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# Smarandache Curves of Spacelike Anti-Salkowski Curve with a Spacelike Principal Normal According to Frenet Frame

*Spacelike Asli Normalli Spacelike Anti-Salkowski Eğrisinin Frenet Çatısına Göre Smarandache Eğrileri*

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**Abstract**
In this study, the $TN$, $TB$, $NB$ and $TNB-$ Smarandache curves constructed by the Frenet vectors of spacelike anti-Salkowski curve with a spacelike principal normal were defined. Later, the Frenet vectors, the curvature and the torsion of this curves were calculated. Finally, the graphics of the curves were drawn with the maple program.

**Keywords**: Minkowski Space, Spacelike Anti-Salkowski Curve, Spacelike Smarandache Curve

***Öz***
*Bu çalışmada, spacelike asli normalli spacelike anti-Salkowski eğrisinin Frenet vektörleri tarafından elde edilen $TN$, $TB$, $NB$ ve $TNB-$ Smarandache eğrileri tanımlandı. Daha sonra, bu eğrilerinin Frenet vektörleri, eğrilik ve torsiyonu hesaplandı. Son olarak, maple proğramı ile eğrilerin grafikleri çizildi.*

***Anahtar kelimeler:*** *Minkowski Uzayı, Spacelike Anti-Salkowski Eğri, Spacelike Smarandache Eğri*

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## 1. Introduction

In the years 1844-1923, anti-Salkowski curves are defined as family of curves with constant torsion but non-constant curvature with an explicit parametrization by E.Salkowski (Salkowski, 1909). In literature, these curves are known as anti-Salkowski curves. The equation of anti-Salkowski curve is given by J. Monterde and he showed that the principal normal vector of this curve makes a constant angle with a constant direction (Monterde, 2009). Similar to the anti-Salkowski curve, some authors have studied the Salkowski curve, Turgut, and Yılmaz, described the Smarandache curves in Minkowski space (Turgut and Yılmaz, 2008a, b). Later, according to the Darboux frame, Bishop frame and Sabban frame, some features of the Smarandache curves are investigated by (Ali, 2010; Şenyut and Sivas, 2013; Bektaş and Yüce, 2013; Çetin et al., 2014; Taşköprü and Tosun, 2014; Çalışkan and Şenyurt, 2015). Timelike anti-Salkowski curve, spacelike anti-Salkowski curve with a spacelike principal normal and spacelike anti-Salkowski curve with a timelike principal normal have given the definition (Ali, 2009, 2010, 2011). Şenyurt and Eren also studied the Smarandache curves obtained from the Frenet vectors of the timelike anti-Salkowski curve (2019a, b).

In this study, $TN$, $TB$, $NB$ and $TNB-$ Smarandache curves, are drawn by unit vector which is obtained from the linear combination of $T$ the unit tangent vector, $N$ unit principal normal vector and $B$ unit binormal vectors of spacelike anti-Salkowski curve with a spacelike principal normal, are defined. The Frenet apparatus of each curve are calculated and the graph of Sanarandache curves is given.

## 2. Preliminaries

The Minkowski 3-space $R_1^3$ be Lorentzian inner product given by

$$\langle , \rangle = -dx_1^2 + dx_2^2 + dx_3^2$$

where, $X = (x_1, x_2, x_3) \in R^3$. The vector product of any vectors $X = (x_1, x_2, x_3)$ and $Y = (y_1, y_2, y_3)$ in $R_1^3$ is defined by

$$X x Y = -\begin{vmatrix} -i & j & k \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix}$$

For an arbitrary vector $X \in R_1^3$, if $\langle X, X \rangle > 0$ or $X = 0$ then $X$ is timelike vector, if $\langle X, X \rangle < 0$, then $X$ is spacelike vector, if $\langle X, X \rangle = 0$, $X \neq 0$, then $X$ is lightlike (or null) vector. The norm of an arbitrary vector $X \in R_1^3$ is $\|X\| = \sqrt{|\langle X, X \rangle|}$.

If $\gamma'(t)$ tangent vector of $\gamma : I \to R_1^3$ curve is timelike vector then $\gamma(t)$ is timelike curve, If $\gamma'(t)$ tangent vector of $\gamma : I \to R_1^3$ curve is timelike vector and spacelike vector then $\gamma(t)$ is timelike curve and spacelike curve, respectively (O'Neill, 1983). The Frenet vectors, the curvatures and the Frenet formula of $\gamma(t)$ spacelike curve with a spacelike principal normal are

$$\begin{aligned} T(t) &= \frac{\gamma'(t)}{\|\gamma'(t)\|}, \\ B(t) &= \frac{\gamma'(t) \wedge \gamma''(t)}{\|\gamma'(t) \wedge \gamma''(t)\|}, \\ N(t) &= B(t) \wedge T(t), \\ \kappa(t) &= \frac{\|\gamma'(t) \wedge \gamma''(t)\|}{\|\gamma'(t)\|^3}, \\ \tau(t) &= \frac{\langle \gamma'(t) \wedge \gamma''(t), \gamma'''(t) \rangle}{\|\gamma'(t) \wedge \gamma''(t)\|^2}, \end{aligned} \qquad (1)$$

$$T' = \kappa N, \ N' = -\kappa T + \tau B, \ B' = \tau N, \qquad (2)$$

respectively. Where $T$ and $N$ are spacelike vectors and $B$ is timelike vector (Ali, 2009).

**Definition 1.** For an arbitrary $m > 1$ and $m \in R$, Let us define the space curve

$$\gamma_m(t) = \frac{n}{4m} \begin{pmatrix} \frac{1-n}{1+2n} \sinh\big((1+2n)t\big) - \frac{1+n}{1-2n} \sinh\big((1-2n)t\big) + 2n \sinh(t), \\ \frac{1-n}{1+2n} \cosh\big((1+2n)t\big) - \frac{1+n}{1-2n} \cosh\big((1-2n)t\big) + 2n \cosh(t), \\ \frac{1}{m}\big(2nt - \sinh(2nt)\big) \end{pmatrix} \qquad (3)$$

where $n = \dfrac{m}{\sqrt{m^2 - 1}}$ (Figure 1). This curve is called anti-Salkowski with a spacelike principal normal. The arc-length of spacelike anti-Salkowski curve with a spacelike principal normal is $s=\dfrac{\sinh(nt)}{m}$ The curvature, the torsion and Frenet frame of spacelike anti-Salkowski with a spacelike principal normal is given as following

$$T(t)=\begin{pmatrix} -n\cosh(t)\sinh(nt)+\sinh(t)\cosh(nt), \\ -n\sinh(t)\sinh(nt)+\cosh(t)\cosh(nt), \\ -\dfrac{n}{m}\sinh(nt) \end{pmatrix},$$

$$N(t)=-\sqrt{1-n^2}\left(\cosh(t),\sinh(t),\frac{n}{\sqrt{n^2-1}}\right), \tag{4}$$

$$B(t)=\begin{pmatrix} \dfrac{n}{m\sqrt{1-n^2}}\left(\sinh(t)\sinh(nt)-n\cosh(t)\cosh(nt)\right), \\ \dfrac{n}{m\sqrt{1-n^2}}\left(\cosh(t)\sinh(nt)-n\sinh(t)\cosh(nt)\right), \\ \sqrt{1-n^2}\cosh(nt) \end{pmatrix},$$

$$\kappa(t)=\coth(nt),\ \tau(t)=1$$

respectively (Ali, 2009). From here, the equation (2) becomes

$$T'=\kappa N,\ \ N'=-\kappa T+B,\ \ B'=N. \tag{5}$$

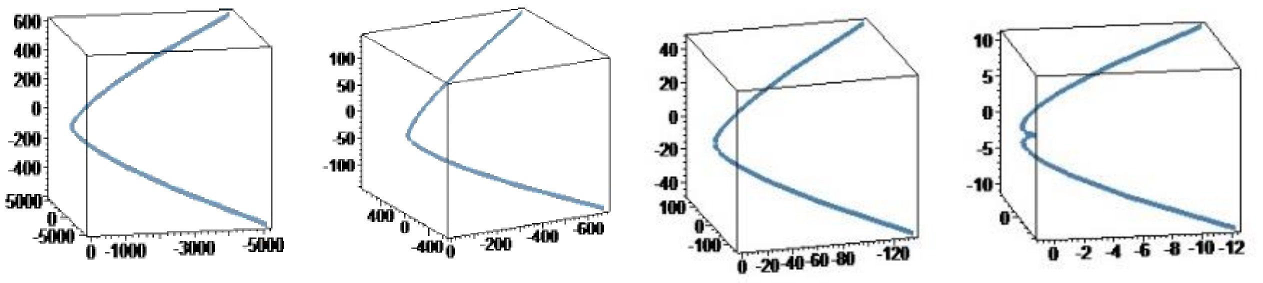

**Figure 1.** $m=\{3,5,8,16\}$ and $t\in[-5,5]$ for spacelike anti-Salkowski curve

## 3. Smarandache curves of spacelike anti-Salkowski curve with a spacelike principal normal according to Frenet frame

In this section, we describe Smarandache curves of spacelike anti-Salkowski curve with a spacelike principal normal according to Frenet frame and we calculate Frenet apparatus of Smarandache curves.

**Definition 2.** Let $\gamma_m(t)$ be a spacelike anti-Salkowski curve with a spacelike principal normal. Then. $\gamma_{TN}(t)$ Smarandache curves of $\gamma_m(t)$ can be defined by the frame vectors of $\gamma_m(t)$ such as:

$$\gamma_{TN}(t)=\frac{1}{\sqrt{2}}\left(T(t)+N(t)\right) \tag{6}$$

where, $T$ and $N$ are spacelike vector (Figure 2). Substituting $T$ and $N$ vectors into the equation (6), we get $\gamma_{TN}(t)$ curve as follow:

$$\gamma_{TN}(t)=\frac{1}{\sqrt{2}}\begin{pmatrix} -n\cosh(t)\sinh(nt) \\ +\sinh(t)\cosh(nt) \\ -\sqrt{1-n^2}\cosh(t), \\ -n\sinh(t)\sinh(nt) \\ +\cosh(t)\cosh(nt) \\ -\sqrt{1-n^2}\sinh(t), \\ -\dfrac{n}{m}\sinh(nt)-\dfrac{n\sqrt{1-n^2}}{\sqrt{n^2-1}} \end{pmatrix} \tag{7}$$

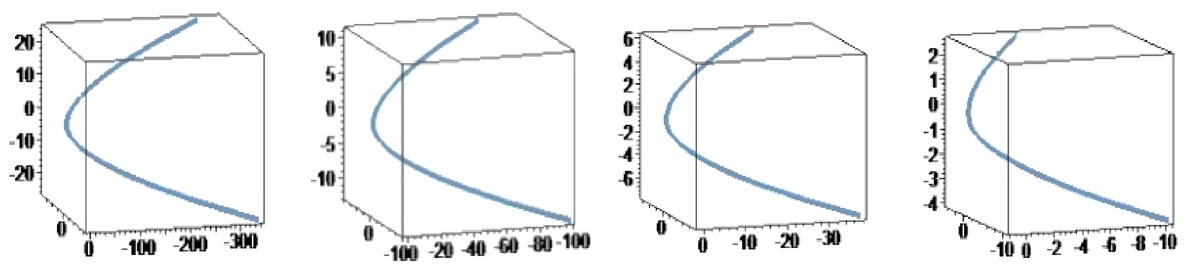

**Figure 2.** $m=\{3,5,8,16\}$ and $t\in[-5,5]$ for $TN-$Smarandache curve

**Theorem 1.** $T_{TN}, N_{TN}$ and $B_{TN}$ Frenet vectors of $\gamma_{TN}(t)$ Smarandache curve are

$$T_{TN} = -\frac{\kappa}{\sqrt{|2\kappa^2-1|}}T + \frac{\kappa}{\sqrt{|2\kappa^2-1|}}N + \frac{1}{\sqrt{|2\kappa^2-1|}}B,$$

$$N_{TN} = \frac{2\kappa^4-\kappa'}{\sqrt{|2\kappa^2-1|\left|\left(-2\kappa^2+\kappa'\right)^2+\kappa'^2-4\kappa^6\right|}}T + \frac{2\kappa^4-2\kappa^2+\kappa'}{\sqrt{|2\kappa^2-1|\left|\left(-2\kappa^2+\kappa'\right)^2+\kappa'^2-4\kappa^6\right|}}N, + \frac{-2\kappa^3+2\kappa\kappa'}{\sqrt{|2\kappa^2-1|\left|\left(-2\kappa^2+\kappa'\right)^2+\kappa'^2-4\kappa^6\right|}}B$$

$$B_{TN} = \frac{-2\kappa^2+\kappa'}{\sqrt{\left|\left(-2\kappa^2+\kappa'\right)^2+\kappa'^2-4\kappa^6\right|}}T + \frac{\kappa'}{\sqrt{\left|\left(-2\kappa^2+\kappa'\right)^2+\kappa'^2-4\kappa^6\right|}}N + \frac{2\kappa^3}{\sqrt{\left|\left(-2\kappa^2+\kappa'\right)^2+\kappa'^2-4\kappa^6\right|}}B,$$

$\left(2\kappa^2-1\right)\neq 0, \left(-2\kappa^2+\kappa'\right)^2+\kappa'^2-4\kappa^6\neq 0,$
respectively.

**Proof:** Considering the equation (5) in derivate of the equation (7), we get

$$\gamma'_{TN}(t) = \frac{1}{\sqrt{2}}\left(-\kappa T+\kappa N+B\right). \tag{8}$$

The norm of this equation is found

$$\|\gamma'_{TN}(t)\| = \frac{1}{\sqrt{2}}\sqrt{|2\kappa^2-1|}. \tag{9}$$

From the equations (8) and (9), the tangent vector of $\gamma_{TN}(t)$ curve is found

$$T_{TN} = -\frac{\kappa}{\sqrt{|2\kappa^2-1|}}T + \frac{\kappa}{\sqrt{|2\kappa^2-1|}}N + \frac{1}{\sqrt{|2\kappa^2-1|}}B, \quad \left(2\kappa^2-1\right)\neq 0. \tag{10}$$

If we take derivate of the equation (8), it gets

$$\gamma''_{TN}(t) = \frac{1}{\sqrt{2}}\left(\left(-\kappa'-\kappa^2\right)T+\left(\kappa'-\kappa^2\right)N+\kappa B\right). \tag{11}$$

From the equations (8) and (11) we found

$$\gamma'_{TN}(t)\wedge\gamma''_{TN}(t) = \frac{1}{2}\left(\left(-2\kappa^2+\kappa'\right)T+\kappa' N+2\kappa^3 B\right). \tag{12}$$

The norm of this equation is

$$\|\gamma'_{TN}(t)\wedge\gamma''_{TN}(t)\| = \frac{1}{2}\sqrt{\left|\left(-2\kappa^2+\kappa'\right)^2+\kappa'^2-4\kappa^6\right|} \tag{13}$$

From the equations (12) and (13), the binormal vector of $\gamma_{TN}(t)$ Smarandache curve is found

$$B_{TN} = \frac{-2\kappa^2+\kappa'}{\sqrt{\left|\left(-2\kappa^2+\kappa'\right)^2+\kappa'^2-4\kappa^6\right|}}T + \frac{\kappa'}{\sqrt{\left|\left(-2\kappa^2+\kappa'\right)^2+\kappa'^2-4\kappa^6\right|}}N + \frac{2\kappa^3}{\sqrt{\left|\left(-2\kappa^2+\kappa'\right)^2+\kappa'^2-4\kappa^6\right|}}B, \quad \left(-2\kappa^2+\kappa'\right)^2+\kappa'^2-4\kappa^6\neq 0. \tag{14}$$

and from the equations (10) and (14), the principal normal vector of $\gamma_{TN}(t)$ Smarandache curve is obtained by

$$N_{TN} = \frac{2\kappa^4-\kappa'}{\sqrt{\left|\left(2\kappa^2-1\right)\left(\left(-2\kappa^2+\kappa'\right)^2+\kappa'^2-4\kappa^6\right)\right|}}T + \frac{2\kappa^4-2\kappa^2+\kappa'}{\sqrt{\left|\left(2\kappa^2-1\right)\left(\left(-2\kappa^2+\kappa'\right)^2+\kappa'^2-4\kappa^6\right)\right|}}N + \frac{-2\kappa^3+2\kappa\kappa'}{\sqrt{\left|\left(2\kappa^2-1\right)\left(\left(-2\kappa^2+\kappa'\right)^2+\kappa'^2-4\kappa^6\right)\right|}}B, \quad \left(2\kappa^2-1\right)\neq 0, \left(\left(-2\kappa^2+\kappa'\right)^2+\kappa'^2-4\kappa^6\right)\neq 0. \tag{15}$$

**Theorem 2.** The curvature and torsion of $\gamma_{TN}(t)$ Smarandache curve are

$$\kappa_{TN} = \frac{\sqrt{2\left|\left(-2\kappa^2+\kappa'\right)^2+\kappa'^2-4\kappa^6\right|}}{\left(2\kappa^2-1\right)\sqrt{|2\kappa^2-1|}}, \quad \tau_{TN} = \frac{\sqrt{2}\left(2\kappa^3\kappa'+2\kappa^2\kappa''-6\kappa\kappa'^2-\kappa\kappa'\right)}{\left|\left(-2\kappa^2+\kappa'\right)^2+\kappa'^2-4\kappa^6\right|}, \tag{16}$$

$\left(2\kappa^2-1\right)\neq 0,\ \left(-2\kappa^2+\kappa'\right)^2+\kappa'^2-4\kappa^6\neq 0,$
receptively.

**Proof:** From the equations (1) (9) and (13), we get $\kappa_{TN}$ the curvature of the $\gamma_{TN}(t)$ curve.
The derivate of the equation (11) is

$$\gamma'''_{TN}(t)=\frac{1}{\sqrt{2}}\begin{pmatrix}\left(\kappa^3-3\kappa\kappa'-\kappa''\right)T\\+\left(-\kappa^3-\kappa-3\kappa\kappa'+\kappa''\right)N\\+\left(-\kappa^2+2\kappa'\right)B\end{pmatrix}\qquad(17)$$

Considering (8), (11), (13) and (17) in the equation (1) we obtain $\tau_{TN}$ the torsion of the $\gamma_{TN}(t)$ curve.

**Definition 3.** Let $\gamma_m(t)$ be a spacelike anti-Salkowski curve with a spacelike principal normal. Then. $\gamma_{TB}(t)$ Smarandache curves of $\gamma_m(t)$ can be defined by the frame vectors of $\gamma_m(t)$ such as:

$$\gamma_{TB}(t)=\frac{1}{\sqrt{2}}\left(T(t)+B(t)\right)\qquad(18)$$

where, $T$ is spacelike vector $B$ is timelike vector (Figure 3). Substituting $T$ and $B$ vectors into (18) the equation, we get $\gamma_{TB}(t)$ curve as following:

$$\gamma_{TB}(t)=\frac{1}{\sqrt{2}}\begin{pmatrix}-n\cosh(t)\sinh(nt)+\sinh(t)\cosh(nt)\\+\dfrac{n}{m\sqrt{1-n^2}}\begin{pmatrix}\sinh(t)\sinh(nt)\\-n\cosh(t)\cosh(nt)\end{pmatrix},\\-n\sinh(t)\sinh(nt)+\cosh(t)\cosh(nt)\\+\dfrac{n}{m\sqrt{1-n^2}}\begin{pmatrix}\cosh(t)\sinh(nt)\\-n\sinh(t)\cosh(nt)\end{pmatrix},\\-\dfrac{n}{m}\sinh(nt)+\sqrt{1-n^2}\cosh(nt)\end{pmatrix}\qquad(19)$$

**Theorem 3.** $\{T_{TB},N_{TB},B_{TB}\}$ Frenet frame of $\gamma_{TB}(t)$ Smarandache curve is given by
$T_{TB}=N$,

$$N_{TB}=-\frac{\kappa}{\sqrt{|1-\kappa^2|}}T+\frac{1}{\sqrt{|1-\kappa^2|}}B,$$

$$B_{TB}=-\frac{1}{\sqrt{|1-\kappa^2|}}T+\frac{\kappa}{\sqrt{|1-\kappa^2|}}B,1-\kappa^2\neq0.$$

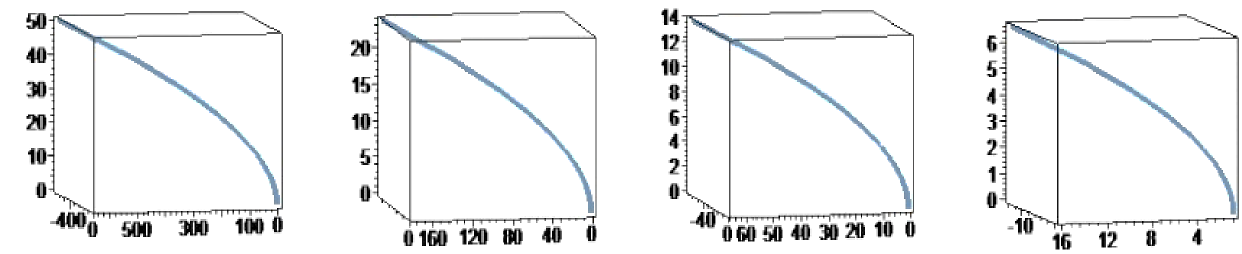

**Figure 3.** $m=\{3,5,8,16\}$ and $t\in[-5,5]$ for $TB-$Smarandache curve

**Proof:** Considering (5) in the derivate of the equation (18), we get

$$\gamma'_{TB}(t)=\frac{1}{\sqrt{2}}(\kappa+1)N\qquad(20)$$

The norm of this equation is found

$$\|\gamma'_{TB}(t)\|=\frac{1}{\sqrt{2}}|\kappa+1|\qquad(21)$$

From the equations (20) and (21), the tangent vector of $\gamma_{TB}(t)$ Smarandache curve is found by

$$T_{TN}=N\qquad(22)$$

If we take derivate of the equation (20), we get

$$\gamma''_{TB}(t)=\frac{1}{\sqrt{2}}\left(\left(-\kappa^2-\kappa\right)T+\kappa'N+(\kappa+1)B\right)\qquad(23)$$

From the equations (20) and (23) we have

$$\gamma'_{TN}(t)\wedge\gamma''_{TN}(t)=\frac{1}{2}\left(-(\kappa+1)^2T+\kappa(\kappa+1)^2B\right)\qquad(24)$$

The norm of this equation is

$$\|\gamma'_{TN}(t)\wedge\gamma''_{TN}(t)\|=\frac{1}{2}(\kappa+1)^2\sqrt{|1-\kappa^2|}\qquad(25)$$

From the equations (24) and (25), the binormal vector of $\gamma_{TB}(t)$ Smarandache curve is found by

$$B_{TB}=-\frac{1}{\sqrt{|1-\kappa^2|}}T+\frac{\kappa}{\sqrt{|1-\kappa^2|}}B,\ 1-\kappa^2\neq0\qquad(26)$$

and from the equations (22) and (26) the principal normal vector of $\gamma_{TB}(t)$ Smarandache curve is obtained by

$$N_{TB} = -\frac{\kappa}{\sqrt{\left|1-\kappa^2\right|}}T + \frac{1}{\sqrt{\left|1-\kappa^2\right|}}B,\ 1-\kappa^2 \neq 0 \tag{27}$$

**Theorem 4.** The curvature and the torsion of $\gamma_{TB}(t)$ Smarandache curve are

$$\kappa_{TB} = \frac{\sqrt{2}\sqrt{\left|1-\kappa^2\right|}}{\left|\kappa+1\right|} \text{ and } \tau_{TB} = \frac{\sqrt{2}\left(2\kappa+1\right)\kappa'}{\left|\left(\kappa+1\right)^3\right|\left|1-\kappa\right|},\ \kappa \neq \pm 1 \tag{28}$$

receptively.

**Proof:** Considering the equations (21) and (25) in the equation (1) we find $\kappa_{TN}$ the curvature of $\gamma_{TB}(t)$ Smarandache curve as

$$\kappa_{TB} = \frac{\sqrt{2}\sqrt{\left|1-\kappa^2\right|}}{\left|\kappa+1\right|},\ \kappa \neq -1$$

If we take derivate of the equation (23), we get

$$\gamma_{TB}'''(t) = \frac{1}{\sqrt{2}}\begin{pmatrix} \left(-3\kappa\kappa'-\kappa'\right)T \\ +\left(-\kappa^3-\kappa^2+\kappa+1+\kappa''\right)N \\ +2\kappa' B \end{pmatrix} \tag{29}$$

From the equations (20), (23), (25) and (29) it obtains $\gamma_{TB}(t)$ the torsion as

$$\tau_{TB} = \frac{\sqrt{2}\left(2\kappa+1\right)\kappa'}{\left|\left(\kappa+1\right)^3\left(1-\kappa\right)\right|},\ \kappa \neq \pm 1$$

**Definition 4.** Let $\gamma_m(t)$ be a spacelike anti-Salkowski curve with a spacelike principal normal. Then. $\gamma_{NB}(t)$ Smarandache curves of $\gamma_m(t)$ can be defined by the frame vectors of $\gamma_m(t)$ such as:

$$\gamma_{NB}(t) = \frac{1}{\sqrt{2}}\left(N(t)+B(t)\right) \tag{30}$$

where, $N$ is spacelike vector and $B$ is timelike vector (Figure 4). Substituting $N$ and $B$ vectors into (30) the equation, we get $\gamma_{NB}(t)$ curve as following:

$$\gamma_{NB}(t) = \frac{1}{\sqrt{2}}\begin{pmatrix} -\sqrt{1-n^2}\cosh(t) \\ +\frac{n}{m\sqrt{1-n^2}}\begin{pmatrix} \sinh(t)\sinh(nt) \\ -n\cosh(t)\cosh(nt) \end{pmatrix}, \\ -\sqrt{1-n^2}\sinh(t) \\ +\frac{n}{m\sqrt{1-n^2}}\begin{pmatrix} \cosh(t)\sinh(nt) \\ -n\sinh(t)\cosh(nt) \end{pmatrix}, \\ \frac{-n\sqrt{1-n^2}}{\sqrt{n^2-1}} + \sqrt{1-n^2}\cosh\left(nt\right). \end{pmatrix} \tag{31}$$

**Theorem 5.** $\left\{T_{NB}, N_{NB}, B_{NB}\right\}$ Frenet frame of $\gamma_{NB}(t)$ Smarandache curve is given by

$$T_{NB} = -\frac{\kappa}{\left|\kappa\right|}T + \frac{1}{\left|\kappa\right|}N + \frac{1}{\left|\kappa\right|}B,$$

$$N_{NB} = \frac{\kappa^3-\kappa-\kappa'}{\left|\kappa\right|\sqrt{\left|\kappa^4+\kappa'^2-\left(\kappa^3-\kappa\right)^2\right|}}T + \frac{\kappa^3-2\kappa^2}{\left|\kappa\right|\sqrt{\left|\kappa^4+\kappa'^2-\left(\kappa^3-\kappa\right)^2\right|}}N - \frac{\kappa^2+\kappa\kappa'}{\left|\kappa\right|\sqrt{\left|\kappa^4+\kappa'^2-\left(\kappa^3-\kappa\right)^2\right|}}B,$$

$$B_{NB} = -\frac{\kappa^2}{\sqrt{\left|\kappa^4+\kappa'^2-\left(\kappa^3-\kappa\right)^2\right|}}T + \frac{\kappa'}{\sqrt{\left|\kappa^4+\kappa'^2-\left(\kappa^3-\kappa\right)^2\right|}}N + \frac{\kappa^3-\kappa}{\sqrt{\left|\kappa^4+\kappa'^2-\left(\kappa^3-\kappa\right)^2\right|}}B,$$

$$\kappa^4+\kappa'^2-\left(\kappa^3-\kappa\right)^2 \neq 0.\ \kappa \neq 0.$$

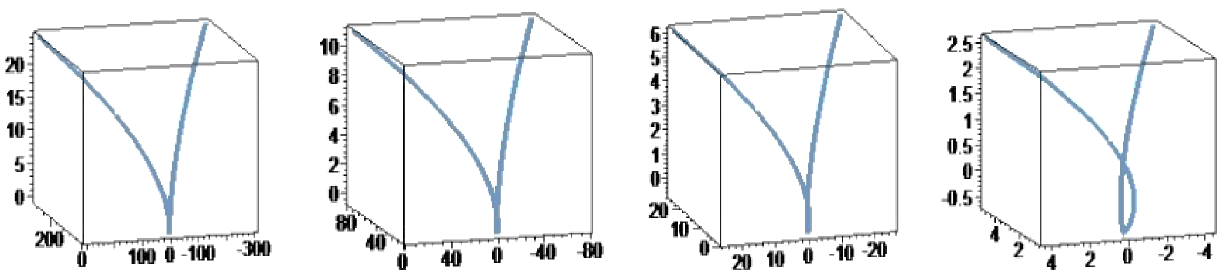

**Figure 4.** $m=\left\{3,5,8,16\right\}$ and $t\in\left[-5,5\right]$ for $NB-$Smarandache curve

**Proof:** The derivate of the equation (30) is

$$\gamma'_{NB}(t)=\frac{1}{\sqrt{2}}\left(-\kappa T+N+B\right) \tag{32}$$

The norm of this equation is found

$$\left\|\gamma'_{NB}(t)\right\|=\frac{1}{\sqrt{2}}|\kappa| \tag{33}$$

From the equations (32) and (33), the tangent vector of $\gamma_{NB}(t)$ Smarandache curve is found

$$T_{NB}=-\frac{\kappa}{|\kappa|}T+\frac{1}{|\kappa|}N+\frac{1}{|\kappa|}B,\ \kappa\neq 0 \tag{34}$$

If we take derivate of the equation (32), we get

$$\gamma''_{NB}(t)=\frac{1}{\sqrt{2}}\left(\left(-\kappa'-\kappa\right)T+\left(1-\kappa^{2}\right)N+B\right) \tag{35}$$

From the equations (32) and (35) it is found

$$\gamma'_{NB}(t)\wedge\gamma''_{NB}(t)=\frac{1}{2}\left(\left(-\kappa^{2}\right)T+\kappa' N+\left(\kappa^{3}-\kappa\right)B\right) \tag{36}$$

The norm of this equation is

$$\left\|\gamma'_{NB}(t)\wedge\gamma''_{NB}(t)\right\|=\frac{1}{2}\sqrt{\left|\kappa^{4}+\kappa'^{2}-\left(\kappa^{3}-\kappa\right)^{2}\right|} \tag{37}$$

From the equations (36) and (37), the tangent vector of $\gamma_{NB}(t)$ curve is found

$$\begin{aligned}B_{NB}=&-\frac{\kappa^{2}}{\sqrt{\left|\kappa^{4}+\kappa'^{2}-\left(\kappa^{3}-\kappa\right)^{2}\right|}}T\\&+\frac{\kappa'}{\sqrt{\left|\kappa^{4}+\kappa'^{2}-\left(\kappa^{3}-\kappa\right)^{2}\right|}}N\\&+\frac{\kappa^{3}-\kappa}{\sqrt{\left|\kappa^{4}+\kappa'^{2}-\left(\kappa^{3}-\kappa\right)^{2}\right|}}B,\\&\kappa^{4}+\kappa'^{2}-\left(\kappa^{3}-\kappa\right)^{2}\neq 0.\end{aligned} \tag{38}$$

From the equations (34) and (38) the principal normal vector of $\gamma_{NB}(t)$ Smarandache curve is obtained by

$$\begin{aligned}N_{NB}=&\frac{\kappa^{3}-\kappa-\kappa'}{|\kappa|\sqrt{\left|\kappa^{4}+\kappa'^{2}-\left(\kappa^{3}-\kappa\right)^{2}\right|}}T\\&+\frac{\kappa^{3}-2\kappa^{2}}{|\kappa|\sqrt{\left|\kappa^{4}+\kappa'^{2}-\left(\kappa^{3}-\kappa\right)^{2}\right|}}N\\&-\frac{\kappa^{2}+\kappa\kappa'}{|\kappa|\sqrt{\left|\kappa^{4}+\kappa'^{2}-\left(\kappa^{3}-\kappa\right)^{2}\right|}}B,\\&\kappa^{4}+\kappa'^{2}-\left(\kappa^{3}-\kappa\right)^{2}\neq 0.\ \kappa\neq 0.\end{aligned} \tag{39}$$

**Theorem 6.** The curvature and torsion of $\gamma_{NB}(t)$ Smarandache curve are

$$\begin{aligned}\kappa_{NB}&=\frac{\sqrt{2}\sqrt{\left|\kappa^{4}+\kappa'^{2}-\left(\kappa^{3}-\kappa\right)^{2}\right|}}{|\kappa|^{3}},\\\tau_{NB}&=\frac{\sqrt{2}\left(-\kappa^{3}+\kappa^{2}\kappa''+\kappa-3\kappa\kappa'^{2}+\kappa'\right)}{\left|\kappa^{4}+\kappa'^{2}-\left(\kappa^{3}-\kappa\right)^{2}\right|},\\&\kappa^{4}+\kappa'^{2}-\left(\kappa^{3}-\kappa\right)^{2}\neq 0.\ \kappa\neq 0,\end{aligned} \tag{40}$$

respectively.

**Proof:** From the equations (1), (33) and (37), we find $\kappa_{NB}$ the curvature of $\gamma_{NB}(t)$ curve. The derivate of the equation (35) is

$$\gamma'''_{TN}(t)=\frac{1}{\sqrt{2}}\begin{pmatrix}\left(\kappa^{3}-\kappa-\kappa''-\kappa'\right)T\\+\left(1-\kappa^{2}-3\kappa\kappa'\right)N+\left(1-\kappa^{2}\right)B\end{pmatrix} \tag{41}$$

From the equations (32), (35), (37) and (41), it obtains the torsion of $\gamma_{NB}(t)$ curve.

**Definition 5.** Let $\gamma_{m}(t)$ be a spacelike anti-Salkowski curve with a spacelike principal normal. Then. $\gamma_{TNB}(t)$ Smarandache curves of $\gamma_{m}(t)$ can be defined by the frame vectors of $\gamma_{m}(t)$ such as:

$$\gamma_{TNB}(t)=\frac{1}{\sqrt{3}}\left(T(t)+N(t)+B(t)\right) \tag{42}$$

where $T$ is spacelike vector, $N$ is spacelike vector and $B$ is timelike vector (Figure 5). Substituting $T$, $N$ and $B$ vectors into (42) the equation, we get $\gamma_{TNB}(t)$ curve as following:

$$\gamma_{TNB}(t)=\frac{1}{\sqrt{3}}\begin{pmatrix} -n\cosh(t)\sinh(nt)+\sinh(t)\cosh(nt) \\ +\dfrac{n}{m\sqrt{1-n^{2}}}\left(\sinh(t)\sinh(nt)-n\cosh(t)\cosh(nt)\right) \\ -\sqrt{1-n^{2}}\cosh(t), \\ -n\sinh(t)\sinh(nt)+\cosh(t)\cosh(nt) \\ +\dfrac{n}{m\sqrt{1-n^{2}}}\left(\cosh(t)\sinh(nt)-n\sinh(t)\cosh(nt)\right) \\ -\sqrt{1-n^{2}}\sinh(t), \\ -\dfrac{n}{m}\sinh(nt)+\sqrt{1-n^{2}}\cosh(nt)-\dfrac{n\sqrt{1-n^{2}}}{\sqrt{n^{2}-1}} \end{pmatrix} \tag{43}$$

**Theorem 7.** $\{T_{TNB},N_{TNB},B_{TNB}\}$ Frenet frame of $\gamma_{TNB}(t)$ Smarandache curve is given by

$$T_{TNB}=-\frac{\kappa}{\sqrt{|2\kappa(\kappa+1)|}}T+\frac{\kappa+1}{\sqrt{|2\kappa(\kappa+1)|}}N+\frac{1}{\sqrt{|2\kappa(\kappa+1)|}}B,$$

$$N_{TNB}=\frac{2\kappa^{3}+2\kappa^{2}}{\sqrt{|2\kappa(\kappa+1)|\left|(-2\kappa^{2}-2\kappa+\kappa')^{2}+\kappa'^{2}-(2\kappa^{3}+2\kappa^{2}+\kappa')^{2}\right|}}T$$
$$+\frac{2\kappa^{4}+2\kappa^{3}-2\kappa^{2}-2\kappa+\kappa\kappa'+\kappa'}{\sqrt{|2\kappa(\kappa+1)|\left|(-2\kappa^{2}-2\kappa+\kappa')^{2}+\kappa'^{2}-(2\kappa^{3}+2\kappa^{2}+\kappa')^{2}\right|}}N$$
$$+\frac{-2\kappa^{3}-4\kappa^{2}-2\kappa+2\kappa\kappa'+\kappa'}{\sqrt{|2\kappa(\kappa+1)|\left|(-2\kappa^{2}-2\kappa+\kappa')^{2}+\kappa'^{2}-(2\kappa^{3}+2\kappa^{2}+\kappa')^{2}\right|}}B,$$

$$B_{TNB}=\frac{-2\kappa^{2}-2\kappa+\kappa'}{\sqrt{\left|(-2\kappa^{2}-2\kappa+\kappa')^{2}+\kappa'^{2}-(2\kappa^{3}+2\kappa^{2}+\kappa')^{2}\right|}}T$$
$$+\frac{\kappa'}{\sqrt{\left|(-2\kappa^{2}-2\kappa+\kappa')^{2}+\kappa'^{2}-(2\kappa^{3}+2\kappa^{2}+\kappa')^{2}\right|}}N$$
$$+\frac{2\kappa^{3}+2\kappa^{2}+\kappa'}{\sqrt{\left|(-2\kappa^{2}-2\kappa+\kappa')^{2}+\kappa'^{2}-(2\kappa^{3}+2\kappa^{2}+\kappa')^{2}\right|}}B,$$

$$(-2\kappa^{2}-2\kappa+\kappa')^{2}+\kappa'^{2}-(2\kappa^{3}+2\kappa^{2}+\kappa')^{2}\neq 0,$$
$$\kappa(\kappa+1)\neq 0.$$

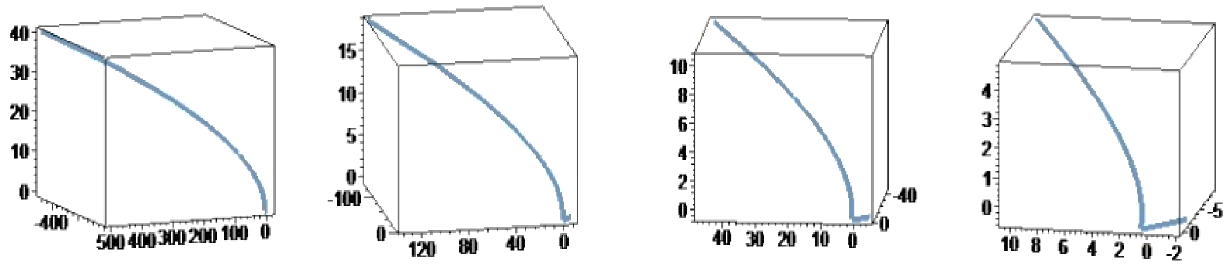

**Figure 5.** $m=\{3,5,8,16\}$ and $t\in[-5,5]$ for $TNB-$Smarandache curve.

**Proof:** The derivate of the equation (42) is

$$\gamma'_{TNB}(t)=\frac{1}{\sqrt{3}}\left(-\kappa T+(\kappa+1)N+B\right) \tag{44}$$

The norm of this equation is found

$$\|\gamma'_{TN}(t)\|=\frac{1}{\sqrt{3}}\sqrt{|2\kappa(\kappa+1)|} \tag{45}$$

From the equations (44) and (45), the tangent vector of $\gamma_{TNB}(t)$ curve is

$$T_{TNB}=-\frac{\kappa}{\sqrt{2|\kappa(\kappa+1)|}}T+\frac{\kappa+1}{\sqrt{2|\kappa(\kappa+1)|}}N+\frac{1}{\sqrt{2|\kappa(\kappa+1)|}}B,\ \kappa(\kappa+1)\neq 0 \tag{46}$$

The derivate of the equation (44) is

$$\gamma''_{TN}(t)=\frac{1}{\sqrt{3}}\begin{pmatrix}(-\kappa'-\kappa-\kappa^{2})T \\ +(\kappa'-\kappa^{2}+1)N+(\kappa+1)B.\end{pmatrix} \tag{47}$$

From the equations (44) and (47), it is found

$$\gamma'_{TNB}(t)\wedge\gamma''_{TNB}(t)=\frac{1}{3}\begin{pmatrix}\left(-2\kappa^2-2\kappa+\kappa'\right)T\\ +\kappa' N\\ +\left(2\kappa^3+2\kappa^2+\kappa'\right)B.\end{pmatrix} \tag{48}$$

The norm of the equation (48) is

$$\left\|\gamma'_{TNB}(t)\wedge\gamma''_{TNB}(t)\right\|=\frac{1}{3}\sqrt{\begin{vmatrix}\left(-2\kappa^2-2\kappa+\kappa'\right)^2\\ +\kappa'^2-\left(2\kappa^3+2\kappa^2+\kappa'\right)^2\end{vmatrix}} \tag{49}$$

From the equations (48) and (49), the binormal vector of $\gamma_{TNB}(t)$ Smarandache curve is found

$$\begin{aligned}B_{TNB}=&\frac{-2\kappa^2-2\kappa+\kappa'}{\sqrt{\left|\left(-2\kappa^2-2\kappa+\kappa'\right)^2+\kappa'^2-\left(2\kappa^3+2\kappa^2+\kappa'\right)^2\right|}}T\\ &+\frac{\kappa'}{\sqrt{\left|\left(-2\kappa^2-2\kappa+\kappa'\right)^2+\kappa'^2-\left(2\kappa^3+2\kappa^2+\kappa'\right)^2\right|}}N\\ &+\frac{2\kappa^3+2\kappa^2+\kappa'}{\sqrt{\left|\left(-2\kappa^2-2\kappa+\kappa'\right)^2+\kappa'^2-\left(2\kappa^3+2\kappa^2+\kappa'\right)^2\right|}}B,\end{aligned} \tag{50}$$

$$\left(-2\kappa^2-2\kappa+\kappa'\right)^2+\kappa'^2-\left(2\kappa^3+2\kappa^2+\kappa'\right)^2\neq 0.$$

From the equations (46) and (50), the principal normal vector of $\gamma_{TNB}(t)$ curve is obtained by

$$\begin{aligned}N_{TNB}=&\frac{2\kappa^3+2\kappa^2}{\sqrt{\left|\left(\kappa+1\right)^2+\kappa^2-1\right|\left|\left(-2\kappa^2-2\kappa+\kappa'\right)^2+\kappa'^2-\left(2\kappa^3+2\kappa^2+\kappa'\right)^2\right|}}T\\ &+\frac{2\kappa^4+2\kappa^3-2\kappa^2-2\kappa+\kappa\kappa'+\kappa'}{\sqrt{\left|\left(\kappa+1\right)^2+\kappa^2-1\right|\left|\left(-2\kappa^2-2\kappa+\kappa'\right)^2+\kappa'^2-\left(2\kappa^3+2\kappa^2+\kappa'\right)^2\right|}}N\\ &+\frac{-2\kappa^3-4\kappa^2-2\kappa+2\kappa\kappa'+\kappa'}{\sqrt{\left|\left(\kappa+1\right)^2+\kappa^2-1\right|\left|\left(-2\kappa^2-2\kappa+\kappa'\right)^2+\kappa'^2-\left(2\kappa^3+2\kappa^2+\kappa'\right)^2\right|}}B,\end{aligned} \tag{51}$$

$$\left(-2\kappa^2-2\kappa+\kappa'\right)^2+\kappa'^2-\left(2\kappa^3+2\kappa^2+\kappa'\right)^2\neq 0,\ \kappa\left(\kappa+1\right)\neq 0.$$

**Theorem 8.** The curvature and the torsion of $\gamma_{TNB}(t)$ Smarandache curve are

$$\kappa_{TNB}=\frac{\sqrt{3}\sqrt{\left(-2\kappa^2-2\kappa+\kappa'\right)^2+\kappa'^2+\left(2\kappa^3+2\kappa^2+\kappa'\right)^2}}{\left|\left(\kappa+1\right)^2+\kappa^2-1\right|\sqrt{\left|\left(\kappa+1\right)^2+\kappa^2-1\right|}},$$

$$\tau_{TNB}=\frac{\sqrt{3}\left(2\kappa\kappa'\left(\kappa^2+2\kappa+1\right)-3\kappa'^2\left(2\kappa+1\right)+2\kappa\kappa''\left(\kappa+1\right)\right)}{\left|\left(-2\kappa^2-2\kappa+\kappa'\right)^2+\kappa'^2-\left(2\kappa^3+2\kappa^2+\kappa'\right)^2\right|}, \tag{52}$$

$$\left(-2\kappa^2-2\kappa+\kappa'\right)^2+\kappa'^2-\left(2\kappa^3+2\kappa^2+\kappa'\right)^2\neq 0,$$

$$\left(\kappa+1\right)^2+\kappa^2-1\neq 0,$$

respectively.

**Proof:** From the equations (1), (45) and (49) we find $\kappa_{TNB}$ the curvature of $\gamma_{TNB}(t)$ curve. The derivate of the equation (47) is

$$\gamma'''_{TN}(t)=\frac{1}{\sqrt{3}}\begin{pmatrix}\left(\kappa^3-\kappa-3\kappa\kappa'-\kappa'-\kappa''\right)T\\ +\left(-\kappa^3-\kappa^2+\kappa-3\kappa\kappa'+1+\kappa''\right)N\\ +\left(-\kappa^2+1+2\kappa'\right)B.\end{pmatrix} \tag{53}$$

From the equations (44), (47), (49) and (53) it is obtained the torsion of $\gamma_{TNB}(t)$ curve.

**Corollary 1.** $TN$, $TB$, $NB$ and $TNB-$ Smarandache curves of spacelike anti-Salkowski curve with a spacelike principal normal are spacelike Smarandache curves with a timelike principal normal.

**Proof:** From the theorems (1), (3), (5) and (7), proof is easily seen.

**Corollary 2.** $TB-$Smarandache curve is evolute of spacelike anti-Salkowski curve with a spacelike principal normal.

**Proof:** From the equations (4) and (22), we get $\langle T,T_{TB}\rangle=\langle T,N\rangle=0$.

In that case, we call that $TB-$Smarandache curve is evolute of spacelike anti-Salkowski curve with a spacelike principal normal.

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