



A Balakrishnan-Rubin type hypersingular integral operator and inversion of Flett potentials

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Abstract

In the present paper we introduce new “truncated” hypersingular integral operators $D_\epsilon^\alpha f$, ($\epsilon > 0$) generated by the modified Poisson semigroup and obtain an explicit inversion formula for the Flett potentials in framework of L_p -spaces.

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1. Introduction

The importance of the potential operators, in particular Riesz and Bessel potentials, lies in the fact that they improve the “smoothness” of a function and in this context they are useful tools in certain function spaces, such as Lipschitz, Sobolev and Hardy spaces.

The Riesz potentials $I^\alpha f$ of a function f are interpreted as the negative fractional powers of the minus Laplacian $(-\Delta) = -\sum_{k=1}^n \frac{\partial^2}{\partial x_k^2}$, and have the following integral representation (see, [19, p.117],[16, p.483],[14, p.215])

$$(I^\alpha f)(x) = \frac{1}{\gamma_n(\alpha)} \int_{\mathbb{R}^n} |y|^{\alpha-n} f(x-y) dy, \quad 0 < \alpha < n, \quad (1.1)$$

where $\gamma_n(\alpha) = \pi^{\frac{n}{2}} 2^\alpha \Gamma(\alpha/2) / \Gamma((n-\alpha)/2)$.

The Bessel potentials $J^\alpha f$ of a function f are interpreted as the negative fractional powers of “the strictly positive” operator $(E - \Delta)$, E is the identity operator, and they have the integral representation (see, [19, p.130],[16, p.540])

$$(J^\alpha f)(x) = \frac{1}{\beta_n(\alpha)} \int_{\mathbb{R}^n} G_\alpha(y) f(x-y) dy, \quad \alpha > 0, \quad (1.2)$$

with the kernel

$$G_\alpha(y) = \int_0^\infty e^{-\xi - \frac{|y|^2}{4\xi}} \xi^{\frac{\alpha-n}{2}-1} d\xi, \quad \beta_n(\alpha) = 2^n \pi^{\frac{n}{2}} \Gamma(\alpha/2).$$

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The Riesz and Bessel potentials have the following “one dimensional” integral representation via the Poisson integral:

$$(I^\alpha f)(x) = \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} (P_t f)(x) dt, \tag{1.3}$$

$$(J^\alpha f)(x) = \frac{\sqrt{\pi}}{\Gamma(\frac{1}{2}\alpha)} \int_0^\infty \left(\frac{t}{2}\right)^{\frac{1}{2}(\alpha-1)} J_{\frac{1}{2}(\alpha-1)}(t) (P_t f)(x) dt, \tag{1.4}$$

where J_ν is the Bessel function of the first kind of order ν and $P_t f$ is the Poisson integral

$$(P_t f)(x) = \int_{\mathbb{R}^n} p(y, t) f(x - y) dy, \quad t > 0, \quad x \in \mathbb{R}^n \tag{1.5}$$

with the Poisson kernel $p(y, t)$

$$p(y, t) = \frac{c_n t}{(t^2 + |y|^2)^{\frac{1}{2}(n+1)}}, \quad c_n = \frac{\Gamma\left(\frac{1}{2}(n+1)\right)}{\pi^{\frac{1}{2}(n+1)}}. \tag{1.6}$$

As seen from (1.3) and (1.4), the Riesz potentials are better suited to the Poisson kernel than Bessel potentials. However, there are another fractional integral operators whose behaviours are midway between that of the Bessel and Riesz potentials. These potentials, that we called the Flett potentials, are introduced by T.M. Flett in the paper [7] (see also [16, p.541]).

The Flett potentials $\mathcal{F}^\alpha f$ of a function f are defined in terms of Fourier transform as

$$(\mathcal{F}^\alpha f)^\wedge(x) = (1 + |x|)^{-\alpha} \widehat{f}(x), \quad x \in \mathbb{R}^n, \quad \alpha > 0, \tag{1.7}$$

with Fourier multipliers $(1 + |x|)^{-\alpha}$, whereas the Fourier multipliers of Riesz and Bessel potentials are $|x|^{-\alpha}$ and $(1 + x^2)^{\alpha/2}$, respectively. These potentials are interpreted as the negative fractional powers of the operator $(E + \Lambda)$, where $\Lambda = (-\Delta)^{\frac{1}{2}}$ and Δ is the Laplacian. They have the following integral representation

$$(\mathcal{F}^\alpha f)(x) = (\phi_\alpha(y) * f)(x) = \int_{\mathbb{R}^n} \phi_\alpha(y) f(x - y) dy, \tag{1.8}$$

where the kernel $\phi_\alpha(y)$ has the representation

$$\phi_\alpha(y) = \frac{1}{\lambda_n(\alpha)} |y|^{\alpha-n} \int_0^\infty \frac{s^\alpha e^{-s|y|}}{(1 + s^2)^{\frac{n+1}{2}}} ds, \quad (\alpha > 0) \tag{1.9}$$

with $\lambda_n(\alpha) = \pi^{(n+1)/2} \Gamma(\alpha) / \Gamma((n+1)/2)$.

One of the important problems for the potential operators is to obtain an explicit inversion formula for them. The hypersingular integral technique, that powerful tool for inversion of the Bessel and Riesz potentials, was introduced and studied by Stein [18], Lizorkin [8], Wheeden [21], Samko [15, 16], Nogin[9, 10], Rubin [11–14], Balakrishnan [6] and many other mathematicians.

We should also mention the papers [1–5, 17], where the explicit inversion formulae for the ordinary and generalized Riesz, Bessel, parabolic and Flett potentials are obtained by making use of the relevant wavelet-type transform.

In this paper, inspired by the techniques in the papers [11–13] we introduce the “truncated” hypersingular integral operators $\mathcal{D}_\varepsilon^\alpha f$, ($\varepsilon > 0$) generated by modified Poisson semi-group (see below (2.12)), and obtain an explicit inversion formula for the Flett potentials.

2. Preliminaries and auxiliary lemmas

$L_p(\mathbb{R}^n)$ is the space of the Lebesgue measurable functions on \mathbb{R}^n with the finite norm

$$\|f\|_p = \left(\int_{\mathbb{R}^n} |f(x)|^p dx \right)^{\frac{1}{p}}, \quad 1 \leq p < \infty ; \quad \|f\|_\infty = \operatorname{ess\,sup}_{x \in \mathbb{R}^n} |f(x)|.$$

Here, $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ and $dx = dx_1 dx_2 \dots dx_n$. The Fourier and inverse Fourier transforms of $f \in L_1(\mathbb{R}^n)$ are defined by

$$\widehat{f}(x) = \int_{\mathbb{R}^n} e^{-ix \cdot t} f(t) dt, \quad x \cdot t = x_1 t_1 + \dots + x_n t_n ; \quad f^\vee(t) = (2\pi)^{-n} \widehat{f}(-t).$$

Denote by $\mathcal{S} = \mathcal{S}(\mathbb{R}^n)$ the space of Schwarz test functions. Namely,

$$\mathcal{S} = \left\{ f \in C^\infty(\mathbb{R}^n) : \sup_{x \in \mathbb{R}^n} \left| x^\alpha \frac{\partial^\beta}{\partial x^\beta} f(x) \right| < \infty, \quad \text{for all } \alpha, \beta \in \mathbb{Z}_+^n \right\}$$

where $x = (x_1, x_2, \dots, x_n)$, $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$, $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \dots x_n^{\alpha_n}$ and $\frac{\partial^\beta}{\partial x^\beta} f(x) = \frac{\partial^{\beta_1}}{\partial x_1^{\beta_1}} \dots \frac{\partial^{\beta_n}}{\partial x_n^{\beta_n}} f(x_1, x_2, \dots, x_n)$.

The notion $C_0 = C_0(\mathbb{R}^n)$ will denote the class of continuous functions on \mathbb{R}^n , vanishing at infinity.

With the aid of the formula (1.9) it is not difficult to show that the kernel ϕ_α has the following properties (see [7] and [16, p.542]):

(i) If $0 < \alpha < n$, then

$$\phi_\alpha(x) \sim c_n(\alpha) |x|^{\alpha-n} \quad \text{as } |x| \rightarrow 0,$$

and

$$\phi_n(x) \sim \frac{c_n}{(n-1)!} \ln \frac{1}{|x|} \quad \text{as } |x| \rightarrow 0,$$

where

$$c_n(\alpha) = \frac{\Gamma\left(\frac{1}{2}(\alpha+1)\right) \Gamma\left(\frac{1}{2}(n-\alpha)\right)}{2\Gamma(\alpha) \pi^{\frac{1}{2}(n+1)}} \quad \text{and} \quad c_n = \frac{\Gamma\left(\frac{1}{2}(n+1)\right)}{\pi^{\frac{1}{2}(n+1)}}.$$

(ii) $\phi_\alpha \in L_1(\mathbb{R}^n)$ and $\|\phi_\alpha\|_1 = 1$, for all $\alpha > 0$.

(iii) The Fourier transform of ϕ_α for $\alpha > 0$ is

$$\int_{\mathbb{R}^n} \phi_\alpha(y) e^{-ix \cdot y} dy = (1 + |x|)^{-\alpha}.$$

(iv)

$$\phi_\alpha(x) \sim \alpha c_n |x|^{-n-1} \quad \text{as } |x| \rightarrow \infty.$$

From (1.8) and (ii) it follows that

$$\|\mathcal{F}^\alpha f\|_p \leq \|f\|_p, \quad \text{for all } \alpha > 0 \text{ and } 1 \leq p \leq \infty. \quad (2.1)$$

Also, using (1.9) and the definition of Poisson integral, we can write equivalently

$$(\mathcal{F}^\alpha f)(x) = \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} e^{-t} (P_t f)(x) dt, \quad f \in L_p, \quad (1 \leq p \leq \infty), \quad (2.2)$$

where $P_t f$ is defined as in (1.5).

The Poisson integral $P_t f$ has the following properties, that will be used later.

Lemma 2.1. (see e.g.[14, p.217])

Let $f \in L_p(\mathbb{R}^n)$, $1 \leq p \leq \infty$, $P_t f$ be the Poisson integral and $p(y, t)$ be defined as in (1.6). Then

(i)
$$\int_{\mathbb{R}^n} p(y, t) dy = 1 \quad , \quad (p(\cdot, t))^\wedge(y) = e^{-t|y|} \quad , \quad \forall t > 0 \quad ; \quad (2.3)$$

(ii)
$$\|P_t f\|_p \leq \|f\|_p \quad ; \quad (2.4)$$

(iii)
$$\sup_{x \in \mathbb{R}^n} |(P_t f)(x)| \leq ct^{-\frac{n}{p}} \|f\|_p \quad , \quad 1 \leq p < \infty \quad , \quad c = c(n, p) \quad ; \quad (2.5)$$

(iv)
$$\sup_{t > 0} |(P_t f)(x)| \leq (Mf)(x) \quad , \quad (2.6)$$

where, (Mf) is the Hardy-Littlewood maximal function;

(v)
$$P_\alpha [P_\beta f(\cdot)](x) = (P_{\alpha+\beta} f)(x) \quad , \quad \alpha > 0, \beta > 0 \quad ; \quad (2.7)$$

(vi)
$$\lim_{t \rightarrow 0} (P_t f)(x) = f(x) \quad , \quad (2.8)$$

where the limit is understood by L_p -norm, pointwise a.e. or uniformly.

2.1. The truncated hypersingular integral operators associated to the modified Poisson integral

The finite difference with order $l \in \mathbb{N}$ and step $\tau \in \mathbb{R}^1$ of the function $g(t)$, $(t \in \mathbb{R}^1)$ is defined by

$$\Delta_\tau^l [g](t) = \sum_{k=0}^l \binom{l}{k} (-1)^k g(t + k\tau). \quad (2.9)$$

In the special case, for $t = 0$,

$$\Delta_\tau^l [g](0) = \sum_{k=0}^l \binom{l}{k} (-1)^k g(k\tau). \quad (2.10)$$

Definition 2.2. Let $f \in L_p(\mathbb{R}^n)$, $1 \leq p \leq \infty$ and the Poisson integral $P_t f$ be as in (1.5). The modified Poisson integral is defined as

$$(\mathfrak{P}_t f)(x) = e^{-t} (P_t f)(x), \quad 0 \leq t < \infty. \quad (2.11)$$

It is clear that the semigroup property

$$(\mathfrak{P}_\alpha (\mathfrak{P}_\beta f))(x) = (\mathfrak{P}_{\alpha+\beta} f)(x)$$

holds and owing to Lemma 2.1-(vi) it is assumed that

$$(e^{-t} \mathfrak{P}_t f)(x) |_{t=0} = f(x) \equiv \mathfrak{P}_0 f.$$

Using this modified Poisson semigroup $\mathfrak{P}_t f$ and the finite difference with order $l \in \mathbb{N}$, we introduce the following truncated integral operators (cf. [14, p.261]).

Definition 2.3. Let $f \in L_p(\mathbb{R}^n)$, $1 \leq p < \infty$, $\alpha > 0$ and $l > \alpha$ ($l \in \mathbb{N}$). Then the construction

$$\begin{aligned} (\mathcal{D}_\varepsilon^\alpha f)(x) &= \frac{1}{\chi_l(\alpha)} \int_\varepsilon^\infty \Delta_\tau^l [(\mathfrak{P}_\cdot f)(x)](0) \frac{d\tau}{\tau^{1+\alpha}} \\ &= \frac{1}{\chi_l(\alpha)} \int_\varepsilon^\infty \left[\sum_{k=0}^l \binom{l}{k} (-1)^k e^{-k\tau} (P_{k\tau} f)(x) \right] \frac{d\tau}{\tau^{1+\alpha}}, \quad \varepsilon > 0, \end{aligned} \quad (2.12)$$

will be called a truncated hypersingular integrals or briefly, a truncated integrals with parameters $\varepsilon > 0$. Here the normalized coefficient $\chi_l(\alpha)$ is defined as

$$\chi_l(\alpha) = \int_0^\infty (1 - e^{-t})^l t^{-1-\alpha} dt. \tag{2.13}$$

By Minkowski integral inequality, it is easy to see that $\mathcal{D}_\varepsilon^\alpha f \in L_p(\mathbb{R}^n)$ for all $\varepsilon > 0$.

The main result of the paper is as follows.

Theorem 2.4. *Let $\varphi \in L_p(\mathbb{R}^n)$, $(1 \leq p < \infty)$ and $\mathcal{F}^\alpha \varphi$ be the Flett potentials of function φ of order $\alpha > 0$. Further, let the integral operators $\mathcal{D}_\varepsilon^\alpha$, $(\varepsilon > 0)$ be defined as in (2.12). Then*

$$\lim_{\varepsilon \rightarrow 0^+} (\mathcal{D}_\varepsilon^\alpha \mathcal{F}^\alpha \varphi)(x) = \varphi(x),$$

where the limit is understood in L_p -norm or pointwise a.e..

3. Proof of the main result

Now we state three lemmas, which play an important role in the proof of Theorem 2.4.

Lemma 3.1. *Let $f \in L_p(\mathbb{R}^n)$, $(1 \leq p < \infty)$ and $\mathfrak{P}_t f$ be the modified Poisson integral defined as in (2.11). Suppose that the Riemann-Liouville fractional integral of a function $h(t)$, $(0 < t < \infty)$ is defined as*

$$I_-^\alpha h(t) = \frac{1}{\Gamma(\alpha)} \int_t^\infty \frac{h(r)}{(r-t)^{1-\alpha}} dr = \frac{1}{\Gamma(\alpha)} \int_0^\infty \frac{h(r+t)}{r^{1-\alpha}} dr, \quad \alpha > 0. \tag{3.1}$$

Then

$$\mathfrak{P}_t [\mathcal{F}^\alpha f](x) = I_-^\alpha [(\mathfrak{P} \cdot f)(x)](t), \tag{3.2}$$

for all $t > 0$ and a.e. $x \in \mathbb{R}^n$. Where $\mathcal{F}^\alpha f$ is the Flett potential operator defined as in (1.8).

Proof. First, let us emphasize that the Fourier transforms of the right and left sides of (3.2) are equal for all $f \in \mathcal{S}$ (Schwartz space). Indeed, if $f \in \mathcal{S}$ then

$$(\mathfrak{P}_t [\mathcal{F}^\alpha f](x))^\wedge(\xi) = e^{-t(1+|\xi|)} (1 + |\xi|)^{-\alpha} \hat{f}(\xi), \tag{3.3}$$

and on the other hand,

$$\begin{aligned} (I_-^\alpha [(\mathfrak{P} \cdot f)(x)](t))^\wedge(\xi) &= \frac{1}{\Gamma(\alpha)} \left[\int_0^\infty (\mathfrak{P}_{r+t} f)(x) r^{\alpha-1} dr \right]^\wedge(\xi) \\ &= \frac{1}{\Gamma(\alpha)} \int_0^\infty e^{-(r+t)} r^{\alpha-1} [(P_{r+t} f)(x)]^\wedge(\xi) dr \\ &= \frac{1}{\Gamma(\alpha)} \int_0^\infty e^{-(r+t)} r^{\alpha-1} e^{-(r+t)|\xi|} \hat{f}(\xi) dr \\ &= \hat{f}(\xi) e^{-t(1+|\xi|)} \frac{1}{\Gamma(\alpha)} \int_0^\infty e^{-r(1+|\xi|)} r^{\alpha-1} dr \\ &= e^{-t(1+|\xi|)} (1 + |\xi|)^{-\alpha} \hat{f}(\xi). \end{aligned} \tag{3.4}$$

Thus, from (3.3) and (3.4) it is easily seen that the equality in (3.2) is true for all Schwartz functions f . On the other hand, since the operators A and B defined as

$$(Af)(x) = \mathfrak{P}_t [\mathcal{F}^\alpha f](x) \text{ and } (Bf)(x) = I_-^\alpha [(\mathfrak{P} \cdot f)(x)](t)$$

are strong type of (p, p) and the Schwarz space \mathcal{S} is dense in $L_p(\mathbb{R}^n)$, the proof is complete. \square

Lemma 3.2. Let $\varphi \in L_p(\mathbb{R}^n)$, $(1 \leq p < \infty)$, $0 < \alpha < \infty$ and the truncated integrals operators $\mathcal{D}_\varepsilon^\alpha$ be defined as in (2.12). If $\mathcal{F}^\alpha \varphi$ are the Flett potentials of $\varphi \in L_p(\mathbb{R}^n)$, and $P_t \varphi$, $(0 < t < \infty)$ is the Poisson integral of φ , then the equality

$$(\mathcal{D}_\varepsilon^\alpha \mathcal{F}^\alpha \varphi)(x) = \int_0^\infty K_\alpha^{(l)}(\eta) e^{-\varepsilon \eta} (P_{\varepsilon \eta} \varphi)(x) d\eta \quad (3.5)$$

is valid for all $\varepsilon > 0$ and pointwise a.e..

Here, the auxiliary kernel function $K_\alpha^{(l)}(\eta)$ is defined as follows:

$$K_\alpha^{(l)}(\eta) = \frac{1}{\Gamma(1+\alpha) \chi_l(\alpha)} \frac{1}{\eta} \sum_{k=0}^l \binom{l}{k} (-1)^k (\eta - k)_+^\alpha, \quad l > \alpha, \quad (3.6)$$

with

$$(\eta - k)_+^\alpha = \begin{cases} (\eta - k)^\alpha, & \text{if } \eta > k \\ 0, & \text{if } \eta \leq k \end{cases}.$$

Proof. By using (3.2), we have

$$\begin{aligned} (\mathcal{D}_\varepsilon^\alpha \mathcal{F}^\alpha \varphi)(x) &= \frac{1}{\chi_l(\alpha)} \int_\varepsilon^\infty \left[\sum_{k=0}^l \binom{l}{k} (-1)^k e^{-k\tau} (\mathfrak{P}_{k\tau} \mathcal{F}^\alpha \varphi)(x) \right] \frac{d\tau}{\tau^{1+\alpha}} \\ &\stackrel{(3.2)}{=} \frac{1}{\chi_l(\alpha)} \int_\varepsilon^\infty \left[\sum_{k=0}^l \binom{l}{k} (-1)^k I_-^\alpha [(\mathfrak{P}_\cdot \varphi)(x)](k\tau) \right] \frac{d\tau}{\tau^{1+\alpha}}. \end{aligned} \quad (3.7)$$

Further,

$$\begin{aligned} &\sum_{k=0}^l \binom{l}{k} (-1)^k I_-^\alpha [(\mathfrak{P}_\cdot \varphi)(x)](k\tau) \\ &\stackrel{(3.1)}{=} \sum_{k=0}^l \binom{l}{k} (-1)^k \frac{1}{\Gamma(\alpha)} \int_{k\tau}^\infty (r - k\tau)^{\alpha-1} (\mathfrak{P}_r \varphi)(x) dr \\ &= \int_0^\infty h_\tau(r) (\mathfrak{P}_r \varphi)(x) dr, \end{aligned} \quad (3.8)$$

where

$$h_\tau(r) = \frac{1}{\Gamma(\alpha)} \sum_{k=0}^l \binom{l}{k} (-1)^k (r - k\tau)_+^{\alpha-1} \quad (3.9)$$

with

$$(r - k\tau)_+^{\alpha-1} = \begin{cases} (r - k\tau)^{\alpha-1}, & \text{if } r > k\tau \\ 0, & \text{if } r \leq k\tau \end{cases}.$$

Now, by taking into account (3.8) in (3.7) we get

$$\begin{aligned} (\mathcal{D}_\varepsilon^\alpha \mathcal{F}^\alpha \varphi)(x) &= \frac{1}{\chi_l(\alpha)} \int_\varepsilon^\infty \frac{1}{\tau^{1+\alpha}} \left(\int_0^\infty h_\tau(r) (\mathfrak{P}_r \varphi)(x) dr \right) d\tau \\ &= \frac{1}{\chi_l(\alpha)} \int_0^\infty (\mathfrak{P}_r \varphi)(x) \left(\int_\varepsilon^\infty \frac{1}{\tau^{1+\alpha}} h_\tau(r) d\tau \right) dr \\ &\quad \text{(changing of variables } r = \varepsilon \eta, 0 < \eta < \infty) \\ &= \frac{\varepsilon}{\chi_l(\alpha)} \int_0^\infty (\mathfrak{P}_{\varepsilon \eta} \varphi)(x) \left(\int_\varepsilon^\infty \frac{1}{\tau^{1+\alpha}} h_\tau(\varepsilon \eta) d\tau \right) d\eta \end{aligned}$$

$$\stackrel{(3.9)}{=} \frac{\varepsilon}{\Gamma(\alpha) \chi_l(\alpha)} \int_0^\infty (\mathfrak{P}_{\varepsilon \eta} \varphi)(x) \left(\sum_{k=0}^l \binom{l}{k} (-1)^k \int_\varepsilon^\infty \frac{1}{\tau^{1+\alpha}} (\varepsilon \eta - k\tau)_+^{\alpha-1} d\tau \right) d\eta. \quad (3.10)$$

Let's evaluate the integrals

$$i_k = \int_0^\infty \tau^{-1-\alpha} (\varepsilon\eta - k\tau)_+^{\alpha-1} d\tau, \quad k = 0, 1, \dots, l$$

those appear in (3.10).

For $k = 0$:

$$i_0 = \int_\varepsilon^\infty \tau^{-1-\alpha} (\varepsilon\eta)^{\alpha-1} d\tau = \frac{1}{\alpha\varepsilon} \eta^{\alpha-1};$$

for $k = 1, 2, \dots, l$:

$$i_k = \begin{cases} \int_\varepsilon^{\frac{\varepsilon\eta}{k}} \tau^{-1-\alpha} (\varepsilon\eta - k\tau)^{\alpha-1} d\tau, & \text{if } \eta > k \\ 0, & \text{if } \eta \leq k \end{cases}.$$

By change of variables $\tau = \frac{\varepsilon\eta}{k} t$, $0 < t < \frac{\eta}{k} - 1$, we get, for $k = 0, 1, \dots, l$,

$$i_k = \begin{cases} \frac{1}{\varepsilon\eta\alpha} (\eta - k)^\alpha, & \text{if } \eta > k \\ 0, & \text{if } \eta \leq k \end{cases} = \frac{1}{\varepsilon\eta\alpha} (\eta - k)_+^\alpha. \tag{3.11}$$

By using this equality in (3.10), we obtain the desired equality

$$(\mathcal{D}_\varepsilon^\alpha \mathcal{F}^\alpha \varphi)(x) = \int_0^\infty K_\alpha^{(l)}(\eta) e^{-\varepsilon\eta} (P_{\varepsilon\eta} \varphi)(x) d\eta,$$

where the kernel function $K_\alpha^{(l)}(\eta)$ is defined as in (3.6). □

Remark 3.3. The function $K_\alpha^{(l)}(\eta)$ has the following properties (see [16, p.125], [14, p.158]):

$$(a) \quad K_\alpha^{(l)}(\eta) \in L_1(0, \infty) \text{ and } \int_0^\infty K_\alpha^{(l)}(\eta) d\eta = 1; \tag{3.12}$$

$$(b) \quad K_\alpha^{(l)}(\eta) = \begin{cases} O(\eta^{\alpha-1}), & \text{as } \eta \rightarrow 0^+ \\ O(\eta^{\alpha-l-1}), & \text{as } \eta \rightarrow \infty \end{cases}. \tag{3.13}$$

Lemma 3.4. [20, p.60] Let $\{T_\varepsilon\}_{\varepsilon>0}$ be a family of linear operators, mapping $L_p(\mathbb{R}^n)$, $1 \leq p \leq \infty$ into the space of measurable functions on \mathbb{R}^n . Define T^*f by setting

$$(T^*f)(x) = \sup_{\varepsilon>0} |(T_\varepsilon f)(x)|, \quad x \in \mathbb{R}^n$$

and denote by $\text{meas}(E)$ the Lebesgue measure of the set $E \subset \mathbb{R}^n$. Suppose that there exist a constant $c > 0$ and real number $q \geq 1$ such that

$$\text{meas} \{x : |(T^*f)(x)| > t\} \leq \left(\frac{c \|f\|_p}{t} \right)^q$$

for all $t > 0$ and $f \in L_p$.

If there exists a dense subset D of L_p such that $\lim_{\varepsilon \rightarrow 0} (T_\varepsilon g)(x)$ exists and is finite a.e. whenever $g \in D$, then for each $f \in L_p$, $\lim_{\varepsilon \rightarrow 0} (T_\varepsilon f)(x)$ exists and is finite a.e..

Proof of Theorem 2.4

By taking into account (3.12), we have

$$\begin{aligned} |(\mathcal{D}_\varepsilon^\alpha \mathcal{F}^\alpha \varphi)(x) - \varphi(x)| &\stackrel{(3.5)}{\leq} \int_0^\infty |K_\alpha^{(l)}(\eta)| |e^{-\varepsilon\eta} (P_{\varepsilon\eta} \varphi)(x) - \varphi(x)| d\eta \\ &\leq \int_0^\infty (1 - e^{-\varepsilon\eta}) |K_\alpha^{(l)}(\eta)| e^{-\varepsilon\eta} |(P_{\varepsilon\eta} \varphi)(x)| d\eta \\ &+ \int_0^\infty |K_\alpha^{(l)}(\eta)| |(P_{\varepsilon\eta} \varphi)(x) - \varphi(x)| d\eta. \end{aligned}$$

Thus,

$$\begin{aligned} \|\mathcal{D}_\varepsilon^\alpha \mathcal{F}^\alpha \varphi - \varphi\|_p &\leq \int_0^\infty (1 - e^{-\varepsilon\eta}) |K_\alpha^{(\ell)}(\eta)| \|P_{\varepsilon\eta}\varphi\|_p d\eta + \\ &\quad + \int_0^\infty |K_\alpha^{(\ell)}(\eta)| \|P_{\varepsilon\eta}\varphi - \varphi\|_p d\eta \\ &= I_1(\varepsilon) + I_2(\varepsilon). \end{aligned} \tag{3.14}$$

By making use of Lemma 2.1, Remark 3.3 and the Lebesgue dominated convergence theorem we can see that

$$\lim_{\varepsilon \rightarrow 0} I_1(\varepsilon) = 0 \text{ and } \lim_{\varepsilon \rightarrow 0} I_2(\varepsilon) = 0.$$

For the convenience of the reader we give here the proof of $\lim_{\varepsilon \rightarrow 0} I_1(\varepsilon) = 0$. By Lemma 2.1-(ii), $\|P_{\varepsilon\eta}\varphi\|_p \leq \|\varphi\|_p$ and therefore,

$$(1 - e^{-\varepsilon\eta}) |K_\alpha^{(\ell)}(\eta)| \|P_{\varepsilon\eta}\varphi\|_p \leq |K_\alpha^{(\ell)}(\eta)| \|\varphi\|_p.$$

Since, the right hand side of the last inequality is integrable and $\lim_{\varepsilon \rightarrow 0^+} (1 - e^{-\varepsilon\eta}) = 0$, the Lebesgue dominated convergence theorem yields that

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0^+} I_1(\varepsilon) &= \lim_{\varepsilon \rightarrow 0^+} \int_0^\infty (1 - e^{-\varepsilon\eta}) |K_\alpha^{(\ell)}(\eta)| \|P_{\varepsilon\eta}\varphi\|_p d\eta \\ &= \int_0^\infty \lim_{\varepsilon \rightarrow 0^+} (1 - e^{-\varepsilon\eta}) |K_\alpha^{(\ell)}(\eta)| \|P_{\varepsilon\eta}\varphi\|_p d\eta = 0. \end{aligned}$$

Similarly, since $\|P_{\varepsilon\eta}\varphi - \varphi\|_p \leq 2\|\varphi\|_p$ and $\lim_{\varepsilon \rightarrow 0} \|P_{\varepsilon\eta}\varphi - \varphi\|_p = 0$ by Lemma 2.1-(vi), we have by the Lebesgue theorem that $\lim_{\varepsilon \rightarrow 0^+} I_2(\varepsilon) = 0$. As a result, we have

$$\lim_{\varepsilon \rightarrow 0} \|\mathcal{D}_\varepsilon^\alpha \mathcal{F}^\alpha \varphi - \varphi\|_p = 0$$

Also, it is not difficult to see that the convergence is uniform for $\varphi \in C_0 \cap L_p$ if we take $p = \infty$ in (3.14). Namely, let now $\varphi \in C_0$. Denote $\|\varphi\|_\infty = \sup_{x \in \mathbb{R}^n} |\varphi(x)|$. Then, we have from (3.14) that

$$\begin{aligned} \|\mathcal{D}_\varepsilon^\alpha \mathcal{F}^\alpha \varphi - \varphi\|_\infty &\leq \int_0^\infty (1 - e^{-\varepsilon\eta}) |K_\alpha^{(\ell)}(\eta)| \|P_{\varepsilon\eta}\varphi\|_\infty d\eta \\ &\quad + \int_0^\infty |K_\alpha^{(\ell)}(\eta)| \|P_{\varepsilon\eta}\varphi - \varphi\|_\infty d\eta = \tilde{I}_1(\varepsilon) + \tilde{I}_2(\varepsilon). \end{aligned}$$

Since $\lim_{\varepsilon \rightarrow 0^+} \|P_{\varepsilon\eta}\varphi - \varphi\|_\infty = 0$, $\|P_{\varepsilon\eta}\varphi\|_\infty \leq \|\varphi\|_\infty$ and $\|P_{\varepsilon\eta}\varphi - \varphi\|_\infty \leq 2\|\varphi\|_\infty$, the Lebesgue dominated convergence theorem yields that $\lim_{\varepsilon \rightarrow 0^+} \tilde{I}_1(\varepsilon) = 0$ and $\lim_{\varepsilon \rightarrow 0^+} \tilde{I}_2(\varepsilon) = 0$, and therefore, $\lim_{\varepsilon \rightarrow 0} \|\mathcal{D}_\varepsilon^\alpha \mathcal{F}^\alpha \varphi - \varphi\|_\infty = 0$

Finally, since we have the following inequality

$$\begin{aligned} \sup_{\varepsilon > 0} |(\mathcal{D}_\varepsilon^\alpha \mathcal{F}^\alpha \varphi)(x)| &\leq \sup_{t > 0} |(P_t \varphi)(x)| \int_0^\infty |K_\alpha^{(\ell)}(\eta)| d\eta \\ &\stackrel{(2.6)}{\leq} c(M\varphi)(x), \end{aligned}$$

and since the Hardy-Littlewood maximal operator M is weak (p, p) , it follows that the maximal operator

$$(D^* \varphi)(x) = \sup_{\varepsilon > 0} |(\mathcal{D}_\varepsilon^\alpha \varphi)(x)|$$

is weak (p, p) .

Also, because of the fact that $(\mathcal{D}_\varepsilon^\alpha \varphi)(x) \rightarrow \varphi(x)$ pointwise (in fact uniformly) as $\varepsilon \rightarrow 0$ for all $\varphi \in C_0 \cap L_p$, and the set $C_0 \cap L_p$ is dense in $L_p(\mathbb{R}^n)$, $(1 \leq p < \infty)$, we get that all the conditions of Lemma 3.4 are fulfilled and therefore the proof is complete.

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