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# A novel method for solving a class of functional differential equations

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## Abstract

*In this work, a novel numerical method based on generalized Laguerre series is introduced. The numerical technique is applied for the solution of a class of functional differential equations with variable delays. This numerical method is substantially related to generalized Laguerre series also its matrix forms as well as collocation points. By error estimation the pertinent features and applicability of the method are demonstrated.*

***Keywords:*** *Generalized Laguerre series, collocation methods, functional differential equations, variable delays.*

# Fonksiyonel diferansiyel denklemlerin bir sınıfının çözümü için yeni bir yöntem

## Öz

*Bu çalışmada, genelleştirilmiş Laguerre serisine dayanan yeni bir sayısal yöntem tanıtıldı. Sayısal teknik, fonksiyonel diferansiyel denklemlerin değişken gecikmeli bir sınıfının çözümü için uygulanır. Bu sayısal yöntem, esas olarak genelleştirilmiş Laguerre serileri ile aynı zamanda matris formları ve sıralama noktaları ile ilgilidir. Hata tahmininde, yöntemin ilgili özellikleri ve uygulanabilirliği gösterilmektedir.*

***Anahtar kelimeler:*** *Genelleştirilmiş Laguerre serileri, sıralama yöntemleri, fonksiyonel diferansiyel denklemler, değişken gecikmeler.*

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## 1. Introduction

In this study, the pantograph type functional differential equations are considered, which involve the functions with hybrid proportional and variable delays, in the form

$$\sum_{k=0}^{m}\sum_{j=0}^{J} P_{kj}(x)y^{(k)}(q_{kj}x+\tau_{kj}(x))=g(x), \quad 0\le a\le x\le b<\infty \tag{1}$$

under the mixed conditions

$$\sum_{k=0}^{m-1}(a_{ki}y^{(k)}(a)+b_{ki}y^{(k)}(b))=\alpha_i, \; i=0,1,2,...,m-1 \tag{2}$$

Here, $P_{kj}(x)$, $\tau_{kj}(x)$ and $g(x)$ are appropriate analytic known functions on the interval $0\le a\le x\le b<\infty$; $q_{kj}$, $a_{ki}$, $b_{ki}$ and $\alpha_{ki}$ are appropriate given constants.

Many real-world phenomena can be modelled by initial or boundary value problems for functional differential equations in the form (1). Generally, these type equations are used to model a wide class of problems in many scientific fields such as engineering, chemical reactions, mathematical physics, biology, ecology, economics, fluid and elastic mechanics, signal processing and industrial processes.

On the other hand, most of such functional differential equations with proportional delays or variable delays cannot be solved exactly. Therefore, it is necessary to design efficient numerical methods to approximate their solutions [1]. The fundamentals and methods for such equations were developed in literature: Oscillation properties [2], global attractivity [3], asymptotic behaviour of solutions [4], stability criteria [5], the existence of positive solutions [6], asymptotic stability [7], the rational approximate method [8], collocation method [5,9], multistep methods [10], Runge Kutta methods [11], block method [12], rational approximation method [13], and one-leg $\theta$- methods [14,15].

In this work, by using the matrix collocation methods which have been developed by Sezer and co-workers for differential-difference equations, delay differential equations with constant delays and pantograph type delay differential equations, a novel method, called "Generalized Laguerre matrix collocation method", is constricted to find the approximate solution of Eq.(1) along with the conditions (2) in the finite generalized Laguerre series form given by

$$y(x)\cong y_N(x)=\sum_{n=0}^{N}a_nL_n(x,\alpha), \quad 0\le a\le x\le b<\infty \tag{3}$$

where $a_n$, $n=0,1,...,N$ are unknown coefficients to be determined and $L_n(x,\alpha)$, $n=0,1,...,N$; $N\ge m$ are the generalized Laguerre polynomials.

In order to find the solutions in the form (3), the collocation points are used which are defined by

$$x_r = a + \frac{b-a}{N} r, \quad r = 0,1,\ldots,N \tag{4}$$

## 2. Some important properties of the generalized Laguerre polynomials

**Definition 2.1:** Let $(a)_n$ is the Pochhammer symbol and ${}_1F_1(-n;\alpha+1;x)$ is a confluent hypergeometric function of the first kind. Generalized Laguerre polynomials $L_n(x,\alpha)$ are orthogonal in the interval $[0,+\infty)$ respecting the weight function $\omega(x,\alpha) = x^{\alpha} e^{-x}$. For $\alpha = 0$, these polynomials become ordinary Laguerre polynomials $L_n(x)$; $L_n(x,0) = L_n(x)$ which are shown by

$$L_n(x,\alpha) = \frac{(\alpha+1)_n}{n!} \, {}_1F_1(-n;\alpha+1;x).$$

Generating function describes the polynomials $L_n(x,\alpha)$ as

$$(1-t)^{-(\alpha+1)} e^{-xt/(1-t)} = \sum_{n=0}^{+\infty} \mathrm{L}_n(x,\alpha) \frac{t^n}{n!}. \tag{5}$$

From the relation (5), recurrence relation of three terms are acquired as

$$(n+1)\mathrm{L}_{n+1}(x,\alpha) = (2n+\alpha+1-x)\mathrm{L}_n(x,\alpha) - (n+\alpha)\mathrm{L}_{n-1}(x,\alpha) \tag{6}$$

with starting values $L_0(x,\alpha) = 1, L_1(x,\alpha) = \alpha + 1 - x$.
Expanding the left side of (5) in powers of $t$, and then comparing coefficients with $t^n$, the explicit representation of $L_n(x,\alpha)$ is obtained as

$$\begin{aligned} \mathrm{L}_n^{\alpha}(x) &= \sum_{k=0}^{n} (-1)^n \binom{n}{k} \frac{\Gamma(\alpha+n+1)}{\Gamma(\alpha+k+1)} \\ &= \sum_{k=0}^{n} (-1)^k \frac{(\alpha+n+1)_{n-k}}{k!(n-k)!} \end{aligned}$$

where

$$(s)_n = s(s+1)(s+2)\ldots(s+n-1) = \frac{\Gamma(s+n)}{\Gamma(s)}$$

is defined by Pochammer symbol and $\Gamma$ is the gamma function [16]. The polynomial $L_n(x,\alpha)$ is a particular solution of the differential equation

$$xy'' + (\alpha+1-x)y' + ny = 0, \; n,\alpha \in \mathbb{N}$$

The polynomial $L_n(x,\alpha)$ has the following some representations,
Explicit representation:

$$\mathrm{L}_n(x,\alpha) = \sum_{k=0}^{n} \frac{(-1)^k}{k!} \binom{n+\alpha}{n\text{-}k} x^k \tag{7}$$

Rodrigues formula:

$$\mathrm{L}_n(x,\alpha) = \frac{x^{-\alpha} e^x}{n!} \frac{d^n}{dx^n} \left( x^{n+\alpha} e^{-x} \right) \tag{8}$$

Derivative relations:

$$\frac{d^k}{dx^k} \mathrm{L}_n^{(\alpha)}(x,\alpha) = (-1)^k \mathrm{L}_{n\text{-}k}^{(\alpha+k)}(x,\alpha) \tag{9}$$

Integral relations:

$$\int_x^\infty e^{-t} \mathrm{L}_n(t,\alpha) dt = e^{-t} \left\{ \mathrm{L}_n(x,\alpha) - \mathrm{L}_{n\text{-}1}(x,\alpha) \right\}$$

$$\mathrm{L}_n(x,\alpha) = \frac{1}{2\pi i} \oint \frac{e^{-xz/(1-z)}}{(1-z)^{\alpha+1} z^{n+1}} dz$$

The first three generalized Laguerre polynomials:

$$\mathrm{L}_0(x,\alpha) = 1$$
$$\mathrm{L}_1(x,\alpha) = \alpha + 1 - x$$
$$\mathrm{L}_2(x,\alpha) = \frac{1}{2!}\left[ x^2 - 2(\alpha+2)x + (\alpha+1)(\alpha+2) \right]$$
$$\mathrm{L}_3(x,\alpha) = \frac{1}{3!}\left[ -x^3 - 3(\alpha+3)x^2 + 3(\alpha+2)(\alpha+3)x - (\alpha+1)(\alpha+2)(\alpha+3) \right].$$

## 3. Fundamental matrix relations and generalized Laguerre matrix-collocation method

Firstly, for solving the problem (1)-(2), the expressions defined in (1), (2) and (3) are converted to matrix forms; then, by means of these matrices, then the generalized Laguerre matrix-collocation method is constructed.

Now, for purpose the truncated Laguerre series (3) in the matrix form is written

$$y(x) \cong y_N(x) = \mathbf{L}(x,\alpha)\mathbf{A} \tag{10}$$

where

$$\mathbf{L}(x,\alpha)=\left[L_0(x,\alpha) \quad L_1(x,\alpha) \quad \ldots \quad L_N(x,\alpha)\right]$$
$$\mathbf{A}=\left[a_0 \quad a_1 \quad \ldots \quad a_N\right]^T.$$

Also, by using the generalized Laguerre polynomials $L_n(x,\alpha)$ defined by (6), (7) or (8), the matrix $L_n(x,\alpha)$ can be written as follows;

$$\mathbf{L}(x,\alpha)=\mathbf{X}(x)\mathbf{M}(\alpha) \tag{11}$$

where

$$\mathbf{X}(x)=\left[1 \quad x^1 \quad \ldots \quad x^N\right]$$

$$\mathbf{M}(\alpha)=\begin{bmatrix} \frac{(-1)^0}{0!}\binom{0+\alpha}{0} & \frac{(-1)^0}{0!}\binom{1+\alpha}{1} & \frac{(-1)^0}{0!}\binom{2+\alpha}{2} & \cdots & \frac{(-1)^0}{0!}\binom{N+\alpha}{N} \\ 0 & \frac{(-1)^1}{1!}\binom{1+\alpha}{0} & \frac{(-1)^1}{1!}\binom{2+\alpha}{1} & \cdots & \frac{(-1)^1}{1!}\binom{N+\alpha}{N-1} \\ 0 & 0 & \frac{(-1)^2}{2!}\binom{2+\alpha}{0} & \cdots & \frac{(-1)^2}{2!}\binom{N+\alpha}{N-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{(-1)^N}{N!}\binom{N+\alpha}{0} \end{bmatrix}$$

From the relations (10) and (11), the matrix form is obtained

$$\left[y_N(x)\right]=\mathbf{X}(x)\mathbf{M}(\alpha)\mathbf{A}. \tag{12}$$

Further admitted relation between the matrix $\mathbf{X}(x)$ and its $k$-th order derivative $\mathbf{X}^{(k)}(x)$ is described as

$$\mathbf{X}^{(k)}(x)=\mathbf{X}(x)\mathbf{B}^k \tag{13}$$

where

$$\mathbf{B}=\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & N \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

By using the relations (12) and (13), the recurrence relations are obtained

$$\begin{aligned}\left[y_N^{(k)}(x)\right] &= \mathbf{X}^{(k)}(x)\mathbf{M}(\alpha)\mathbf{A}\\ &= \mathbf{X}(x)\mathbf{B}^k\mathbf{M}(\alpha)\mathbf{A},\ \ k=0,1,\ldots,m.\end{aligned} \tag{14}$$

By putting $x \to q_{kj}x + \tau_{kj}(x)$ into (14),

$$\begin{aligned}\left[y^{(k)}\left(q_{kj}x+\tau_{kj}(x)\right)\right] &= \mathbf{X}^{(k)}(q_{kj}x+\tau_{kj}(x))\mathbf{B}^k\mathbf{M}(\alpha)\mathbf{A}\\ &= \mathbf{X}(x)\mathbf{T}(q_{kj}x+\tau_{kj}(x))\mathbf{B}^k\mathbf{M}(\alpha)\mathbf{A}\end{aligned} \tag{15}$$

is obtained where

$$\mathbf{T}\left(q_{kj},\tau_{kj}(x)\right)=\begin{bmatrix}\binom{0}{0}\left(q_{kj}\right)^0\left(\tau_{kj}(x)\right)^0 & \binom{1}{0}\left(q_{kj}\right)^0\left(\tau_{kj}(x)\right)^1 & \cdots & \binom{N}{0}\left(q_{kj}\right)^0\left(\tau_{kj}(x)\right)^N\\ 0 & \binom{1}{1}\left(q_{kj}\right)^1\left(\tau_{kj}(x)\right)^0 & \cdots & \binom{N}{1}\left(q_{kj}\right)^1\left(\tau_{kj}(x)\right)^{N-1}\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & \binom{N}{N}\left(q_{kj}\right)^N\left(\tau_{kj}(x)\right)^0\end{bmatrix}.$$

Note that the matrix $\mathbf{X}(q_{kj}x+\tau_{kj}(x))$ can be written as

$$\mathbf{X}(q_{kj}x+\tau_{kj}(x)) = \mathbf{X}(x)\mathbf{T}(q_{kj},\tau_{kj}(x)).$$

By substituting (15) into Eq. (1) and then, by placing the collocation points $x_r$ defined by (4), respectively, the matrix equation

$$\sum_{k=0}^{m}\sum_{j=0}^{J}\mathbf{P}_{kj}(x)\mathbf{X}(x)\mathbf{T}(q_{kj}x+\tau_{kj}(x))\mathbf{B}^k\mathbf{M}(\alpha)\mathbf{A} = g(x)$$

is gained and the system of matrix equations, for $r=0,1,\ldots,N$

$$\sum_{k=0}^{m}\sum_{j=0}^{J}\mathbf{P}_{kj}(x_r)\mathbf{X}(x_r)\mathbf{T}(q_{kj}x_r+\tau_{kj}(x_r))\mathbf{B}^k\mathbf{M}(\alpha)\mathbf{A} = g(x_r)\,. \tag{16}$$

The compact form of the system (16), which is the fundamental matrix equation for Eq. (1), can be written as

$$\left(\sum_{k=0}^{m}\sum_{j=0}^{J}\mathbf{P}_{kj}(x_r)\overline{\mathbf{X}}\overline{\mathbf{T}}\overline{\mathbf{B}}^k\,\overline{\mathbf{M}}(\alpha)\right)\mathbf{A} = \mathbf{G} \tag{17}$$

where

$$\mathbf{P}_{kj} = diag\begin{bmatrix} P_{kj}(x_0) & P_{kj}(x_1) & \dots & P_{kj}(x_N) \end{bmatrix},$$
$$\overline{\mathbf{X}} = diag\begin{bmatrix} \mathbf{X}(x_0) & \mathbf{X}(x_1) & \dots & \mathbf{X}(x_N) \end{bmatrix},$$
$$\overline{\mathbf{T}_{kj}} = diag\begin{bmatrix} \mathbf{T}(q_{kj}x_0 + \tau_{kj}(x_0)) & \mathbf{T}(q_{kj}x_1 + \tau_{kj}(x_1)) & \dots & \mathbf{T}(q_{kj}x_N + \tau_{kj}(x_N)) \end{bmatrix},$$
$$\overline{\mathbf{B}}^k = diag\begin{bmatrix} \mathbf{B}^k & \mathbf{B}^k & \dots & \mathbf{B}^k \end{bmatrix},$$
$$\overline{\mathbf{M}}(\alpha) = \begin{bmatrix} \mathbf{M}(\alpha) \\ \mathbf{M}(\alpha) \\ \vdots \\ \mathbf{M}(\alpha) \end{bmatrix}, \mathbf{G} = \begin{bmatrix} g(x_0) \\ g(x_1) \\ \vdots \\ g(x_N) \end{bmatrix}, \mathbf{A} = \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_N \end{bmatrix}.$$

In Eq.(17), the full dimensions of the matrices $\mathbf{P}_{kj}$, $\overline{\mathbf{X}}$, $\overline{\mathbf{T}_{kj}}$, $\overline{\mathbf{B}}^k$, $\overline{\mathbf{M}}(\alpha)$, $\mathbf{A}$ and $\mathbf{G}$, respectively, are $(N+1)\times(N+1)$, $(N+1)\times(N+1)^2$, $(N+1)^2\times(N+1)^2$, $(N+1)^2\times(N+1)^2$, $(N+1)^2\times(N+1)$, $(N+1)\times 1$ and $(N+1)\times 1$.
The fundamental matrix equation (17) is denoted in the form

$$\mathbf{WA} = \mathbf{G} \quad \text{or} \quad [\mathbf{W};\mathbf{G}] \tag{18}$$

where

$$\mathbf{W} = \left[\omega_{pq}\right] = \sum_{k=0}^{m}\sum_{j=0}^{J} \mathbf{P}_{kj}\overline{\mathbf{X}}\overline{\mathbf{T}_{kj}}\overline{\mathbf{B}}^k\overline{\mathbf{M}}(\alpha), \quad p,q = 0,1,\dots,N$$

By means of the relation (14), equivalent matrix forms of the conditions (2) are obtained as [17]

$$\mathbf{V}_i\mathbf{A} = \alpha_i \quad \text{or} \quad [\mathbf{V}_i;\alpha_i] \tag{19}$$

where

$$\begin{aligned}\mathbf{V}_i &= \sum_{k=0}^{m-1}\left(a_{ki}\mathbf{X}(a) + b_{ki}\mathbf{X}(b)\right)\mathbf{B}^k\mathbf{M}(\alpha) \\ &= \begin{bmatrix} v_{i0} & v_{i1} & \dots & v_{iN} \end{bmatrix}, \quad i = 0,1,2,\dots,m-1\end{aligned}$$

Eventually, in order to solve the problem (1)-(2) *m* rows of the matrix (19) by the last *m* rows of (18) are substituted. So that the following new augumented matrix

$$\left[\tilde{\mathbf{W}};\tilde{\mathbf{G}}\right] \quad \text{or} \quad \tilde{\mathbf{W}}\mathbf{A} = \tilde{\mathbf{G}} \tag{20}$$

is obtained. If $rank\left(\tilde{\mathbf{W}}\right) = rank\left[\tilde{\mathbf{W}};\tilde{\mathbf{G}}\right] = N+1$, then

$$\mathbf{A} = \left(\tilde{\mathbf{W}}\right)^{-1}\tilde{\mathbf{G}}$$

is written. Thus the matrix $\mathbf{A}$, thereby the coefficients $a_n$, $n=0,1,...,N$, is uniquely determined; problem (1)-(2) has one-of-a-kind solution. On the other hand, if $rank\left(\tilde{\mathbf{W}}\right) \neq rank\left[\tilde{\mathbf{W}};\tilde{\mathbf{G}}\right]$, the problem (1)-(2) has not any solution.

## 4. Error analysis

The accuracy of the solutions is checked by resulting equation. The function $y_N(x)$ and derivatives of the function are alternated in Eq. (1). It is already known that the truncated Laguerre series (3) is approximate solution of (1). So that, the resulting equation must be provided approximately.

Namely, for $x = x_i \in [a,b]$, $i = 0,1,2,...$

$$R_N(x_i) = \sum_{k=0}^{m}\sum_{j=0}^{J} P_{kj}(x_i)\, y^{(k)}(q_{kj}x_i + \tau_{kj}(x_i)) = g(x_i) \cong 0$$

or

$R_N(x_i) \leq 10^{-k_i}$, ($k_i$ is any positive integer)

If $\max 10^{-k_i} = 10^{-k}$ ($k \in \mathbb{Z}^+$), then the truncation limit $N$ is rised. This behavior continues till the difference $R_N(x_i)$ grows into the smaller value than $10^{-k}$ at the each of the points. However, using the residual function defined by $R_N(x)$ and the mean value of the function $\left|R_N(x)\right|$ on the interval $[a,b]$, the accuracy of the solution can be controlled and the error can be estimated. If $R_N(x) \to 0$ when $N$ is sufficiency large enough, then the error declines. Moreover, Mean Value Theorem is used for estimating the upper bound of the mean error $\bar{R}_N$ as

$$\left|\int_a^b R_N(x)dx\right| \leq \int_a^b \left|R_N(x)\right| dx \text{ and } \int_a^b R_N(x)dx = (b-a)R_N(c),\ a \leq c \leq b$$

$$\Rightarrow \left|\int_a^b R_N(x)dx\right| = (b-a)\left|R_N(c)\right|$$

$$\Rightarrow (b-a)\left|R_N(c)\right| \leq \int_a^b \left|R_N(x)\right| dx$$

$$\left|R_N(c)\right| \leq \frac{\int_a^b \left|R_N(x)\right| dx}{b-a} = \bar{R}_N, \quad a \leq c \leq b.$$

## 5. Illustrative examples

In this section, some illustrations will be introduced to establish the efficiency of our method on the high-order linear pantograph-type functional differential equations with mixed proportional and variable delays. All the problems have been calculated by using Maple18 and the graphics have been plotted by MatlabR2014b.

**Example 5.1:** First deal with a high-order linear pantograph-type functional differential equation

$$y''(x)+y'(x-\sin(x))+2xy(x)=2x^3-2\sin(x)+2,\ \ 0\le x\le 1 \tag{21}$$

with initial conditions

$$y(0)=-1,\ \ y'(0)=0.$$

where $P_{01}=2x,\ P_{10}=1,\ P_{20}=1,\ g(x)=2x^3-2\sin(x)+2$. Now, let us seek the solution $y(x)$ as a truncated generalized Laguerre series by putting $N=2$ in Eq. (3). For this purpose, the set of collocation points (4) is calculated for $N=2$.

$$\{x_0=0, x_1=1/2, x_2=1\}$$

and from Eq. (21), the fundamental matrix equation of the problem is gained as

$$\left\{\mathbf{P}_{01}\overline{\mathbf{X}}\overline{\mathbf{M}}(\alpha)+\mathbf{P}_{10}\overline{\mathbf{X}}\overline{\mathbf{T}}_{10}\overline{\mathbf{B}}\overline{\mathbf{M}}(\alpha)+\mathbf{P}_{20}\overline{\mathbf{X}}\overline{\mathbf{B}}^{2}\overline{\mathbf{M}}(\alpha)\right\}\mathbf{A}=\mathbf{G}\,.$$

Then, by applying the procedure in Section 3, the fundamental matrix relations for the equation and conditions are computed and the Laguerre coefficients are found. The same procedure is repeated for $N=5$ and $N=30$. The results are outlined by the graphics. Figure 1 displays the results for comparing exact and approximate solutions for $N=2,5,30$ values. Moreover, Figure 2 displays the results for comparing error functions for the same $N$ values in previous figure.

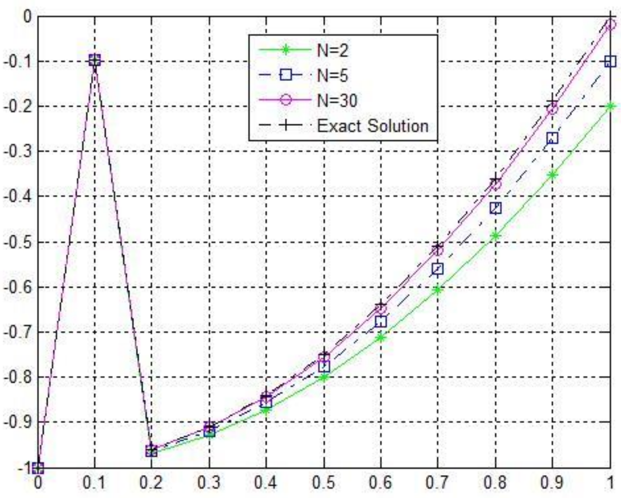


Figure 1. Numerical and exact solution $y(x)=x^2-1$ of the Example 5.1. for $N=2,5,30$.

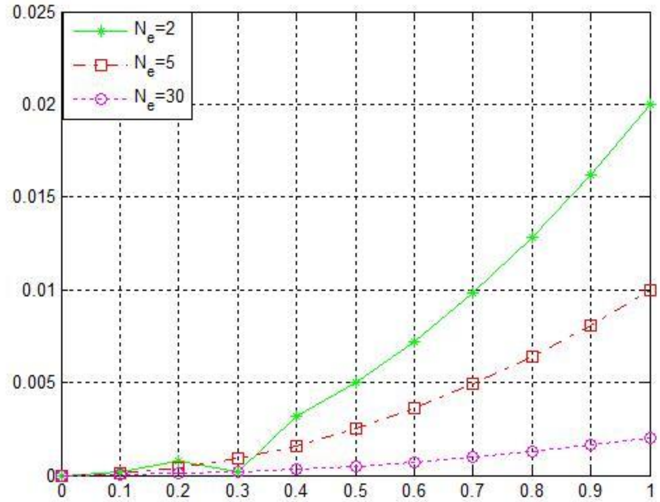


Figure 2. Comparison of the absolute errors of the Example 5.1. for $N = 2,5,30$.

**Example 5.2:** Consider high-order linear pantograph-type functional differential equation with mixed proportional and variable delay

$$y'''(x) - e^{\left(-\frac{2x}{3}\right)} y'\left(\frac{x}{3}\right) - 2e^{x} y\left(x - e^{-x}\right) = -2e^{e^{-x}} - 2\sin(x) + 2, \;\; 0 \le x \le 1 \tag{22}$$

with initial conditions

$y(0) = 1, \;\; y'(0) = -1, \;\; y''(0) = 1$.

and the exact solution of Example 5.2. is; $y(x) = e^{-x}$.

Table 1 shows the comparison between absolute errors of novel method based on generalized Laguerre polynomial solutions (NMLPS) and Taylor collocation method (TCM) for the different $N$ values of Example 5.2.

Table 1. Comparison of the absolute errors for $N = 5, 20$, NMLPS and TCM methods of Example 5.2.

| The absolute errors | | | | |
|---|---|---|---|---|
| | TCM | NMLPS | TCM | NMLPS |
| $x$ | $R_5$ | $R_5$ | $R_{20}$ | $R_{20}$ |
| 0.0 | 0.70500E-05 | 4.000000E-06 | 0.10000E-18 | 0.0000000000 |
| 0.1 | 0.14952E-05 | 3.520951E-06 | 0.15020E-08 | 1.004450E-10 |
| 0.2 | 0.65413E-05 | 2.624813E-06 | 0.63456E-08 | 4.123521E-10 |
| 0.3 | 0.62344E-05 | 4.642222E-06 | 0.14223E-10 | 9.844450E-12 |
| 0.4 | 0.42322E-04 | 5.195428E-05 | 0.25746E-07 | 1.020651E-09 |
| 0.5 | 0.35462E-04 | 2.456654E-05 | 0.38552E-07 | 2.510012E-09 |
| 0.6 | 0.54236E-04 | 1.395122E-04 | 0.54621E-07 | 3.951237E-09 |
| 0.7 | 0.95175E-03 | 3.753951E-04 | 0.74500E-06 | 4.511933E-08 |
| 0.8 | 0.94269E-03 | 7.751483E-04 | 0.94066E-06 | 6.110281E-08 |
| 0.9 | 0.62451E-03 | 1.215996E-03 | 0.11599E-06 | 8.088510E-08 |
| 1.0 | 0.42116E-03 | 2.951517E-03 | 0.14511E-06 | 9.711220E-08 |

**Example 5.3:** Consider first-order differential equation with variable delays

$$y'(x) - xy(x) - xe^{3x^2/4} y\left(x - y\left(\frac{x}{2}\right)\right) = 0, \ 0 \le x \le 1$$

with initial condition $y(0) = 1$. Table 2 shows the comparison between absolute errors of novel method based on generalized Laguerre polynomial solutions (NMLPS) for the different $N$ values of Example 5.3.

Table 2. Comparison of the residual functions of Example 5.3.

| Error analysis $\lvert R_N(x)\rvert$ | | | | |
|---|---|---|---|---|
| $x$ | Exact Solution | $\lvert \bar{R}_6 \rvert$ | $\lvert \bar{R}_8 \rvert$ | $\lvert \bar{R}_{10} \rvert$ |
| 0.0 | 1.000000000 | 5.000000E-06 | 0.10000E-10 | 0.0000000000 |
| 0.1 | 1.010050167 | 3.125400E-06 | 0.22300E-08 | 1.112950E-10 |
| 0.2 | 1.040810774 | 7.852951E-06 | 0.64402E-08 | 7.445622E-10 |
| 0.3 | 1.094174284 | 1.781943E-06 | 0.14408E-08 | 7.712320E-10 |
| 0.4 | 1.173510871 | 9.155120E-05 | 0.22653E-07 | 3.020052E-09 |
| 0.5 | 1.284025417 | 1.777659E-05 | 0.26698E-07 | 2.516922E-09 |
| 0.6 | 1.433329415 | 3.122235E-04 | 0.41123E-07 | 2.598633E-09 |
| 0.7 | 1.632316220 | 3.111912E-04 | 0.74344E-06 | 1.325566E-08 |
| 0.8 | 1.896480879 | 5.667833E-04 | 0.45962E-06 | 2.775483E-08 |
| 0.9 | 2.247907987 | 7.218861E-03 | 0.11135E-06 | 2.599174E-08 |
| 1.0 | 2.718281828 | 7.112030E-03 | 0.11148E-06 | 3.885955E-08 |

**Example 5.4:** Consider first-order linear pantograph-type functional differential equation with variable delays

$$y'(x) + xy\left(x - \ln\left(x + \varepsilon\right)\right) + y(x) = x^2 e^{-x}, \ 0 \le x \le 1$$

with initial condition

$$y(0) = 1.$$

and the exact solution of Example 5.4. is; $y(x) = e^{-x}$. Figure 3 shows us correlation between exact and approximate solutions for $N = 5,13,17$ values. Figure 4 shows the error functions of Example 5.4. for the same $N$ values.

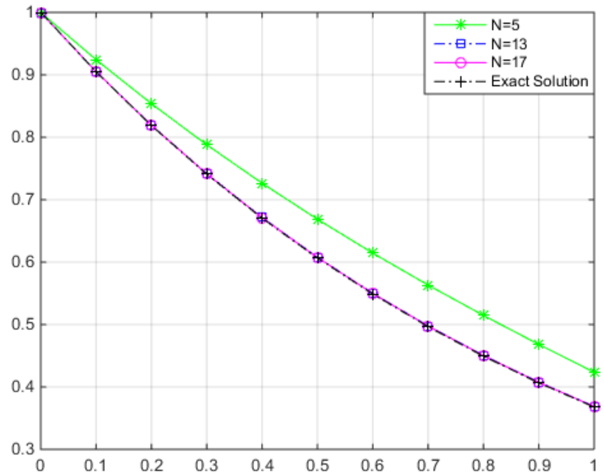


Figure 3. Numerical and exact solution $y(x)=e^{-x}$ of the Example 5.4. for $N=5,13,17$.

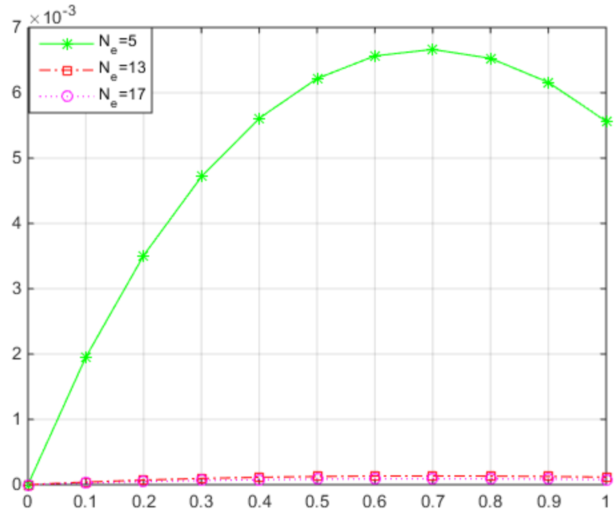


Figure 4. Comparison of the absolute errors of the Example 5.4. for $N=5,13,17$.

## 6. Conclusions

In this task, the novel method based on generalized Laguerre polynomials have been presented and illustrated to obtain the numerical solutions of high-order linear pantograph-type functional differential equations with mixed proportional and variable delays. This is commonly demanding assignment to find analytical solutions of the model. Also they play main role on biology, ecology, economics and fluid and elastic mechanics, etc. [18], [19].

Anyhow, finding the approximate solutions of these type of problems are needed. Accordingly, the presented novel method can be planned. The technique is built on the computation of the coefficients in the Laguerre expansion of solution of the high-order linear pantograph-type functional differential equations with mixed proportional and variable delays [20-23].

As a result, the novel method can also be lengthened and widen to any other type models [24]. However, some alterations are essential.

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## References

[1] Gürbüz, B., Sezer, M., Laguerre polynomial approach for solving Lane-Emden type functional differential equations, **Applied Mathematics and Computation**, 242, 255-264, (2014).

[2] Dix, J. G., Asymptotic behavior of solutions to a first-order differential equation with variable delays, **Computers & Mathematics with Applications**, 50, 10-12, 1791-1800, (2005).

[3] Graef, J. R., Qian, C., Global attractivity in differential equations with variable delays, **The ANZIAM Journal**, 41, 4, 568-579, (2000).

[4] Syski, R., Saaty, T. L., **In Modern Nonlinear Equations**, McGraw-Hill, New York, (1967).

[5] Ishiwata, E., Muroya, Y., Brunner, H., A super-attainable order in collocation methods for differential equations with proportional delay, **Applied Mathematics and Computation**, 198, 1, 227-236, (2008).

[6] Caraballo, T., Langa, J. A., Robinson, J. C., Attractors for differential equations with variable delays, **Journal of Mathematical Analysis and Applications**, 260, 2, 421-438, (2001).

[7] Diblík, J., Svoboda, Z., Šmarda, Z., Explicit criteria for the existence of positive solutions for a scalar differential equation with variable delay in the critical case, **Computers & Mathematics with Applications**, 56, 2, 556-564, (2008).

[8] Bellen, A., Zennaro, M., **Numerical methods for delay differential equations**, Oxford University Press, (2013).

[9] Reutskiy, S. Y., A new collocation method for approximate solution of the pantograph functional differential equations with proportional delay, **Applied Mathematics and Computation**, 266, 642-655, (2015).

[10] Hu, P., Huang, C., Wu, S., Asymptotic stability of linear multistep methods for nonlinear neutral delay differential equations, **Applied Mathematics and Computation**, 211, 1, 95-101, (2009).

[11] Wang, W., Zhang, Y., Li, S., Stability of continuous Runge-Kutta-type methods for nonlinear neutral delay-differential equations, **Applied Mathematical Modelling**, 33, 8, 3319-3329, (2009).

[12] Ishak, F., Suleiman, M. B., Majid, Z. A., Block method for solving pantograph-type functional differential equations, **In Proceedings of the World Congress on Engineering**, 2, (2013).

[13] Ishiwata, E., Muroya, Y., Rational approximation method for delay differential equations with proportional delay, **Applied Mathematics and Computation**, 187, 2, 741-747, (2007).

[14] Wang, W. S., Li, S. F., On the one-leg θ-methods for solving nonlinear neutral functional differential equations, **Applied Mathematics and Computation**, 193, 1, 285-301, (2007).

[15] Wang, W., Qin, T., Li, S., Stability of one-leg θ-methods for nonlinear neutral differential equations with proportional delay, **Applied Mathematics and Computation**, 213, 1, 177-183, (2009).

[16] Arfken, G. B., Weber, H. J., Mathematical methods for physicists, Elsevier Inc., (1999).

[17] Gürbüz, B., Sezer, M., A numerical solution of parabolic-type Volterra partial integro-differential equations by Laguerre collocation method, **International Journal of Applied Physics and Mathematics**, 7, 1, 49, (2017).

[18] Liu, X. G., Tang, M. L., Martin, R. R., Periodic solutions for a kind of Liénard equation, **Journal of Computational and Applied Mathematics**, 219, 1, 263-275, (2008).

[19] Schley, D., Shail, R., Gourley, S. A., Stability criteria for differential equations with variable time delays, **International Journal of Education in Mathematics, Science and Technology**, 33, 3, 359-375, (2002).

[20] Yıldızhan, I., Kürkçü, O. K., Sezer, M., A numerical approach for solving pantograph-type functional differential equations with mixed delays using Dickson polynomials of the second kind, **Journal of Science and Arts**, 18, 3, 667-680, (2018).

[21] Zhang, B., Fixed points and stability in differential equations with variable delays, **Nonlinear Analysis, Theory, Methods and Applications**, 63, 5-7, 233-242, (2005).

[22] Özer, S., An effective numerical technique for the Rosenau-KdV-RLW equation, **Balıkesir Üniversitesi Fen Bilimleri Enstitüsü Dergisi**, 20, 3, 1-14, (2018).

[23] Görgülü, M. Z., Irk, D., Numerical solution of modified regularized long wave equation by using cubic trigonometric B-spline functions, **Balıkesir Üniversitesi Fen Bilimleri Enstitüsü Dergisi**, 21, 1, 126-138, (2019).

[24] Düşünceli F, Çelik E., Numerical solution for high-order linear complex differential equations with variable coefficients, **Numerical Methods for Partial Differential Equations**, 34, 5, 1645-58, (2018).