,0) = &= \exp(-(x+y)). \end{aligned} \tag{4.1}$$

Applying the above mention method as in (3.5) step by step, we get the following terms of the series solution

$$\begin{aligned} \mathrm{u}_0(x,y,t) &= \exp[-(x+y)]+(x+y+1)\frac{t^{\theta}}{\Gamma(\theta+1)},\\ \mathrm{u}_1(x,y,t) &= 2\exp[-(x+y)]\frac{t^{\theta}}{\Gamma(\theta+1)},\\ \mathrm{u}_2(x,y,t) &= 4\exp[-(x+y)]\frac{t^2\theta}{\Gamma(2\theta+1)},\\ \mathrm{u}_3(x,y,t) &= 8\exp[-(x+y)]\frac{t^3\theta}{\Gamma(3\theta+1)},\\ \mathrm{u}_4(x,y,t) &= 16\exp[-(x+y)]\frac{t^4\theta}{\Gamma(4\theta+1)} \end{aligned} \tag{4.2}$$

and so on and other terms may be calculated in this way. Hence the required series solution is written in the form of infinite series as

$$\begin{aligned} \mathrm{u}(x,y,t) &= \exp[-(x+y)]+(x+y+1)\frac{t^{\theta}}{\Gamma(\theta+1)}+2\exp[-(x+y)]\frac{t^{\theta}}{\Gamma(\theta+1)}\\ &+ 4\exp[-(x+y)]\frac{t^2\theta}{\Gamma(2\theta+1)}+8\exp[-(x+y)]\frac{t^3\theta}{\Gamma(3\theta+1)}\\ &+ 16\exp[-(x+y)]\frac{t^4\theta}{\Gamma(4\theta+1)}+\dots. \end{aligned}$$

Here we plot approximate series solutions up to four terms corresponding to different fractional order at $t=0.5$ as given in Figure 1.

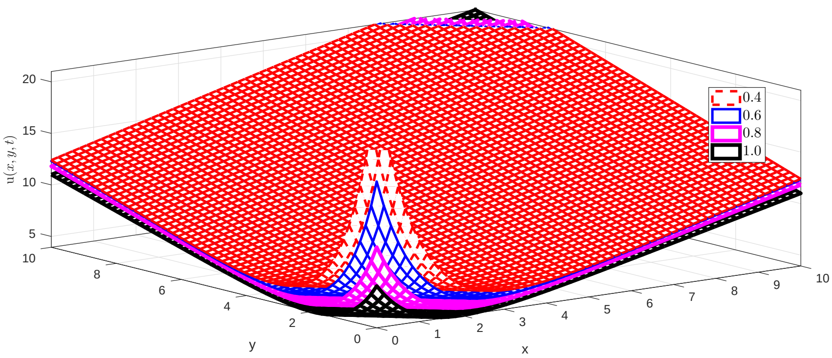

Fig. 1 Plot of approximate solution in 3D corresponding to different fractional order θ and at given values of $t = 0.5$ of Example 4.1.

Example 4.2. Let us take the given heat equation under the initial condition as

$$\begin{aligned} \mathcal{D}_t^{\theta}\mathrm{u}(x,y,t) &= \mathcal{D}_x^2\mathrm{u}(x,y,t) + \mathcal{D}_y^2\mathrm{u}(x,y,t) + x + y + t^2,\ 0 < \theta \leq 1, \\ \mathrm{u}(x,y,0) = &= \sin(x+y). \end{aligned} \tag{4.3}$$

Applying the above mention method step by step we get the following terms

$$\begin{aligned} \mathrm{u}_0(x,y,t) &= \sin(x+y) + (x+y)\frac{t^{\theta}}{\Gamma(\theta+1)} + \frac{t^{\theta+1}}{\Gamma(\theta+2)}, \\ \mathrm{u}_1(x,y,t) &= -2\sin(x+y)\frac{t^{\theta}}{\Gamma(\theta+1)}, \\ \mathrm{u}_2(x,y,t) &= 4\sin(x+y)\frac{t^{2}\theta}{\Gamma(2\theta+1)}, \\ \mathrm{u}_3(x,y,t) &= -8\sin(x+y)\frac{t^{3}\theta}{\Gamma(3\theta+1)} \end{aligned} \tag{4.4}$$

and so on and the other terms may be computed in this way.
The obtained solution is written in the form of infinite series as

$$\begin{aligned} \mathrm{u}(x,y,t) &= \sin(x+y) + (x+y)\frac{t^{\theta}}{\Gamma(\theta+1)} + \frac{t^{\theta}+1}{\Gamma(\theta+2)} - 2\sin(x+y)\frac{t^{\theta}}{\Gamma(\theta+1)} \\ &+ 4\sin(x+y)\frac{t^{2}\theta}{\Gamma(2\theta+1)} - 8\sin(x+y)\frac{t^{3}\theta}{\Gamma(3\theta+1)} + \dots. \end{aligned}$$

The plot of first four term of approximate solution is given in Figure 2.

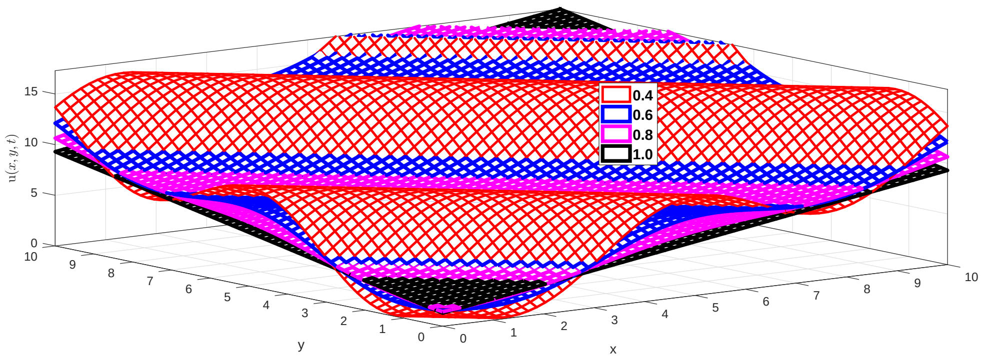

Fig. 2 Plot of approximate solution in 3D corresponding to different fractional order θ and at given values of $t = 0.5$ of Example 4.2.

Example 4.3. Consider the given heat equation under the initial condition as

$$\begin{aligned} \mathcal{D}_t^{\theta}\mathrm{u}(x,y,t) &= \mathcal{D}_x^2\mathrm{u}(x,y,t) + \mathcal{D}_y^2\mathrm{u}(x,y,t) + x + y + t^4,\ 0 < \theta \leq 1, \\ \mathrm{u}(x,y,0) = &= \exp(x+y)\sin(x+y). \end{aligned} \tag{4.5}$$

Applying the above mention method step by step we get the following terms

$$\begin{aligned} \mathrm{u}_0(x,y,t) &= \exp(x+y)\sin(x+y) + (x+y)\frac{t^{\theta}}{\Gamma(\theta+1)} + \frac{t^{\theta+4}}{\Gamma(\theta+5)}, \\ \mathrm{u}_1(x,y,t) &= 4\exp(x+y)\cos(x+y)\frac{t^{\theta}}{\Gamma(\theta+1)}, \qquad (4.6) \\ \mathrm{u}_2(x,y,t) &= -16\exp(x+y)\sin(x+y)\frac{t^2\theta}{\Gamma(2\theta+1)}, \\ \mathrm{u}_3(x,y,t) &= -64\exp(x+y)\cos(x+y)\frac{t^3\theta}{\Gamma(3\theta+1)}, \\ \mathrm{u}_4(x,y,t) &= 128\exp(x+y)\sin(x+y)\frac{t^4\theta}{\Gamma(4\theta+1)} \qquad (4.7) \end{aligned}$$

and so on. The other terms may be computed in this way.
The obtained solution is written in the form of infinite series from (4.6) as

$$\begin{aligned} \mathrm{u}(x,y,t) &= 4\exp(x+y)\cos(x+y)\frac{t^{\theta}}{\Gamma(\theta+1)} - 16\exp(x+y)\sin(x+y)\frac{t^2\theta}{\Gamma(2\theta+1)} \\ &- 64\exp(x+y)\cos(x+y)\frac{t^3\theta}{\Gamma(3\theta+1)} + 128\exp(x+y)\sin(x+y)\frac{t^4\theta}{\Gamma(4\theta+1)} + \dots. \end{aligned}$$

The plot of first four term of approximate solution is given in Figure 3.

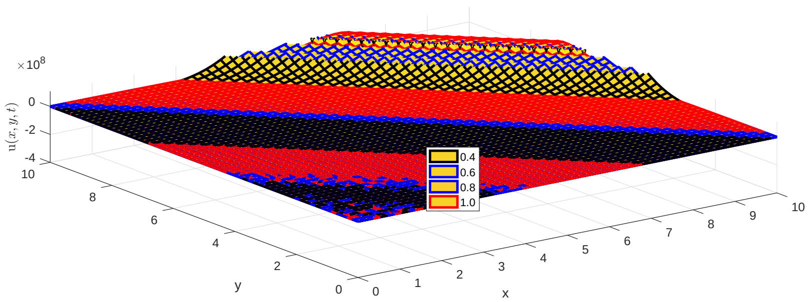

Fig. 3 Plot of approximate solution in 3D corresponding to different fractional order θ and at given values of $t = 1.0$ of Example 4.3.

5. Conclusion and Discussion

We have successfully established series type solutions to general two dimensions heat equations in the presence of external source term. The obtained results have been testified by an interesting example. Also we have given plots of the approximate solutions at various fractional orders. We have ploted approximate solutions for all three examples in Figures 1-3 respectively. From the Figures it is clear that as the fractional order θ is tending to its integer value the plots are going to close with the curve at classical order 1.

Acknowledgment

This research work has been supported by HED under grant No: HEREF-46 and HEC of Pakistan under grant No: NRPU-10039.

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