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On Fibonacci Ideal Convergence of Double Sequences in Intuitionistic Fuzzy Normed Linear Spaces

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Abstract. The aim of this article is to introduce and study the notion of Fibonacci $\mathcal{I}_2$-convergence on intuitionistic fuzzy normed linear space. We define the Fibonacci $\mathcal{I}_2$-Cauchy sequences and the Fibonacci $\mathcal{I}_2$ completeness with respect to an intuitionistic fuzzy normed linear space.

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1. Introduction and Background

Statistical convergence of sequences of points was introduced by Fast [12]. In [46], Schoenberg established some basic properties of statistical convergence and also studied the concept as a summability method.

The notion of $\mathcal{I}$-convergence was studied at initial stage by Kostyrko et al. [35]. They gave some of basic properties of $\mathcal{I}$-convergence and dealt with extremal $\mathcal{I}$-limit points. Later on it was studied by Salat et al. [45], Tripathy and Hazarika [48] and many others.

Fridy and Orhan [14] introduced the concept of lacunary statistical convergence. Some work on lacunary statistical convergence can be found in [5, 15, 36, 38]. The notion of lacunary ideal convergence of real sequences was introduced in [22, 49].

Fibonacci sequences were initiated in the book Liber Abaci of Fibonacci. The sequence had been described earlier as Virahanka numbers in Indian mathematics [18]. In Liber Abaci, the sequence starts with 1, nowadays the sequence begins either with $f_0 = 0$ or with $f_1 = 1$. The numbers in the bottom row are called Fibonacci numbers, and the number sequence $1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, ...$ is the Fibonacci sequence [34]. The Fibonacci numbers are a sequence of numbers (f_n) for $n = 1, 2,...$ defined by the linear recurrence equation $f_n = f_{n+1} - f_{n-2}$, $n \geq 2$.

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Some of the fundamental properties of Fibonacci numbers are given as follows:

$$\lim_{n\to\infty} \frac{f_{n+1}}{f_n} = \frac{1+\sqrt{5}}{2} = \alpha, \text{ (Golden ratio)}$$

$$\sum_{k=0}^{n} f_k = f_{n+2} - 1 \ \ (n \in \mathbb{N}),$$

$$\sum_{k} \frac{1}{f_k} \text{ converges,}$$

$$f_{n-1}f_{n+1} - f_n^2 = (-1)^{n+1} \ \ (n \geq 1) \ \text{(Cassini formula)}$$

It yields $f_{n-1}^2 + f_n f_{n-1} - f_n^2 = (-1)^{n+1}$, if we can substituting for f_{n+1} in Cassini's formula.

The Fibonacci sequence was firstly used in the theory of sequence spaces by Kara and Başarır [27]. Afterward, Kara [28] defined the Fibonacci difference matrix $\widehat{F}$ by using the Fibonacci sequence (f_n) for $n \in \{1, 2, 3, ...\}$ and introduced the new sequence spaces related to the matrix domain of $\widehat{F}$. In [31], the definition of statistical convergence with Fibonacci sequence was given and also various approximation results concerning the classical Korovkin theorem via Fibonacci type statistical convergence were proved.

Let f_n be the nth Fibonacci number for every $n \in \mathbb{N}$. Then, we define the infinite matrix $\widehat{F} = (\widehat{f_{nk}})$ [28] by

$$\widehat{f_{nk}} = \begin{cases} -\frac{f_{n+1}}{f_n}, & (k = n-1) \\ \frac{f_n}{f_{n+1}}, & (k = n) \\ 0, & (0 \leq k < n-1 \text{ or } k > n) \end{cases}$$

Following the introduction of fuzzy set theory by Zadeh [51], there has been extensive research to find applications and fuzzy analogues of the classical theories. Fuzzy logic has become an important area of research in various branches of mathematics such as metric and topological spaces [11,16,26], theory of functions [25,50] and approximation theory [1]. Fuzzy set theory has also found applications for modeling uncertainty and vagueness in various fields of science and engineering, such as computer programming [17], nonlinear dynamical systems [23], population dynamics [7], control of chaos [13], and quantum physics [37].

Hazarika [19, 20] introduced the lacunary ideal convergent sequences of fuzzy real numbers and studied some basic properties of this notion. Hazarika [21] introduced the notion of lacunary ideal convergent double sequences of fuzzy real numbers. Bakery and Mohammed [6] introduced lacunary mean ideal convergence in generalized random n-normed spaces.

As a generalization of fuzzy sets, the concept of intuitionistic fuzzy sets was introduced by Atanassov [3] in 1986, it has been extensively used in decision-making problems [4] and in E-infinity theory of high-energy physics [10]. The concept of an intuitionistic fuzzy metric space was introduced by Park [43]. Furthermore, Saadati and Park [44] gave the notion of an intuitionistic fuzzy normed space. Karakus et al [29] defined statistical convergence in intuitionistic fuzzy normed space (IFNS for short) and Mursaleen et al [39] investigated statistical convergence of double sequences in IFNS. Some works related to the convergence of sequences in several normed linear spaces in a fuzzy setting can be found in [8, 24, 32, 33, 40–42, 47].

Kirişci [30], studied the concept of Fibonacci statistical convergence on intuitionisitic fuzzy normed space. He defined the Fibonacci statistically Cauchy sequences with respect to an intuitionisitic fuzzy normed space and introduced the Fibonacci statistical completenes with respect to an intuitionisitic fuzzy normed space.

First we remind some basic definitions used in the paper.

A family of sets $\mathcal{I} \subseteq 2^{\mathbb{N}}$ is called an ideal if and only if

(i) for each $A, B \in \mathcal{I}$ we have $A \cup B \in \mathcal{I}$,

(ii) for each $A \in \mathcal{I}$ and each $B \subseteq A$ we have $B \in \mathcal{I}$.

An ideal is called non-trivial if $\mathbb{N} \notin \mathcal{I}$ and non-trivial ideal is called admissible if $\{n\} \in \mathcal{I}$ for each $n \in \mathbb{N}$.

A nontrivial ideal $\mathcal{I}_2$ of $\mathbb{N} \times \mathbb{N}$ is called strongly admissible if $\{i\} \times \mathbb{N}$ and $\mathbb{N} \times \{i\}$ belong to $\mathcal{I}_2$ for each $i \in \mathbb{N}$. It is evident that a strongly admissible ideal is admissible also. Throughout the paper we take $\mathcal{I}_2$ as a strongly admissible ideal in $\mathbb{N} \times \mathbb{N}$.

Let

$$\mathcal{I}_2^0 = \{A \subset \mathbb{N} \times \mathbb{N} : \exists m(A) \in \mathbb{N} \text{ such that } i, j \geq m(A) \ \Rightarrow \ (i, j) \notin A\}.$$

Then, $\mathcal{I}_2^0$ is a nontrivial strongly admissible ideal and clearly an ideal $\mathcal{I}_2$ is strongly admissible if and only if $\mathcal{I}_2^0 \subset \mathcal{I}_2$.

A family of sets $\mathcal{F} \subseteq 2^{\mathbb{N}}$ is a filter in $\mathbb{N}$ if and only if

(i) $\emptyset \notin \mathcal{F}$,

(ii) for each $A, B \in \mathcal{F}$ we have $A \cap B \in \mathcal{F}$,

(iii) for each $A \in \mathcal{F}$ and each $B \supseteq A$ we have $B \in \mathcal{F}$.

If $\mathcal{I}$ is a nontrivial ideal in $\mathbb{N}$ (i.e., $\mathbb{N} \notin \mathcal{I}$), then the family of sets

$$\mathcal{F}(\mathcal{I}) = \{M \subset \mathbb{N} : \exists A \in \mathcal{I} : M = \mathbb{N} \setminus A\}$$

is a filter of $\mathbb{N}$ and it is called the filter associated with the ideal $\mathcal{I}$.

$\mathcal{I}_2$-convergence and $\mathcal{I}_2$-Cauchy sequences have been studied in [9], [2].

Let (X, ρ) be a metric space A double sequence $x = (x_{mn})$ in X is said to be $\mathcal{I}_2$-convergent to $L \in X$, if for any $\varepsilon > 0$ we have $A(\varepsilon) = \{(m, n) \in \mathbb{N} \times \mathbb{N} : \rho(x_{mn}, L) \geq \varepsilon\} \in \mathcal{I}_2$. In this case, we say that x is $\mathcal{I}_2$-convergent and we write

$$\mathcal{I}_2 - \lim_{m,n\to\infty} x_{mn} = L.$$

Definition 1.1 ([44]). The 5-tuple $(X, \mu, \nu, *, \diamond)$ is said to be intuitionistic fuzzy normed linear space (in short, IFNLS) if X is a linear space, $*$ is a continuous t-norm, $\diamond$ is a continuous t-conorm, and μ and ν are fuzzy sets on $X \times (0, \infty)$ satisfying the following conditions for every $x, y \in X$ and $s, t > 0$:

(a) $\mu(x, t) + \nu(x, t) \leq 1$,
(b) $\mu(x, t) > 0$,
(c) $\mu(x, t) = 1$ if and only if $x = 0$,
(d) $\mu(\alpha x, t) = \mu\left(x, \frac{t}{|\alpha|}\right)$ for each $\alpha \neq 0$,
(e) $\mu(x, t) * \mu(y, s) \leq \mu(x + y, t + s)$,
(f) $\mu(x, t) : (0, \infty) \to [0, 1]$ is continuous in t,
(g) $\lim_{t\to\infty} \mu(x, t) = 1$ and $\lim_{t\to 0} \mu(x, t) = 0$,
(h) $\nu(x, t) < 1$,
(i) $\nu(x, t) = 0$ if and only if $x = 0$,
(j) $\nu(\alpha x, t) = \nu\left(x, \frac{t}{|\alpha|}\right)$ for each $\alpha \neq 0$,
(k) $\nu(x, t) \diamond \nu(y, s) \geq \nu(x + y, t + s)$,
(l) $\nu(x, t) : (0, \infty) \to [0, 1]$ is continuous in t,
(m) $\lim_{t\to\infty} \nu(x, t) = 0$ and $\lim_{t\to 0} \nu(x, t) = 1$.
In this case (μ, ν) is called an intuitionistic fuzzy norm.

Remark 1.2. Hosseini et al. [24] have given a more complete definition of an intuitionistic fuzzy normed space. The results of this paper can be investigated as a more general case in this new setting. However, the present definition eases up the computational aspects.

Definition 1.3 ([44]). Let $(X, \mu, \nu, *, \diamond)$ be an IFNLS. A sequence $x = (x_k)$ in X is said to be convergent to $\xi \in X$ with respect to the intuitionistic fuzzy norm (μ, ν) if, for every $\varepsilon \in (0, 1)$ and $t > 0$, there exists $k_0 \in \mathbb{N}$ such that $\mu(x_k - \xi, t) > 1 - \varepsilon$ and $\nu(x_k - \xi, t) < \varepsilon$ for all $k \geq k_0$. It is denoted by $(\mu, \nu) - \lim x = \xi$.

Definition 1.4 ([44]). Let $(X, \mu, \nu, *, \diamond)$ be an IFNLS. A sequence $x = (x_k)$ in X is said to be Cauchy sequence with respect to the intuitionistic fuzzy norm (μ, ν) if, for every $\varepsilon \in (0, 1)$ and $t > 0$, there exists $k_0 \in \mathbb{N}$ such that $\mu(x_k - x_m, t) > 1 - \varepsilon$ and $\nu(x_k - x_m, t) < \varepsilon$ for all $k, m \geq k_0$.

Definition 1.5 ([44]). Let $(X, \mu, \nu, *, \diamond)$ be an IFNLS. For $t > 0$, the open ball $B(x, r, t)$ with center $x \in X$ and radius $r \in (0, 1)$ is defined as

$$B(x, r, t) = \{y \in X : \mu(x - y, t) > 1 - r \text{ and } \nu(x - y, t) < r\}.$$

Definition 1.6. Let $(X, \mu, \nu, *, \diamond)$ be an IFNLS. A subset F of X is said to be closed if any sequence (x_k) in X converging to the some $x \in X$ with respect to the intuitionistic fuzzy norm (μ, ν) implies that $x \in F$. A subset Y of X is said to be the closure of $A \subset X$ if, for any $x \in Y$, then there exists a sequence (x_k) in A converging to x with respect to the intuitionistic fuzzy norm (μ, ν). We denote the set Y by $\overline{A}$.

Definition 1.7 ([44]). Consider the IFNLS $(X,\mu,\nu,*,\diamond)$. Let $\varepsilon, t>0$. If there exists $N\in\mathbb{N}$ such that $\mu(x_k-\xi,t)>1-\varepsilon$ and $\nu(x_k-\xi,t)<\varepsilon$ for all $k\geq N$, then, in $IFN(\mu,\nu)$, a sequence (x_k) in X is said to be convergent to $\xi\in X$. It is denoted by $(\mu,\nu)-\lim x=\varepsilon$.

Definition 1.8 ([30]). Take an IFNS $(X,\mu,\nu,*,\diamond)$. A sequence (x_k) is said to be Fibonacci statistical convergence with respect to IFN (μ,ν) (briefly, FSC-IFN), if there is a number $\xi\in X$ such that for every $\varepsilon>0$ and $t>0$, the set

$$K_\varepsilon(\widehat{F}):=\left\{k\leq n:\mu\left(\widehat{F}x_k-\xi,t\right)\leq 1-\varepsilon \text{ or } \nu\left(\widehat{F}x_k-\xi,t\right)\geq\varepsilon\right\}$$

has natural density zero., i.e., $d\left(K_\varepsilon(\widehat{F})\right)=0$. That is,

$$\lim_{n\to\infty}\frac{1}{n}\left|k\leq n:\mu\left(\widehat{F}x_k-\xi,t\right)\leq 1-\varepsilon \text{ or } \nu\left(\widehat{F}x_k-\xi,t\right)\geq\varepsilon\right|=0.$$

In this case, we write $d(\widehat{F})_{IFN}-\lim x_k=\xi$ or $x_k\to\xi\left(S(\widehat{F})_{IFN}\right)$. The set of FSC-IFN will denoted by $S(\widehat{F})_{IFN}$.

Definition 1.9 ([30]). Take an IFNS $(X,\mu,\nu,*,\diamond)$. A sequence (x_k) is said to be Fibonacci statistical Cauchy with respect to IFN (μ,ν) (briefly, FSCa-IFN), if for every $\varepsilon>0$ and $t>0$, there exists $N=N(\varepsilon)$ such that

$$K_\varepsilon(\widehat{F}):=\left\{k\leq n:\mu\left(\widehat{F}x_k-\widehat{F}x_N,t\right)\leq 1-\varepsilon \text{ or } \nu\left(\widehat{F}x_k-\widehat{F}x_N,t\right)\geq\varepsilon\right\}$$

has natural density zero., i.e., $d\left(K_\varepsilon(\widehat{F})\right)=0$. That is,

$$\lim_{n\to\infty}\frac{1}{n}\left|k\leq n:\mu\left(\widehat{F}x_k-\widehat{F}x_N,t\right)\leq 1-\varepsilon \text{ or } \nu\left(\widehat{F}x_k-\widehat{F}x_N,t\right)\geq\varepsilon\right|=0.$$

2. Main Results

In this section, we introduce the notion of Fibonacci $\mathcal{I}_2$-convergence in an intuitionistic fuzzy normed space and study some its properties.

Definition 2.1. Let $(X,\mu,\nu,*,\diamond)$ be an IFNLS and $\mathcal{I}_2\subset P(\mathbb{N}\times\mathbb{N})$ be a nontrivial ideal. A double sequence $x=(x_{ki})$ in X is said to be Fibonacci $\mathcal{I}_2$-convergent with respect to the intuitionistic fuzzy norm (μ,ν) (briefly, F$\mathcal{I}_2$C-IFN), if there is a number $\xi\in X$ such that for every $t>0$ and $\varepsilon\in(0,1)$, the set

$$K_\varepsilon(\widehat{F}):=\left\{(k,i)\in\mathbb{N}\times\mathbb{N}:\mu\left(\widehat{F}x_{ki}-\xi,t\right)\leq 1-\varepsilon \text{ or } \nu\left(\widehat{F}x_{ki}-\xi,t\right)\geq\varepsilon\right\}\in\mathcal{I}_2.$$

In this case, we write $\mathcal{I}_2^{(\mu,\nu)}-\lim x_{ki}=\xi$. The element ξ is called the Fibonacci $\mathcal{I}_2$-limit of the sequence (x_{ki}) with respect to the intuitionistic fuzzy norm (μ,ν). The set of F$\mathcal{I}_2$C-IFN will be denoted by $\mathcal{I}_2(\widehat{F})_{IFN}$.

Example 2.2. If we take $\mathcal{I}_2$=$\{A\subset\mathbb{N}\times\mathbb{N}:\delta(A)=0\}$, then it is clear that $\mathcal{I}_2$ is an admissible ideal in $\mathbb{N}\times\mathbb{N}$ and the corresponding Fibonacci $\mathcal{I}_2$-convergence coincides with Fibonacci statistical convergence in an IFNLS.

Lemma 2.3. *Let $(X,\mu,\nu,*,\diamond)$ be an IFNLS, and $x=(x_{ki})$ be a double sequence in X. Then, for every $\varepsilon>0$ and $t>0$, the following statements are equivalent.*

(a) $\mathcal{I}_2^{(\mu,\nu)}-\lim x_{ki}=\xi$;

(b) $\left\{(k,i)\in\mathbb{N}\times\mathbb{N}:\mu\left(\widehat{F}x_{ki}-\xi,t\right)\leq 1-\varepsilon\right\}\in\mathcal{I}_2$ *and* $\left\{(k,i)\in\mathbb{N}\times\mathbb{N}:\nu\left(\widehat{F}x_{ki}-\xi,t\right)\geq\varepsilon\right\}\in\mathcal{I}_2$;

(c) $\left\{(k,i)\in\mathbb{N}\times\mathbb{N}:\mu\left(\widehat{F}x_{ki}-\xi,t\right)>1-\varepsilon \text{ and } \nu\left(\widehat{F}x_{ki}-\xi,t\right)<\varepsilon\right\}\in\mathcal{F}(\mathcal{I}_2)$,

(d) $\left\{(k,i)\in\mathbb{N}\times\mathbb{N}:\mu\left(\widehat{F}x_{ki}-\xi,t\right)>1-\varepsilon\right\}\in\mathcal{F}(\mathcal{I}_2)$ *and* $\left\{(k,i)\in\mathbb{N}\times\mathbb{N}:\ \nu\left(\widehat{F}x_{ki}-\xi,t\right)<\varepsilon\right\}\in\mathcal{F}(\mathcal{I}_2)$
and

(e) $\mathcal{I}_2^{(\mu,\nu)}-\lim\mu\left(\widehat{F}x_{ki}-\xi,t\right)=1$ *and* $\mathcal{I}_2^{(\mu,\nu)}-\lim\nu\left(\widehat{F}x_{ki}-\xi,t\right)=0$.

Proof. It is not hard to prove the equivalence of (a), (b), (c), (d). So we only prove the equivalence of (b) and (e). Suppose that (b) holds. Since for every $\varepsilon>0$ and $t>0$, we have

$$\left\{(k,i)\in\mathbb{N}\times\mathbb{N}:\left|\mu\left(\widehat{F}x_{ki}-\xi,t\right)-1\right|\geq\varepsilon\right\}$$

$$=\left\{(k,i)\in\mathbb{N}\times\mathbb{N}:\mu\left(\widehat{F}x_{ki}-\xi,t\right)\geq 1+\varepsilon\right\}\cup\left\{k\in\mathbb{N}:\mu\left(\widehat{F}x_{ki}-\xi,t\right)\leq 1-\varepsilon\right\}$$

and for every $\varepsilon > 0$ the set $\left\{(k,i) \in \mathbb{N}\times\mathbb{N} : \mu\left(\widehat{F}x_{ki} - \xi, t\right) \geq 1+\varepsilon\right\} = \varnothing \in \mathcal{I}_2$, it follows together with (b) that $\left\{(k,i) \in \mathbb{N}\times\mathbb{N} : \left|\mu\left(\widehat{F}x_{ki} - \xi, t\right) - 1\right| \geq \varepsilon\right\} \in \mathcal{I}_2$. Hence, we have $\mathcal{I}_2^{(\mu,\nu)} - \lim \mu\left(\widehat{F}x_{ki} - \xi, t\right) = 1$. Similarly, the fact that for every $\varepsilon > 0$ and $t > 0$,

$$\left\{(k,i) \in \mathbb{N}\times\mathbb{N} : \left|\nu\left(\widehat{F}x_{ki} - \xi, t\right) - 0\right| \geq \varepsilon\right\}$$

$$= \left\{(k,i) \in \mathbb{N}\times\mathbb{N} : \nu\left(\widehat{F}x_{ki} - \xi, t\right) \geq \varepsilon\right\} \cup \left\{k \in \mathbb{N} : \nu\left(\widehat{F}x_{ki} - \xi, t\right) \leq -\varepsilon\right\}$$

and $\left\{(k,i) \in \mathbb{N}\times\mathbb{N} : \mu\left(\widehat{F}x_{ki} - \xi, t\right) \leq -\varepsilon\right\} = \varnothing \in \mathcal{I}_2$, implies that $\mathcal{I}_2^{(\mu,\nu)} - \lim \nu\left(\widehat{F}x_{ki} - \xi, t\right) = 0$. Also it is clear that (e) implies (b). □

Theorem 2.4. *Let $(X, \mu, \nu, *, \diamond)$ be an IFNLS, and $x = (x_{ki})$ be a double sequence in X. If (x_{ki}) is Fibonacci $\mathcal{I}_2$-convergent with respect to the intuitionistic fuzzy norm (μ,ν), then $\mathcal{I}_2^{(\mu,\nu)} - \lim x$ is unique.*

Proof. Suppose that there exist two distinct elements $\xi_1, \xi_2 \in X$ such that $\mathcal{I}_2^{(\mu,\nu)} - \lim x_k = \xi_1$ and $\mathcal{I}_2^{(\mu,\nu)} - \lim x_k = \xi_2$. Given $\varepsilon \in (0,1)$, choose $\gamma > 0$ such that $(1-\gamma)*(1-\gamma) > 1-\varepsilon$ and $\gamma \diamond \gamma < \varepsilon$. Then, for any $t > 0$, define the following sets:

$$K_{\mu,1}(\gamma,t) = \left\{(k,i) \in \mathbb{N}\times\mathbb{N} : \mu\left(\widehat{F}x_{ki} - \xi_1, \tfrac{t}{2}\right) \leq 1-\gamma\right\},$$

$$K_{\nu,1}(\gamma,t) = \left\{(k,i) \in \mathbb{N}\times\mathbb{N} : \nu\left(\widehat{F}x_{ki} - \xi_1, \tfrac{t}{2}\right) \geq \gamma\right\},$$

$$K_{\mu,2}(\gamma,t) = \left\{(k,i) \in \mathbb{N}\times\mathbb{N} : \mu\left(\widehat{F}x_{ki} - \xi_2, \tfrac{t}{2}\right) \leq 1-\gamma\right\},$$

$$K_{\nu,2}(\gamma,t) = \left\{(k,i) \in \mathbb{N}\times\mathbb{N} : \nu\left(\widehat{F}x_{ki} - \xi_2, \tfrac{t}{2}\right) \geq \gamma\right\}.$$

and $K_{\mu,\nu}(\gamma,t) = \left(K_{\mu,1}(\gamma,t) \cup K_{\mu,2}(\gamma,t)\right) \cap \left(K_{\nu,1}(\gamma,t) \cup K_{\nu,2}(\gamma,t)\right)$.

Since $\mathcal{I}_2^{(\mu,\nu)} - \lim x_{ki} = \xi_1$ and $\mathcal{I}_2^{(\mu,\nu)} - \lim x_{ki} = \xi_2$, therefore all the sets $K_{\mu,1}(\gamma,t), K_{\nu,1}(\gamma,t), K_{\mu,2}(\gamma,t), K_{\nu,2}(\gamma,t)$ and $K_{\mu,\nu}(\gamma,t)$ belongs to $\mathcal{I}$. This implies that its complement $K^c_{\mu,\nu}(\gamma,t)$ is a non-empty set in $\mathcal{F}(\mathcal{I}_2)$. Let $(m,n) \in K^c_{\mu,\nu}(\gamma,t)$. Then we have $(m,n) \in K^c_{\mu,1}(\gamma,t) \cap K^c_{\mu,2}(\gamma,t)$ or $(m,n) \in K_{\nu,1}(\gamma,t) \cap K_{\nu,2}(\gamma,t)$.

Case (i) Suppose that $(m,n) \in K^c_{\mu,1}(\gamma,t) \cap K^c_{\mu,2}(\gamma,t)$. Then we have $\mu\left(\widehat{F}x_{mn} - \xi, \tfrac{t}{2}\right) > 1-r$, $\mu\left(\widehat{F}x_{mn} - \xi_2, \tfrac{t}{2}\right) > 1-r$ and therefore

$$\begin{aligned}\mu(\xi_1 - \xi_2, t) &\geq \mu\left(\widehat{F}x_{mn} - \xi, \frac{t}{2}\right) * \mu\left(\widehat{F}x_{mn} - \xi_2, \frac{t}{2}\right) \\ &> (1-\gamma)*(1-\gamma) > 1-\varepsilon.\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we get $\mu(\xi_1 - \xi_2, t) = 1$ for all $t > 0$, which yields $\xi_1 = \xi_2$.

Case (ii) Suppose that $(m,n) \in K^c_{\nu,1}(\gamma,t) \cap K^c_{\nu,2}(\gamma,t)$. Then, we have $\nu\left(\widehat{F}x_{mn} - \xi_1, \tfrac{t}{2}\right) < \gamma$, $\nu\left(\widehat{F}x_{mn} - \xi_2, \tfrac{t}{2}\right) < r$ and therefore

$$\begin{aligned}\nu(\xi_1 - \xi_2, t) &< \nu\left(\widehat{F}x_{mn} - \xi_1, \frac{t}{2}\right) \diamond \nu\left(\widehat{F}x_{mn} - \xi_2, \frac{t}{2}\right) \\ &< \gamma \diamond \gamma < \varepsilon.\end{aligned}$$

As $\varepsilon > 0$ was selected arbitrary, we have $\nu(\xi_1 - \xi_2, t) = 0$ for all $t > 0$. This implies that $\xi_1 = \xi_2$. Hence, in all cases, we conclude that $\mathcal{I}_2^{(\mu,\nu)} - \lim x$ is unique. This completes the proof of the theorem. □

Theorem 2.5. *Let $(X, \mu, \nu, *, \diamond)$ be an IFNLS, and $x = (x_{ki})$, $y = (y_{ki})$ be two sequences in X.*

(a) If $(\mu,\nu) - \lim x_{ki} = \xi$, then $\mathcal{I}_2^{(\mu,\nu)} - \lim \widehat{F}x_{ki} = \xi$.
(b) If $\mathcal{I}_2^{(\mu,\nu)} - \lim \widehat{F}x_{ki} = \xi_1$ and $\mathcal{I}_2^{(\mu,\nu)} - \lim \widehat{F}y_{ki} = \xi_2$, then $\mathcal{I}_2^{(\mu,\nu)} - \lim\left(\widehat{F}x_{ki} + \widehat{F}y_{ki}\right) = (\xi_1 + \xi_2)$;
(b) If $\mathcal{I}_2^{(\mu,\nu)} - \lim \widehat{F}x_{ki} = \xi$ and α be any real number, then $\mathcal{I}_2^{(\mu,\nu)} - \lim \alpha\widehat{F}x_{ki} = \alpha\xi$.

Proof. (a) As $(\mu,\nu)-\lim x_{ki}=\xi$, so for each $\varepsilon>0$ and $t>0$ there exists $r_0\in\mathbb{N}$ such that $\mu(x_{ki}-\xi,t)>1-\varepsilon$ and $\nu(x_{ki}-\xi,t)<\varepsilon$ for all $k,i\geq r_0$. Since

$$A=\{(k,i)\in\mathbb{N}\times\mathbb{N}:\mu(x_{ki}-\xi,t)\leq 1-\varepsilon \text{ or } \nu(x_{ki}-\xi,t)\geq\varepsilon\}$$

is contained in $\{1,2,...,r_0-1\}$.

Then,

$$\left\{(k,i)\in\mathbb{N}\times\mathbb{N}:\mu\left(\widehat{F}x_{ki}-\xi,t\right)\leq 1-\varepsilon \text{ or } \nu\left(\widehat{F}x_{ki}-\xi,t\right)\geq\varepsilon\right\}\in\mathcal{I}_2,$$

because the ideal $\mathcal{I}_2$ is admissible. This shows that $\mathcal{I}_2^{(\mu,\nu)}-\lim x_{ki}=\xi$.

(b) Let $\varepsilon>0$ be given. Choose $\gamma>0$ such that $(1-\gamma)*(1-\gamma)>1-\varepsilon$ and $\gamma\diamond\gamma<\varepsilon$. For $t>0$, define

$$K_{\mu,1}(\gamma,t)=\left\{(k,i)\in\mathbb{N}\times\mathbb{N}:\mu\left(\widehat{F}x_{ki}-\xi_1,\tfrac{t}{2}\right)\leq 1-\gamma\right\},$$

$$K_{\nu,1}(\gamma,t)=\left\{(k,i)\in\mathbb{N}\times\mathbb{N}:\nu\left(\widehat{F}x_{ki}-\xi_1,\tfrac{t}{2}\right)\geq \gamma\right\},$$

$$K_{\mu,2}(\gamma,t)=\left\{(k,i)\in\mathbb{N}\times\mathbb{N}:\mu\left(\widehat{F}y_{ki}-\xi_2,\tfrac{t}{2}\right)\leq 1-\gamma\right\},$$

$$K_{\nu,2}(\gamma,t)=\left\{(k,i)\in\mathbb{N}\times\mathbb{N}:\nu\left(\widehat{F}y_{ki}-\xi_2,\tfrac{t}{2}\right)\geq \gamma\right\}.$$

and $K_{\mu,\nu}(\gamma,t)=\left(K_{\mu,1}(\gamma,t)\cup K_{\mu,2}(\gamma,t)\right)\cup\left(K_{\nu,1}(\gamma,t)\cup K_{\nu,2}(\gamma,t)\right)$. Since $\mathcal{I}_2^{(\mu,\nu)}-\lim x_k=\xi_1$ and $\mathcal{I}_2^{(\mu,\nu)}-\lim y_k=\xi_2$, so for $t>0$, all the sets $K_{\mu,1}(\gamma,t)$, $K_{\nu,1}(\gamma,t)$, $K_{\mu,2}(\gamma,t)$, $K_{\nu,2}(\gamma,t)$ and $K_{\mu,\nu}(\gamma,t)$ belongs to $\mathcal{I}_2$. It follows that $K^c_{\mu,\nu}(\gamma,t)$ is a non-empty set in $\mathcal{F}(\mathcal{I}_2)$. Next we shall show that

$$K^c_{\mu,\nu}(\gamma,t)\subset\left\{\begin{array}{c}(k,i)\in\mathbb{N}\times\mathbb{N}:\mu\left(\widehat{F}(x_{ki}+y_{ki})-(\xi_1+\xi_2),t\right)>1-\varepsilon \text{ and}\\ \nu\left(\widehat{F}(x_{ki}+y_{ki})-(\xi_1+\xi_2),t\right)<\varepsilon\end{array}\right\}.$$

For this let $(m,n)\in K^c_{\mu,\nu}(\gamma,t)$. Then we have

$$\begin{aligned}\mu\left(\widehat{F}x_{mn}-\xi_1,\frac{t}{2}\right) &> 1-r,\ \mu\left(\widehat{F}y_{mn}-\xi_2,\frac{t}{2}\right)>1-r\\ \nu\left(\widehat{F}x_{mn}-\xi_1,\frac{t}{2}\right) &< r,\ \nu\left(\widehat{F}y_{mn}-\xi_2,\frac{t}{2}\right)<r.\end{aligned}$$

Now we have,

$$\begin{aligned}\mu\left(\widehat{F}(x_{mn}+y_{mn})-(\xi_1+\xi_2),t\right) &\geq \mu\left(\widehat{F}x_{mn}-\xi,\frac{t}{2}\right)*\mu\left(\widehat{F}y_{mn}-\xi_2,\frac{t}{2}\right)\\ &> (1-\gamma)*(1-\gamma)>1-\varepsilon\end{aligned}$$

and

$$\begin{aligned}\nu\left(\widehat{F}(x_{mn}+y_{mn})-(\xi_1+\xi_2),t\right) &\leq \nu\left(\widehat{F}x_{mn}-\xi,\frac{t}{2}\right)\diamond\nu\left(\widehat{F}y_{mn}-\xi_2,\frac{t}{2}\right)\\ &< \gamma\diamond\gamma<\varepsilon.\end{aligned}$$

This shows that

$$K^c_{\mu,\nu}(\gamma,t)\subset\left\{\begin{array}{c}(k,i)\in\mathbb{N}\times\mathbb{N}:\mu\left(\widehat{F}(x_{ki}+y_{ki})-(\xi_1+\xi_2),t\right)>1-\varepsilon \text{ and}\\ \nu\left(\widehat{F}(x_{ki}+y_{ki})-(\xi_1+\xi_2),t\right)<\varepsilon\end{array}\right\}.$$

Since $K^c_{\mu,\nu}(\gamma,t)\in\mathcal{F}(\mathcal{I}_2)$. Hence $\mathcal{I}_2^{(\mu,\nu)}-\lim\left(\widehat{F}x_{ki}+\widehat{F}y_{ki}\right)=(\xi_1+\xi_2)$.

(c) Case-(i) If $\alpha=0$, then for each $\varepsilon>0$ and $t>0$, $\mu\left(\widehat{F}0x_{ki}-0\xi,t\right)=\mu(0,t)=1>1-\varepsilon$ and $\nu\left(\widehat{F}0x_{ki}-0\xi,t\right)=\nu(0,t)=0<\varepsilon$. This implies that $(\mu,\nu)-\lim 0x_{ki}=\theta$, and so by part (i), we have $\mathcal{I}_2^{(\mu,\nu)}-\lim\widehat{F}0x_{ki}=\theta$.

Case-(ii) If $\alpha\neq 0$. As $\mathcal{I}_2^{(\mu,\nu)}-\lim x_{ki}=\xi$ so for each $\varepsilon>0$ and $t>0$,

$$A=\left\{(k,i)\in\mathbb{N}\times\mathbb{N}:\mu\left(\widehat{F}x_{ki}-\xi,t\right)>1-\varepsilon \text{ and } \nu\left(\widehat{F}x_{ki}-\xi,t\right)<\varepsilon\right\}\in\mathcal{F}(\mathcal{I}_2).$$

Next to prove the result it is sufficient to prove that for each $\varepsilon > 0$ and $t > 0$,

$$A \subset \left\{(k,i) \in \mathbb{N}\times\mathbb{N} : \mu\left(\alpha\widehat{F}x_{ki} - \alpha\xi, t\right) > 1-\varepsilon \text{ and } \nu\left(\alpha\widehat{F}x_{ki} - \alpha\xi, t\right) < \varepsilon\right\}.$$

For this let $(m,n) \in A$. Then we have $\mu\left(\widehat{F}x_{m,n} - \xi, t\right) > 1-\varepsilon$ and $\nu\left(\widehat{F}x_{mn} - \xi, t\right) < \varepsilon$. Now,

$$\mu\left(\alpha\widehat{F}x_{mn} - \alpha\xi, t\right) = \mu\left(\left(\widehat{F}x_{mn} - \xi\right), \tfrac{t}{|\alpha|}\right) \geq \mu\left(\widehat{F}x_{mn} - \xi, t\right) * \mu\left(0, \tfrac{t}{|\alpha|} - t\right)$$

$$= \mu\left(\widehat{F}x_{mn} - \xi, t\right) * 1 = \mu\left(\widehat{F}x_{mn} - \xi, t\right) > 1-\varepsilon$$

and

$$\nu\left(\alpha\widehat{F}x_{mn} - \alpha\xi, t\right) = \nu\left(\left(\widehat{F}x_{mn} - \xi\right), \tfrac{t}{|\alpha|}\right) \leq \nu\left(\widehat{F}x_{mn} - \xi, t\right) \diamond \mu\left(0, \tfrac{t}{|\alpha|} - t\right)$$

$$= \mu\left(\widehat{F}x_{mn} - \xi, t\right) \diamond 0 = \mu\left(\widehat{F}x_{mn} - \xi, t\right) < \varepsilon$$

Hence, we have $A \subset \left\{(k,i) \in \mathbb{N}\times\mathbb{N} : \mu\left(\alpha\widehat{F}x_{ki} - \alpha\xi, t\right) > 1-\varepsilon \text{ and } \nu\left(\alpha\widehat{F}x_{ki} - \alpha\xi, t\right) < \varepsilon\right\}$. But then (2) shows that $\mathcal{I}_2^{(\mu,\nu)} - \lim \alpha\widehat{F}x_{ki} = \alpha\xi$. □

Before giving the next theorem we recall the following: Let $(X, \mu, \nu, *, \diamond)$ be an IFNLS. The open ball $B(x, \varepsilon, t)$ with center x and radius $0 < \varepsilon < 1$ is given as

$$B(x,\varepsilon,t) = \{y \in X : \mu(x-y,t) > 1-\varepsilon \text{ and } \nu(x-y,t) < \varepsilon\}$$

where $t > 0$. A subset A of X is said to be IF-bounded if there exists $t > 0$ and $0 < \varepsilon < 1$ such that $\mu(x,t) > 1-\varepsilon$ and $\nu(x,t) < \varepsilon$ for each $x \in A$.

Let $l^{\infty}_{(\mu,\nu)}(X)$ denotes the space of all IF-bounded double sequences whereas by $\mathcal{I}^{\infty}_{2,(\mu,\nu)}(X)$ we shall denote the space of all IF-bounded and $\mathcal{I}_2$-convergent double sequences in an intuitionistic fuzzy normed linear space $(X, \mu, \nu, *, \diamond)$. Now we have the following theorem.

Theorem 2.6. *Let $(X, \mu, \nu, *, \diamond)$ be an IFNLS. Then $\mathcal{I}^{\infty}_{2,(\mu,\nu)}(X)$ is a closed linear subspace of $l^{\infty}_{(\mu,\nu)}(X)$.*

Proof. It is clear by Theorem 2 that $\mathcal{I}^{\infty}_{2,(\mu,\nu)}(X)$ is a subspace of $l^{\infty}_{(\mu,\nu)}(X)$. Next we prove the closedness of $\mathcal{I}^{\infty}_{2,(\mu,\nu)}(X)$. As $\mathcal{I}^{\infty}_{2,(\mu,\nu)}(X) \subset \overline{\mathcal{I}^{\infty}_{2,(\mu,\nu)}(X)}$ always, so we prove that $\overline{\mathcal{I}^{\infty}_{2,(\mu,\nu)}(X)} \subset \mathcal{I}^{\infty}_{2,(\mu,\nu)}(X)$. For this, let $x \in \overline{\mathcal{I}^{\infty}_{2,(\mu,\nu)}(X)}$. Then $B(x,r,t) \cap \mathcal{I}^{\infty}_{2,(\mu,\nu)}(X) \neq \emptyset$, for each open ball $B(x,r,t)$ centered at x. Let $y \in B(x,r,t) \cap \mathcal{I}^{\infty}_{2,(\mu,\nu)}(X)$. Let $t > 0$ and $\varepsilon \in (0,1)$. Choose $\gamma \in (0,1)$ such that $(1-\gamma)*(1-\gamma) > 1-\varepsilon$ and $\gamma\diamond\gamma < \varepsilon$. As $y \in B(x,r,t) \cap \mathcal{I}^{\infty}_{2,(\mu,\nu)}(X)$ so there exists a subset $K \subset \mathbb{N}\times\mathbb{N}$ such that $K \in \mathcal{F}(\mathcal{I}_2)$ and for all $(k,i) \in K$ we have $\mu\left(\widehat{F}x_{ki} - \widehat{F}y_{ki}, \frac{t}{2}\right) > 1-\gamma$, $\nu\left(\widehat{F}x_{ki} - \widehat{F}y_{ki}, \frac{t}{2}\right) < \gamma$, $\mu\left(\widehat{F}y_{ki}, \frac{t}{2}\right) > 1-\gamma$, $\nu\left(\widehat{F}y_{ki}, \frac{t}{2}\right) < \gamma$. But then for every $(k,i) \in K$ we have

$$\mu\left(\widehat{F}x_{ki}, t\right) = \mu\left(\widehat{F}x_{ki} - \widehat{F}y_{ki} + \widehat{F}y_{ki}, t\right)$$

$$\geq \mu\left(\widehat{F}x_{ki} - \widehat{F}y_{ki}, \tfrac{t}{2}\right) * \mu\left(\widehat{F}y_{ki}, \tfrac{t}{2}\right) > (1-\gamma)*(1-\gamma) > 1-\varepsilon$$

and

$$\nu\left(\widehat{F}x_{ki}, t\right) = \nu\left(\widehat{F}x_{ki} - \widehat{F}y_{ki} + \widehat{F}y_{ki}, t\right)$$

$$\leq \nu\left(\widehat{F}x_{ki} - \widehat{F}y_{ki}, \tfrac{t}{2}\right) \diamond \mu\left(\widehat{F}y_{ki}, \tfrac{t}{2}\right) < \gamma\diamond\gamma < \varepsilon.$$

This implies that $K \subset \left\{(k,i) \in \mathbb{N}\times\mathbb{N} : \mu\left(\widehat{F}x_{ki}, t\right) > 1-\varepsilon \text{ and } \nu\left(\widehat{F}x_{ki}, t\right) < \varepsilon\right\}$.

Since $K \in \mathcal{F}(\mathcal{I}_2)$, it follows that $\left\{(k,i) \in \mathbb{N}\times\mathbb{N} : \mu\left(\widehat{F}x_{ki}, t\right) > 1-\varepsilon \text{ and } \nu\left(\widehat{F}x_{ki}, t\right) < \varepsilon\right\} \in \mathcal{F}(\mathcal{I}_2)$. Hence, we have $x \in \mathcal{I}^{\infty}_{2,(\mu,\nu)}(X)$. □

Definition 2.7. Let $(X, \mu, \nu, *, \diamond)$ be an IFNLS and $\mathcal{I}_2 \subset P(\mathbb{N}\times\mathbb{N})$ be a nontrivial ideal. A double sequence $x = (x_{ki})$ in X is said to be Fibonacci $\mathcal{I}_2$-Cauchy with respect to the intuitionistic fuzzy norm (μ,ν) if, for every $\varepsilon > 0$ and $t > 0$, there exists $N = N(\varepsilon)$ and $M = M(\varepsilon)$ such that, for all $k, p \geq N$, $i, q \geq M$,

$$K_{\varepsilon}(\widehat{F}) := \left\{(k,i) \in \mathbb{N}\times\mathbb{N} : \mu\left(\widehat{F}x_{ki} - \widehat{F}x_{pq}, t\right) \leq 1-\varepsilon \text{ or } \nu\left(\widehat{F}x_{ki} - \widehat{F}x_{pq}, t\right) \geq \varepsilon\right\} \in \mathcal{I}_2.$$

Theorem 2.8. *Let $(X, \mu, \nu, *, \diamond)$ be an IFNLS. Then a double sequence $x = (x_{ki})$ in X Fibonacci $\mathcal{I}_2$-convergent with respect to the intuitionistic fuzzy norm (μ, ν) if and only if it is Fibonacci $\mathcal{I}_2$-Cauchy with respect to the intuitionistic fuzzy norm (μ, ν).*

Proof. Necessity. Let $x = (x_{k,i})$ in X Fibonacci $\mathcal{I}_2$-convergent to ξ with respect to the intuitionistic fuzzy norm (μ, ν), i.e., $\mathcal{I}_2^{(\mu,\nu)} - \lim x_{k,i} = \xi$. For a given $\varepsilon > 0$, choose $\varepsilon > 0$ such that $(1-\gamma) * (1-\gamma) > 1-\varepsilon$ and $\gamma \diamond \gamma < \varepsilon$. Then, for $t > 0$, we have,

$$K(\widehat{F}) = \left\{(k,i) \in \mathbb{N}\times\mathbb{N} : \mu\left(\widehat{F}x_{ki} - \xi, t\right) \leq 1-\gamma \text{ or } \nu\left(\widehat{F}x_{ki} - \xi, t\right) \geq \gamma\right\} \in \mathcal{I}_2 \tag{1}$$

which implies that

$$\varnothing \neq K^c(\widehat{F}) = \left\{(k,i) \in \mathbb{N}\times\mathbb{N} : \mu\left(\widehat{F}x_{ki} - \xi, t\right) > 1-\gamma \text{ or } \nu\left(\widehat{F}x_{ki} - \xi, t\right) < \gamma\right\} \in \mathcal{F}(\mathcal{I}_2).$$

Let $(m,n) \in K^c_{(\varepsilon,t)}(\widehat{F})$. But then for every $t > 0$ we have, $\mu\left(\widehat{F}x_{mn} - \xi, t\right) > 1-\gamma$ or $\nu\left(\widehat{F}x_{mn} - \xi, t\right) < \gamma$. If we take

$$B(\widehat{F}) = \left\{(k,i) \in \mathbb{N}\times\mathbb{N} : \mu\left(\widehat{F}x_{ki} - \widehat{F}x_{mn}, t\right) \leq 1-\varepsilon \text{ or } \nu\left(\widehat{F}x_{ki} - \widehat{F}x_{mn}, t\right) \geq \varepsilon\right\}; t \geq 0,$$

then to prove the result it is sufficient to prove $B(\widehat{F})$ is contained in $K(\widehat{F})$. Let $(k,i) \in B(\widehat{F})$, then we have $\mu\left(\widehat{F}x_{ki} - \widehat{F}x_{mn}, \frac{t}{2}\right) \leq 1-\varepsilon$ or $\nu\left(\widehat{F}x_{ki} - \widehat{F}x_{mn}, \frac{t}{2}\right) \geq \varepsilon$, for $t > 0$. We have two possible cases.

Case (i) We first consider that $\mu\left(\widehat{F}x_{ki} - \widehat{F}x_{mn}, t\right) \leq 1-\varepsilon$. Then, we have $\mu\left(\widehat{F}x_{ki} - \xi, \frac{t}{2}\right) \leq 1-\gamma$ and therefore, $(k,i) \in K(\widehat{F})$. As otherwise i.e., if $\mu\left(\widehat{F}x_{ki} - \xi, \frac{t}{2}\right) > 1-\gamma$, then we have

$$1-\varepsilon \geq \mu\left(\widehat{F}x_{ki} - \widehat{F}x_{mn}, t\right) \geq \mu\left(\widehat{F}x_{ki} - \xi, \frac{t}{2}\right) * \mu\left(\widehat{F}x_{mn} - \xi, \frac{t}{2}\right)$$

$$> (1-\gamma) * (1-\gamma) > 1-\varepsilon$$

which is not possible. Hence, $B(\widehat{F}) \subset K(\widehat{F})$.

Case (ii) If $\nu\left(\widehat{F}x_{ki} - \widehat{F}x_{mn}, t\right) \geq \varepsilon$, then we have $\nu\left(\widehat{F}x_{ki} - \xi, \frac{t}{2}\right) > \gamma$ and therefore $(k,i) \in K(\widehat{F})$. As otherwise i.e., if $\nu\left(\widehat{F}x_{ki} - \xi, \frac{t}{2}\right) < \gamma$, then we have

$$\varepsilon \leq \nu\left(\widehat{F}x_{ki} - \widehat{F}x_{mn}, t\right) \geq \nu\left(\widehat{F}x_{ki} - \xi, \frac{t}{2}\right) \diamond \nu\left(\widehat{F}x_{mn} - \xi, \frac{t}{2}\right)$$

$$< \gamma \diamond \gamma < \varepsilon;$$

which is not possible. Hence $B(\widehat{F}) \subset K(\widehat{F})$. Thus, in all case we get $B(\widehat{F}) \subset K(\widehat{F})$. By (1) $B(\widehat{F}) \in \mathcal{I}_2$. This shows that (x_{ki}) in X is a Fibonacci $\mathcal{I}_2$-Cauchy sequence.

Sufficiency. Let $x = (x_{ki})$ in X be Fibonacci $\mathcal{I}_2$-Cauchy with respect to the intuitionistic fuzzy norm (μ, ν) but not Fibonacci $\mathcal{I}_2$-convergent with respect to the intuitionistic fuzzy norm (μ, ν). Then there exist (p,q) such that

$$A_{(\varepsilon,t)}(\widehat{F}) := \left\{(k,i) \in \mathbb{N}\times\mathbb{N} : \mu\left(\widehat{F}x_{ki} - \widehat{F}x_{pq}, t\right) \leq 1-\varepsilon \text{ or } \nu\left(\widehat{F}x_{ki} - \widehat{F}x_{pq}, t\right) \geq \varepsilon\right\} \in \mathcal{I}$$

and

$$B_{(\varepsilon,t)}(\widehat{F}) = \left\{(k,i) \in \mathbb{N}\times\mathbb{N} : \mu\left(\widehat{F}x_{ki} - \xi, \frac{t}{2}\right) > 1-\varepsilon \text{ or } \nu\left(\widehat{F}x_{ki} - \xi, \frac{t}{2}\right) < \varepsilon\right\} \in \mathcal{I}$$

equivalently, $B_{(\varepsilon,t)}(\widehat{F}) \in \mathcal{F}(\mathcal{I}_2)$. Since

$$\mu\left(\widehat{F}x_{ki} - \widehat{F}x_{pq}, t\right) \geq 2\mu\left(\widehat{F}x_{ki} - \xi, \frac{t}{2}\right) > 1-\varepsilon,$$

and

$$\nu\left(\widehat{F}x_{ki} - \widehat{F}x_{pq}, t\right) \leq 2\nu\left(\widehat{F}x_{ki} - \xi, \frac{t}{2}\right) < \varepsilon,$$

If $\mu\left(\widehat{F}x_{ki} - \xi, \frac{t}{2}\right) > \frac{(1-\varepsilon)}{2}$ and $\nu\left(\widehat{F}x_{ki} - \xi, \frac{t}{2}\right) < \frac{\varepsilon}{2}$, respectively, we have $A^c_{(\varepsilon,t)}(\widehat{F}) \in \mathcal{I}_2$, and so $A_{(\varepsilon,t)}(\widehat{F}) \in \mathcal{F}(\mathcal{I}_2)$, which is a contradiction, as $x = (x_{ki})$ was Fibonacci $\mathcal{I}_2$-Cauchy with respect to the intuitionistic fuzzy norm (μ, ν). Hence, $x = (x_{ki})$ must be Fibonacci $\mathcal{I}_2$-convergent with respect to the intuitionistic fuzzy norm (μ, ν). □

Definition 2.9. Let $(X, \mu, \nu, *, \diamond)$ be an IFNLS. A sequence $x = (x_{ki})$ in X is said to be Fibonacci $\mathcal{I}^*$-convergent to $\xi \in X$ with respect to the intuitionistic fuzzy norm (μ, ν) if there exists a subset

$$M = \{(k_m, i_m) : k_1 < k_2 < ...; i_1 < i_2 < ...\}$$

of $\mathbb{N} \times \mathbb{N}$ such that $M \in \mathcal{F}(\mathcal{I}_2)$ and $(\mu, \nu) - \lim_{m\to\infty} x_{k_m i_m} = \xi$. The element ξ is called the Fibonacci $\mathcal{I}_2^*$-limit of the sequence (x_{ki}) with respect to the intuitionistic fuzzy norm (μ, ν) and we write $\mathcal{I}_2^{*,(\mu,\nu)} - \lim x_{ki} = \xi$.

Theorem 2.10. *Let $(X, \mu, \nu, *, \diamond)$ be an IFNLS and $\mathcal{I}_2 \subset P(\mathbb{N} \times \mathbb{N})$ be a nontrivial ideal. If $\mathcal{I}_2^{*,(\mu,\nu)} - \lim x_{ki} = \xi$ then $\mathcal{I}_2^{(\mu,\nu)} - \lim x_{ki} = \xi$.*

Proof. Suppose that $\mathcal{I}_2^{*,(\mu,\nu)} - \lim x_{ki} = \xi$. Then $M = \{(k_m, i_m) : k_1 < k_2 < ...; i_1 < i_2 < ...\} \in \mathcal{F}(\mathcal{I}_2)$ such that $(\mu, \nu) - \lim_{m\to\infty} x_{k_m i_m} = \xi$. But then for each $\varepsilon > 0$ and $t > 0$ there exists a positive integer N such that $\mu(x_{k_m i_m} - \xi, t) > 1 - \varepsilon$ and $\nu(x_{k_m i_m} - \xi, t) < \varepsilon$ for all $m > N$. Since

$$\{n \in \mathbb{N} : \mu(x_{k_m i_m} - \xi, t) > 1 - \varepsilon \text{ or } \nu(x_{k_m i_m} - \xi, t) < \varepsilon\} \in \mathcal{I}_2.$$

Hence,

$$\left\{k \in \mathbb{N} : \mu\left(\widehat{F}x_{ki} - \xi, t\right) > 1 - \varepsilon \text{ or } \nu\left(\widehat{F}x_{ki} - \xi, t\right) < \varepsilon\right\}$$

$$\subseteq H \cup \{(k_m, i_m) : k_1 < k_2 < ... < k_{N-1}; i_1 < i_2 < ... < i_{N-1}\} \in \mathcal{I}_2.$$

for all $\varepsilon > 0$ and $t > 0$. Therefore, we conclude that $\mathcal{I}_2^{(\mu,\nu)} - \lim x_{ki} = \xi$. □

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

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