

Modeling of Duhem hysteresis with Riemann-Liouville fractional derivative

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Abstract

In the view of memory effect of hysteresis, this work aims to interpret hysteresis nonlinearities in terms of Riemann-Liouville fractional derivative which is a singular operator with memory and hereditary properties. For this purpose, Duhem hysteresis, a model defined by a first order differential equation, is considered and adapted to a fractional order differential equation. Since the fractional order Duhem hysteresis cannot be solved by an analytical scheme, Grünwald-Letnikov approximation is used to obtain numerical solutions. Thus, the effect of fractional order derivative to Duhem hysteresis is demonstrated with graphics obtained by this approximation and plotting using MATLAB. As a result, it is observed that the fractional order model exhibits hysteresis behavior for the orders that are smaller than 1.

Keywords: *Duhem hysteresis, Riemann-Liouville fractional derivative, fractional order differential equations, Grünwald-Letnikov approximation.*

Duhem histeresisin Riemann-Liouville kesirli türevi ile modellenmesi

Öz

Bu çalışma, histeresisin hafıza etkisini göz önüne alarak, doğrusal olmayan histeresis davranışının hafıza ve kalıtım özelliğine sahip tekil olmayan Riemann-Liouville kesirli türevi açısından yorumlamayı amaçlamaktadır. Bunun için, birinci mertebeden diferansiyel denklem ile tanımlanan bir model olan Duhem histeresis göz önüne alınmış ve kesirli mertebeden bir diferansiyel denkleme uyarlanmıştır. Kesirli mertebeden Duhem histeresis analitik bir yöntem ile çözülemeyeceğinden, nümerik çözümleri elde etmek için Grünwald-Letnikov yaklaşımı kullanılmıştır. Böylece, kesirli mertebeden

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türevin modele etkisi bu yaklaşım göre elde edilen ve MATLAB kullanılarak çizdirilen grafikler ile gösterilmiştir. Sonuç olarak, kesirli mertebeden modelin 1 den küçük mertebeler için histerisis etkisi gösterdiği gözlemlenmiştir.

Anahtar kelimeler: *Duhem histerisis, Riemann-Liouville kesirli türevi, kesirli mertebeden diferansiyel denklemler, Grünwald-Letnikov yaklaşımı.*

1. Introduction

Hysteresis phenomenon is an important task of science and technology because of its wide range of applications such as ferromagnetic hysteresis in physics, plastic hysteresis in mechanics, soil-moisture hysteresis in hydrology and etc. Hysteresis is a general name of the memory-based nonlinear relation between input and output signals. “Input” and “output” are the terms of system terminology, therefore their sense varies via the application areas. For instance, magnetic field is the input and magnetic induction or magnetization is the output of ferromagnetic hysteresis; force is the input and displacement is the output for mechanic hysteresis; and strain is the input and stress is the output of elastic hysteresis. These examples can be extended to other types of hysteresis.

Increasing importance of hysteresis bring about the necessity of its mathematical modeling and so substantial amount of book, review articles and papers have been published at this area, see [1-4]. In the mathematical studies, hysteresis nonlinearities are defined by operators between function spaces. Basic hysteresis operators relay, play and stop are defined for simple hysteresis nonlinearities and composition of these operators with some experimental functions, as Preisach or Prandtl-Ishlinskii models, gives hysteresis models for complex systems. Another commonly used model of hysteresis is developed with differential equation known as Duhem model [5]. However, the studies still continue to obtain the most accurate model which is necessary to explain the physical systems exactly. Recently, hysteresis nonlinearities are analyzed in fractional dynamics, see [6-16].

Fractional order modeling of hysteresis has been more often analyzed for piezoelectric materials in sense of Riemann-Liouville fractional derivative. Fractional derivatives have been used to model hysteresis of ferroelectric materials in [17] in which a fractional derivative term of polarization have been added to take into account the dynamical effects cannot be described by integer order derivative. The model leads to good matching between measured and simulation curves on a large frequency bandwidth. Zhu and Zhou [18] presented a linearized hysteresis force for piezoelectric actuated fast tool servomechanism modeled by a fractional order differential equation. Also, Zhu et al. [19] proposed a differential model of hysteresis which models nonlinear component of displacement for piezoelectric actuators. Finally, Ding et al. [20] characterized the hysteresis of piezoelectric actuators in time and frequency domain by a fractional order model and validated the effectiveness of the model by simulations and experiments. Caputo and Carcione [21] introduced to model the memory formalism in the constitutive equation of anelastic media represents hysteresis and fatigue phenomena by using Caputo fractional derivative. More recently, a new fractional order operator with nonsingular kernel known as Caputo Fabrizio derivative has been also used to model hysteresis nonlinearities in ferromagnetic materials in [22]. Moreover, not only

nonlocal operators but also local operators with fractional order are considered to model hysteresis behavior. Naser and Ikhouane [23] characterized the Duhem model of hysteresis with a local operator and identified the consistency class of the model with hysteresis behavior via the range of fractional order parameter λ . According to this classification, they concluded that the generalized Duhem model with $\lambda > 1$ and the generalized semilinear Duhem model with $0 < \lambda < 1$ are not in agreement with the hysteresis property. However, they theoretically showed that all other types are compatible with hysteresis behavior.

In this work, we analyze Duhem model of hysteresis nonlinearity with fractional calculus. We choose Duhem model, because it is a general form of elastic hysteresis, in which the fractional dynamics naturally arise, and suitable for ferromagnetic hysteresis [24,25]. To define a fractional order model of hysteresis, we reconstruct the model with fractional differential equation in sense of Riemann-Liouville derivative. The sketch of the paper can be briefed as follows. Section 1 is dedicated to preliminaries for fractional calculus. Integer order model of Duhem hysteresis is reminded in Section 3. Fractional order Duhem model is presented and solved numerically in Section 4 which also includes simulation results for different values of α . Finally, the concluding remarks are given in Section 5.

2. Preliminaries

Fractional calculus, concerns the generalization of derivative and integral concepts to non-integer order, is as old as classical calculus. However, intense interest by the scientific community to fractional calculus dates back to the last fifty years in which its wide range of application areas have been explored such as viscoelasticity, diffusion phenomena, signal processing, control design and so on. Since different non-integer order derivative and integral operators have been introduced to model different phenomena in science and engineering, fractional calculus is now recognized as an effective mathematical tool.

In this work, we use Riemann-Liouville fractional derivative (RLFD) to model hysteresis phenomena which can be defined for a time dependent function $x(\cdot)$ as

$${}_0D_t^\alpha x(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_0^t (t-\tau)^{n-\alpha-1} x(\tau) d\tau, \quad (1)$$

where α is order of derivative such that $n-1 \leq \alpha < n$, ($n \in \mathbb{N}^+$) and $\Gamma(\cdot)$ is Euler's gamma function, [26-28].

Also, we use the Grünwald-Letnikov (GL) definition of fractional derivative which can be interpreted as the discrete version of RLFD. For a time dependent function $x(\cdot)$ the GL derivative is defined as

$${}^{\text{GL}}D_t^\alpha x(t) = \lim_{h \rightarrow 0} \frac{1}{h^\alpha} \sum_{k=0}^{\left[\frac{t}{h} \right]} (-1)^k \binom{\alpha}{k} x(t - kh), \quad (2)$$

where

$$\binom{\alpha}{k} = \frac{\Gamma(\alpha + 1)}{\Gamma(k + 1)\Gamma(\alpha - k + 1)}, \quad (3)$$

h represents the time increment and $\left[\frac{t}{h} \right]$ means the integer parts of $\frac{t}{h}$.

The finite approximation of the GL derivative will be used to discretize the RLFD in calculation steps (see [28]).

3. An integer order Duhem model for hysteresis

Duhem model of hysteresis is defined by an ordinary differential equation constructed by depending on the fact that the output changes its character when the input changes its direction. In this paper, we assume that the input represented by x and the output represented by y are real valued functions of time t and also both of them have piecewise continuous derivatives. Then, the Duhem model is given by

$$\frac{dy}{dx} = \rho [\zeta(x) - y] \left| \frac{dx}{dt} \right| + \eta(x) \frac{dx}{dt}. \quad (4)$$

where ρ is a positive constant, η and ζ are material functions. In detail, η corresponds to the mean slopes of hysteresis branches and ζ corresponds to the mean difference of the output on hysteresis branches. These functions must also satisfy the following conditions for good agreement with the experimental data, [29]:

- ζ is an odd, monotone increasing, piecewise smooth, real valued function with a derivative ζ' that has a finite limit $\zeta'(\infty)$ for large input.
- η is an even, piecewise continuous, real valued function with a limit, for large input, obeying $\zeta(\infty) = \eta(\infty)$.

Analytical solution of Eq.(4) under the initial conditions $y(t_0) = y_0$ and $x(t_0) = x_0$ can be easily obtained by dividing the both side of the equation with $\frac{dx}{dt}$ and using the chain rule. Thus, the solution of the acquired linear differential equation is

$$y(x) = \zeta(x) + [y_0 - \zeta(x_0)] e^{\rho \text{sign}\left(\frac{dx}{dt}\right)(x-x_0)} - e^{-\rho \text{sign}\left(\frac{dx}{dt}\right)x} \int_{x_0}^x [\zeta'(\tau) - \eta(\tau)] e^{\rho \text{sign}\left(\frac{dx}{dt}\right)\tau} d\tau. \quad (5)$$

The Duhem hysteresis is a generalized form of mechanics hysteresis backlash which is used for elastic models. Colemann and Hodgdon [24,25] showed that it is also useful to model ferromagnetic hysteresis by choosing the function ζ and η to match the experimental data.

4. Fractional order Duhem model for hysteresis

To obtain a fractional order model of hysteresis, we replace the integer order derivative of the input x and the output y in Eq.(4) with RLFD operator. We choose order of the RLFD as $0 < \alpha \leq 1$. Therefore; we get the following fractional order differential equation:

$${}_0D_t^\alpha y(t) = \rho [\zeta(x) - y] |{}_0D_t^\alpha x(t)| + \eta(x) {}_0D_t^\alpha x(t) \tag{6}$$

subjected to initial conditions

$$y(t_0) = y_0 \text{ and } x(t_0) = x_0.$$

Note that the functions ζ and η have the same properties given for integer order models in Section 3. Since the chain rule is not valid for RLFD, we prefer to solve the fractional Duhem model numerically. For this purpose, GL approximation of RLFD is used. The solution procedure can be summarized as below. Firstly, the time interval $[0, T]$ is divided into N equal parts with size of $h = \frac{1}{N}$ and the nodes are labeled as $0, 1, 2, \dots, N$. Secondly, the fractional derivative terms are approximated by GL approach. For a time dependent function of z , we can evaluate RLFD of order α with GL approach at node M as

$${}_0D_t^\alpha z(hM) = \frac{1}{h^\alpha} \sum_{j=0}^M w_j^{(\alpha)} z(hM - jh), \tag{7}$$

where $w_j^{(\alpha)} = (-1)^j \binom{\alpha}{j}$. The coefficients $w_j^{(\alpha)}$ can be computed by the following recurrence relationships

$$\begin{aligned} w_0^{(\alpha)} &= 1 \\ w_j^{(\alpha)} &= \left(1 - \frac{\alpha + 1}{j}\right) w_{j-1}^{(\alpha)}, \text{ for } j = 1, 2, \dots, N \end{aligned} \tag{8}$$

Using Eq.(7), the numerical solution of Eq.(6) is obtained as

$$y(hM) = \frac{\rho \text{sign}({}_0D_t^\alpha x(hM)) \zeta(hM) + \eta(hM) - \frac{1}{h^\alpha} \sum_{j=1}^M w_j^{(\alpha)} y(hM - jh)}{\frac{1}{h^\alpha} w_0^{(\alpha)} + \rho |{}_0D_t^\alpha x(hM)|}. \quad (9)$$

For graphical representation, we consider zero initial conditions and two types of ζ and η functions, one is used for elastic hysteresis and the other is used for ferromagnetic hysteresis. Although the positive constant ρ is calculated from experimental data to match the hysteresis nonlinearity exactly, it can be observed that ρ is often equal to 1 in both the experimental and the theoretical studies [29, 30]. Therefore, we select $\rho = 1$ to focus the effect of fractional order on the hysteresis nonlinearity and we do not aim to explore different values of this parameter to keep confusion away. Also we choose step size $h = 0.01$ and all figures are plotted for sinusoidal input functions.

In Figure 1, we demonstrate the behavior of the fractional order model of hysteresis via variations of α values. For this aim, we choose $\zeta(x) = 0.03$ and $\eta(x) = 0.345$ to model elastic hysteresis, and the input function $x(t) = 4.5 \sin(2.3t)$. It can be seen from Figure 1 that decreasing values of α lead to a rectangular hysteresis nonlinearity. In terms of the applicability of fractional order hysteresis, it is important to emphasize that the fractional order model has smooth corners means it is differentiable.

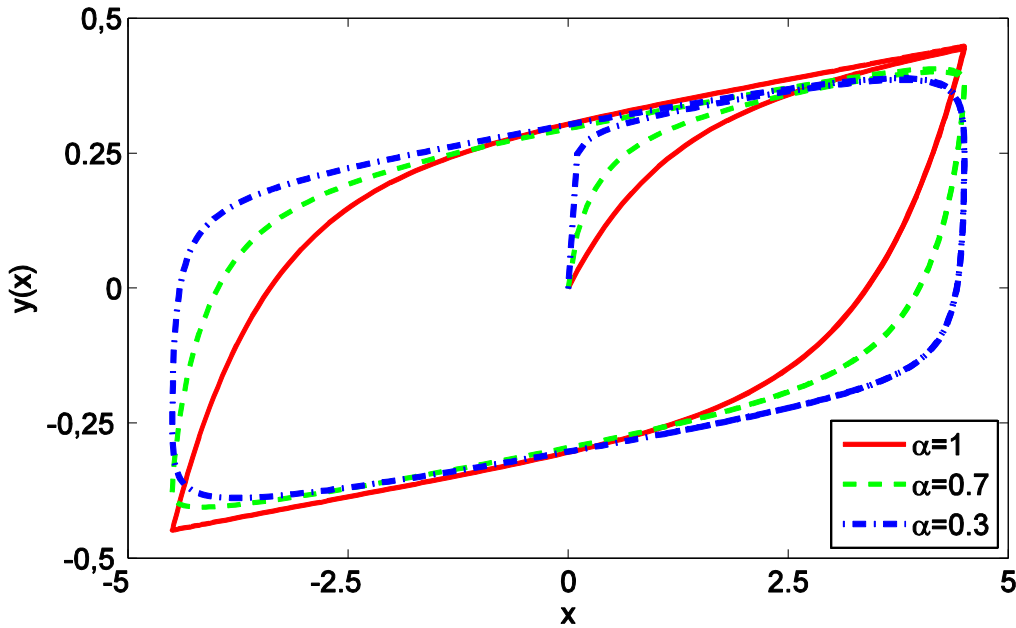


Figure 1. Fractional Duhem model of elastic hysteresis.

Similarly, we investigate fractional order model of ferromagnetic hysteresis with respect to α in Figure 2. Therefore, the ζ and η functions can be chosen as the following, (see [29]):

$$\zeta(x) = \begin{cases} a \tan(bx), & |x| < x^* \\ a \tan(bx^*) + (x - x^*)/s, & x > x^* \\ -a \tan(bx^*) + (x + x^*)/s, & x < -x^* \end{cases}$$

and

$$\eta(x) = \begin{cases} \zeta'(x) \left[1 - c \exp\left(\frac{-d|x|}{x^* - |x|}\right) \right], & |x| < x^* \\ \zeta'(x), & |x| > x^* \end{cases}$$

where $a=1.02$, $b=14.26$, $c=0.95$, $d=1.2$, and $s=0.002$, $x^*=0.09$. The input function $x(t)=0.09\sin(t)$ is applied for numerical computation. We conclude from Figure 2 exhibited for $\alpha=0.3, 0.6, 0.9$ that variations of α affect the shape of hysteresis. We conclude that decreasing values of α lead to narrow hysteresis.

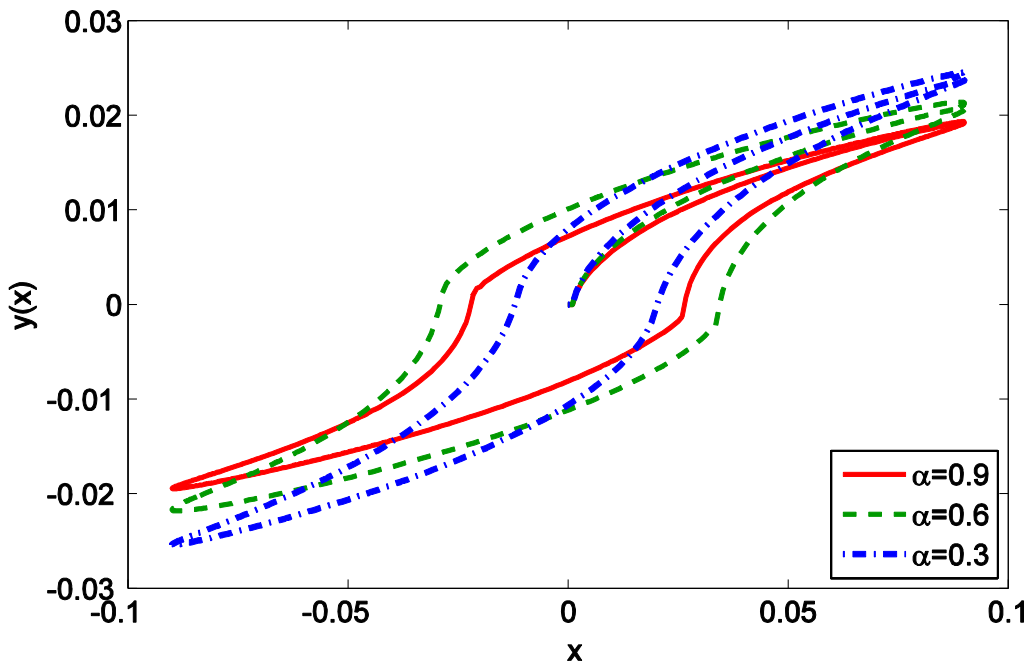


Figure2. Fractional Duhem model of ferromagnetic hysteresis.

5. Conclusions

We investigated the fractional order model of hysteresis nonlinearity in the sense of Riemann-Liouville fractional derivative. Therefore, the Duhem model of hysteresis was defined by a fractional differential equation. Although the integer model analytically could be solved by using the chain rule of the classical derivative, the same solution strategy could not be applied to the fractional model since the fractional derivative does not supply the chain rule. Thus, a numerical approximation which is Grünwald-Letnikov approach was used to solve fractional Duhem hysteresis. Fractional order

models were plotted for two kinds of hysteresis nonlinearities which are used to model elastic and magnetic hysteresis via sinusoidal inputs. It was concluded that the model still shows hysteresis behavior for different values of the order. Also, it was observed from the figures that the fractional derivative affected the shape of hysteresis for both elastic hysteresis and ferromagnetic hysteresis. Moreover, the fractional derivative led to smooth hysteresis nonlinearity for elastic hysteresis.

References

- [1] Krasnosel'skii, M.A., and Pokrovskii, A.V., **Systems with hysteresis**, Springer Verlag, (1989).
- [2] Mayergoyz, I.D., **Mathematical models of hysteresis**, Springer Verlag, Berlin, (1991).
- [3] Macki, J.W., Nistri, P., and Zecca, P., Mathematical models for hysteresis, **Siam Review**, 35, 1, 9-123, (1993).
- [4] Visintin, A., **Differential models of hysteresis**, Springer, Berlin. (1994).
- [5] Duhem, P., Die dauernden aenderungen und die thermodynamik, **Zeitschrift fur Physikalische Chemie**, 22, 543-589, (1897).
- [6] Bagley, R.L., and Torvik, P.J., On the fractional calculus model of viscoelastic behavior, **Journal of Rheology**, 30, 1, 133—155, (1986).
- [7] Padovan, J., and Sawicki, J. T., Diophantine type fractional derivative representation of structural hysteresis, **Computational Mechanics**, 19, 335-340, (1997).
- [8] Machado, J.A., Analysis and design of fractional order digital control systems, **Systems Analysis Modelling Simulation**, 27, 107-122, 1997.
- [9] Darwish, M.A., and El-Bary, A.A., Existence of fractional integral equation with hysteresis, **Applied Mathematics and Computation**, 176, 684-687, (2006).
- [10] Schafer, I., and Kruger, K., Modeling of coils using fractional derivatives, **Journal of Magnetism and Magnetic Materials**, 307, 91-98, (2006).
- [11] Duarte, F. and Machado, J.A., Fractional dynamics in the describing function analysis of nonlinear friction, Proceedings of the **2nd IFAC Workshop on Fractional Differentiation and Its Applications**, Porto, Portugal, 39, 11, 218-223, (2006).
- [12] Deng, W. and Lü, J., Generating multi-directional multi-scroll chaotic attractors via a Fractional Differential Hysteresis system, **Physics Letters A**, 369, 438-443, (2007).
- [13] Duarte, F. and Machado, J.A., Describing function of two masses with backlash, **Nonlinear Dynamics**, 56, 409-413, (2009).
- [14] Özdemir N. and İskender, B.B., Fractional order control of fractional diffusion systems subject to input hysteresis, **Journal of Computational and Nonlinear Dynamics, ASME**, 5, 2, 021002 (6 pages), (2010).
- [15] İskender, B. B., Özdemir, N. and N., Karaoglan, A.D., **Parameter optimization of fractional order $PI^\lambda D^\mu$ controller using response surface methodology**, Discontinuity and Complexity in Nonlinear Physical Systems, Series: Nonlinear Systems and Complexity, Vol. 6, Machado, J. A. T, Baleanu, D., Luo, A.C.J. (Eds.), Chapter 5, ISBN 978-3-319-01411-1, (2014).
- [16] Spanos, P.D., Di Matteo, A. and Pirodda, A., Steady-state dynamic response of various hysteretic systems endowed with fractional derivative elements, **Nonlinear Dynamics** 98,4,3113-3124, (2019).

- [17] Guyomar, D., Ducharne, B. and Sebald, G., Dynamical hysteresis model of ferroelectric ceramics under electric field using fractional derivatives, **Journal of Physics D: Applied Physics**, 40, 6048-6054, (2007).
- [18] Zhu, Z. and Zhou, X., A novel fractional order model for dynamic hysteresis of piezoelectrically actuated fast tool servo, **Materials**, 5, 2465-2485. (2012).
- [19] Zhu, Z., To, S., Li, Y., Zhu, W-L. and Bian, L., External force estimation of a piezo-actuated compliant mechanism based on a fractional order hysteresis model, **Mechanical Systems and Signal Processing**, 110, 296-306, (2018).
- [20] Ding, C., Cao, J. and Chen, Y.Q., Fractional-order model and experimental verification for broadband hysteresis in piezoelectric actuators, **Nonlinear Dynamics**, 98,3143-3153, (2019).
- [21] Caputo, M., and Carcione, J.M., Hysteresis cycles and fatigue criteria using anelastic models based on fractional derivatives, **Rheologica Acta**, 50, 107-115, (2011).
- [22] Caputo, M. and Fabrizio, M., On the notion of fractional derivative and applications to the hysteresis phenomena, **Meccanica**, 52, 3043-3052, (2017).
- [23] Naser, M.F.M. and Ikhouane, F., Consistency of the Duhem model with hysteresis, **Mathematical Problems in Engineering**, 2013, 586130, (16 pages), (2013).
- [24] Colemann, B.D., and Hodgdon, M.L., A constitutive relation for rate-independent hysteresis in ferromagnetically soft materials, **International Journal of Engineering Science**, 24, 897-919, (1986).
- [25] Colemann, B.D. and Hodgdon, M.L., On a class of constitutive relations for ferromagnetic hysteresis, **Archive for Rational Mechanics and Analysis**, 99, 375-396, (1987).
- [26] Oldham, K. B. and Spanier, J., **The fractional calculus**, Academic Press, New York, (1974).
- [27] Miller, K.S. and Ross B., **An introduction to the fractional calculus and fractional differential equations**, Wiley, New York. (1993).
- [28] Podlubny I., **Fractional differential equations**, Academic Press, San Diego, (1999).
- [29] Hodgdon, M.L., Applications of a theory of ferromagnetic hysteresis, **IEEE Transactions on Magnetics**, 24, 218-221, (1988).
- [30] Su, C.Y., Stepanenko, Y., Svoboda, J. and Leung, T.P., Robust and adaptive control of a class of nonlinear systems with unknown backlash-like hysteresis, **IEEE Transactions on Automatic Control**, 45, 2427-2432, (2000).