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Coefficient Estimates for Certain Subclasses of Analytic Functions Defined by New Differential Operator

Sibel Yalçın[1], Hasan Bayram[2]*

[1,2] Department of Mathematics, Faculty of Arts and Science, Bursa Uludag University, Bursa, Turkey
*hbayram@uludag.edu.tr
*Orcid No: 0000-0001-8106-6834

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Abstract

The study of operators plays an essential role in Mathematics, especially in Geometric Function Theory in Complex Analysis and its related fields. Many derivative and integral operators can be written in terms of convolution of certain analytic functions. The class of analytic functions, which has an essential place in the theory of geometric functions, has been studied by many researchers before. This topic still maintains its popularity today. In this paper, we investigate certain subclasses of analytic functions defined by generalized differential operators involving binomial series. Also, we obtain coefficient estimates involving of the nonhomogeneous Cauchy-Euler differential equation of order r.

Keywords: Analytic functions, coefficient bounds, differential operator, subordination.

1. Introduction

Let A denote the class of all analytic functions in the open unit disc

$$\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$$

and having the form for $z \in \mathbb{D}$,

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \quad (1)$$

A function $f \in A$ is said to belong to the class $S^*(\alpha)$ of starlike functions of order α if and only if

$$\operatorname{Re}\left(1 + \frac{zf'(z)}{f(z)}\right) > \alpha \quad (2)$$

for $z \in \mathbb{D}$ and $0 \le \alpha < 1$.

Also, a function $f \in A$ is said to belong to the class $K(\alpha)$ of convex functions of order α if and only if

$$\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha \quad (3)$$

for $z \in \mathbb{D}$ and $0 \le \alpha < 1$.

The classes $S^*(\alpha)$ and $K(\alpha)$ considered by Silverman [9]. We consider that $S^*(0)$ and $K(0)$ are respectively, the classes of starlike functions and convex functions.

Let the functions $f, g \in A$ be analytic in $\mathbb{D}$. Then f is said to be subordinate g if there exists a Schwarz function $w(z)$ on $\mathbb{D}$ with $w(0) = 0$, $|w(z)| < 1$, such that $f(z) = g(w(z))$ for $z \in \mathbb{D}$. We denote this subordination by $f(z) \prec g(z)$ for $z \in \mathbb{D}$. In particular, if the function g is univalent in $\mathbb{D}$, then we get

$$f(z) \prec g(z) \Leftrightarrow f(0) = g(0) \text{ and } f(\mathbb{D}) \subset g(\mathbb{D}).$$

Making use of the binomial series for $k \in \mathbb{N} := \{1,2,3,\cdots\}$, $m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$

$$(1-\lambda)^k = \sum_{m=0}^{k} \binom{k}{m} (-1)^m \lambda^m$$

and for $f \in A$, $\lambda \in \mathbb{R}$, $\mu \ge 0$ with $\lambda + \mu > 0$ and $\delta \in \mathbb{N}_0$, Wanas [11] introduced the differential operator $W_{\lambda,\mu}^{k,\delta} f(z)$ which is defined as follows:

$$W_{\lambda,\mu}^{k,0} f(z) = f(z)$$

$$W_{\lambda,\mu}^{k,1} f(z) = \frac{[1-(1-\lambda)^k]f(z) + [1-(1-\mu)^k]zf'(z)}{2-(1-\lambda)^k-(1-\mu)^k}$$

$$\vdots$$

$$W_{\lambda,\mu}^{k,\delta} f(z) = W_{\lambda,\mu}^{k,1}\left(W_{\lambda,\mu}^{k,\delta-1} f(z)\right). \quad (4)$$

If f is given by (1), then from (4) we see that

$$W_{\lambda,\mu}^{k,\delta} f(z) = z + \sum_{n=2}^{\infty}\left[\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\lambda^m + n\mu^m}{\lambda^m+\mu^m}\right)\right]^{\delta} a_n z^n. \quad (5)$$

If we choose the parameters λ, μ, k and δ special, we obtain operators which studied by various authors;

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(i) $W_{\lambda,\mu}^{k,1} = I_{\lambda,\mu}^{\delta}$ ([10]),

(ii) $W_{\lambda,1}^{1,\delta} = I_{\lambda}^{\delta}$; $\alpha > -1$ ([3,4]),

(iii) $W_{1-\mu,\mu}^{1,\delta} = D_{\mu}^{\delta}$; $\mu \geq 0$ ([2]),

(iv) $W_{0,1}^{1,\delta} = S^{\delta}$ ([8]).

In references ([12-20]) can be found similar studies on analytical functions in recent years.

Herefrom we introduce the following subclasses of analytic functions with using $W_{\lambda,\mu}^{k,\delta}$.

1.1 Definition

Let $p: \mathbb{D} \to \mathbb{C}$ be a convex function such that $p(0) = 1$ and $Re\{p(z)\} > 0$, $z \in \mathbb{D}$. We denote by $S_g^{*,\delta}(\lambda, \mu, k; \alpha)$ the subclass of A given by

$$S_g^{*,\delta}(\lambda, \mu, k; \alpha) = \left\{ f : f \in A \text{ and } \frac{1}{1-\alpha}\left(\frac{z\left(W_{\lambda,\mu}^{k,\delta} f(z)\right)'}{W_{\lambda,\mu}^{k,\delta} f(z)} - \alpha \right) \in g(\mathbb{D}) \right\} \tag{6}$$

where $z \in \mathbb{D}$, $\lambda \in \mathbb{R}$, $\mu \geq 0$ with $\lambda + \mu > 0$ and $\delta \in \mathbb{N}_0$, $k \in \mathbb{N}$, and $\alpha \in [0,1)$.

1.2 Definition

A function $f \in A$ in the class $C_g^{\delta}(\lambda, \mu, k, t; r)$ if it satisfies the following non-homogeneous Cauchy-Euler differential equation of order t;

$$z^t \frac{d^t w}{dz^t} + \binom{t}{1}(r + t - 1) z^{t-1} \frac{d^{t-1} w}{dz^{t-1}} + \cdots + \binom{t}{t} w \prod_{i=0}^{t-1}(r + i) = g(z) \prod_{i=0}^{t-1}(r + i + 1), \tag{7}$$

where $w = f(z)$, $f \in A$, $g(z) \in S_g^{*,\delta}(\lambda, \mu, k; \alpha)$, $r \in \mathbb{R} \backslash (-\infty, -1]$ and $t \in \mathbb{N}_2 = \mathbb{N} - \{1\} = \{2,3, \cdots\}$.

Clearly, by suitably specializing parameters for

$$g(z) = \frac{1 + Az}{1 + Bz} \quad (-1 \leq B \leq A \leq 1, z \in \mathbb{D})$$

and

$$g(z) = \frac{1 + (1 - 2\alpha)z}{1 - z} \quad (0 \leq \alpha < 1, z \in \mathbb{D}),$$

$S_g^{*,\delta}(\lambda, \mu, k; \alpha)$ reduces to the various subclasses of analytic functions (see [9]). Motivated from the recent work of Al-Hawary [1] (see also, for example [5,6,7]) the main object of our investigation is to obtain some coefficient bounds for functions in the subclasses $S_g^{*,\delta}(\lambda, \mu, k; \alpha)$ and $C_g^{\delta}(\lambda, \mu, k, t; r)$ of analytic functions of order α by using the subordination principle between analytic functions.

To prove our main results, we recall 1.3 Lemma.

1.3 Lemma

Let

$$g(z) = \sum_{n=1}^{\infty} b_n z^n.$$

This function where $z \in \mathbb{D}$ be convex in $\mathbb{D}$.

Let

$$f(z) = \sum_{n=1}^{\infty} a_n z^n.$$

This function where $z \in \mathbb{D}$ be analytic in $\mathbb{D}$. If $f(z) \prec g(z)$ where $z \in \mathbb{D}$ then

$$|a_n| \leq |b_n|$$

for $n \in \mathbb{N}$.

2. Coefficient Bounds for the Classes $S_g^{*,\delta}(\lambda, \mu, k; \alpha)$ and $C_g^{\delta}(\lambda, \mu, k, t; r)$

We start by acquiring coefficient bounds for functions in the class $S_g^{*,\delta}(\lambda, \mu, k; \alpha)$.

2.1. Theorem

Let the function $f \in A$ be given by (**1**). If $f \in S_g^{*,\delta}(\lambda, \mu, k; \alpha)$, then

$$|a_n| \leq \frac{\prod_{j=0}^{n-2}\left(j + (1 - \alpha)|g'(0)|\right)}{(n-1)! \left|\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\lambda^m + n\mu^m}{\lambda^m + \mu^m}\right)\right|^{\delta}} \tag{8}$$

for $n \in \mathbb{N}_2$.

Proof.

By the equation (**5**), the function $W_{\lambda,\mu}^{k,\delta} f(z)$ has the Taylor-Maclaurin series expansion

$$W_{\lambda,\mu}^{k,\delta} f(z) = z + \sum_{n=2}^{\infty} A_n z^n$$

for $z \in \mathbb{D}$ where

$$A_n = \left[\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\lambda^m + n\mu^m}{\lambda^m + \mu^m}\right)\right]^{\delta} \tag{9}$$

for $n \in \mathbb{N}_2$.

We see that $W_{\lambda,\mu}^{k,\delta} f(z)$ is analytic in $\mathbb{D}$ with

$$W_{\lambda,\mu}^{k,\delta} f(0) = \left(W_{\lambda,\mu}^{k,\delta} f\right)'(0) - 1 = 0.$$

Now, from Definition 1.1 we have

$$\frac{1}{1-\alpha}\left(\frac{z\left(W_{\lambda,\mu}^{k,\delta} f(z)\right)'}{W_{\lambda,\mu}^{k,\delta} f(z)} - \alpha \right) \in g(\mathbb{D}).$$

Let us define the function $p(z)$ by

$$p(z) = \frac{1}{1-\alpha}\left(\frac{z\left(W_{\lambda,\mu}^{k,\delta} f(z)\right)'}{W_{\lambda,\mu}^{k,\delta} f(z)} - \alpha\right). \qquad \mathbf{(10)}$$

We deduce that $p(0) = g(0) = 1$ and $p(z) \in g(\mathbb{D})$ $(z \in \mathbb{D})$. Therefore, we have $p(z) \prec g(z)$, $(z \in \mathbb{D})$. Thus according to Lemma 1.3, we obtain

$$\left|\frac{p^{(n)}(0)}{n!}\right| \leq |g'(0)| \quad n \in \mathbb{N} \qquad \mathbf{(11)}$$

where $p(z) = 1 + p_1(z) + p_2(z) + \cdots$ is analytic in $\mathbb{D}$. From $\mathbf{(10)}$, we easily get

$$z\left(W_{\lambda,\mu}^{k,\delta} f(z)\right)' - \alpha W_{\lambda,\mu}^{k,\delta} f(z) = (1-\alpha)p(z) W_{\lambda,\mu}^{k,\delta} f(z) \qquad \mathbf{(12)}$$

for $z \in \mathbb{D}$.
Since $A_1 = 1$, from $\mathbf{(12)}$, it follows that $(n-\alpha)A_n = (1-\alpha)(p_{n-1} + p_{n-2}A_2 + \cdots + p_1 A_{n-1})$.
Especially, for $n = 2,3,4,\cdots$, we have

$$|A_2| \leq (1-\alpha)|g'(0)|,$$

$$|A_3| \leq \frac{(1-\alpha)|g'(0)|(1+(1-\alpha)|g'(0)|)}{2!}$$

and

$$|A_4| \leq \frac{(1-\alpha)|g'(0)|(1+(1-\alpha)|g'(0)|)(2+(1-\alpha)|g'(0)|)}{3!}$$

respectively.
Thus, by appealing to the principle of mathematical induction, we obtain

$$|A_n| \leq \frac{\prod_{j=0}^{n-2}(j+(1-\alpha)|g'(0)|)}{(n-1)!} \qquad \mathbf{(13)}$$

for $n \in \mathbb{N}_2$.
We now immediately find from $\mathbf{(9)}$ that

$$|a_n| \leq \frac{\prod_{j=0}^{n-2}(j+(1-\alpha)|g'(0)|)}{(n-1)!\left|\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\lambda^m + n\mu^m}{\lambda^m+\mu^m}\right)\right|^{\delta}}.$$

This completes the proof.
Next, we give coefficient bounds for functions In the $C_g^{\delta}(\lambda,\mu,k,t;r)$.

2.2. Theorem
Let the function $f \in A$ be given by $(\mathbf{1})$. If $f \in C_g^{\delta}(\lambda,\mu,k,t;r)$, then

$$|a_n| \leq \frac{\prod_{j=0}^{n-2}(j+(1-\alpha)|g'(0)|)\prod_{i=0}^{t-1}(r+i+1)}{(n-1)!\left|\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\lambda^m + n\mu^m}{\lambda^m+\mu^m}\right)\right|^{\delta}\prod_{i=0}^{t-1}(r+i+n)} \qquad \mathbf{(14)}$$

for $t, n \in \mathbb{N}_2$ where $r \in \mathbb{R}\backslash(-\infty,-1]$.
Proof.
Let the function $f \in A$ be given by $(\mathbf{1})$ and

$$g(z) = \sum_{n=1}^{\infty} b_n z^n \in S_g^{*,\delta}(\lambda,\mu,k;\alpha).$$

Then from $(\mathbf{7})$, we get

$$a_n = \frac{\prod_{i=0}^{t-1}(r+i+1)}{\prod_{i=0}^{t-1}(r+i+n)} b_n$$

for $n \in \mathbb{N}_2$, $r \in \mathbb{R}\backslash(-\infty,-1]$. Hence from Theorem 2.1 we obtain inequality $(\mathbf{14})$. This completes the proof.

Author's Contributions

All authors contributed equally to this manuscript and all authors reviewed the final manuscript.

Ethics

There are no ethical issues after the publication of this manuscript.

References

1. Al-Hawary, T., Frasin, B. A., Yousef, F. 2018. Coefficients estimates for certain classes of analytic functions, *Afrika Mathematika*, 29, 1265-1271.

2. Al-Oboudi, F. M. 2004. On univalent functions defined by a generalized Salagean operator, *Int. J. Math. Math. Sci.* 27, 1429-1436.

3. Cho, N. M., Kim, T. H. 2003. Multiplier transformations and strongly close to convex functions, *Bull. Korean Math. Soc*. 40-3, 399-410.

4. Cho, N. M., Srivastava, H. M. 2003. Argument estimates of certain analytic functions defined by a class of multiplier transformations, *Math. Comput. Modelling* 37-1-2, 39-49.

5. Miller, S. S., Mocanu, P. T., Differential subordination, Monographs and Textbooks in Pure and Applied Mathematics,. Marcel Dekker Inc. New York, 2000; pp 225.

6. Robertson, M. S. 1936. On the theory of univalent functions, *Annals of Mathematics* 37, 374-408.

7. Rogosinski, W. 1943. On the coefficients of subordinate functions, *Proc. Lond. Math. Soc*. (Ser 2) 48, 48-82.

8. Salagean, G. 1983. Subclasses of univalent functions, *Lecture Notes in Math., Springer Verlag, Berlin* 1013, 362-372.

9. Silverman, H. 1978. Subclasses of starlike functions, *Rev. Roum. Math. Pures et Appl*. 23, 1093-1099.

10. Swamy, S. R. 2012. Inclusion properties of certain subclasses of analytic functions, *Int. Math. Forum* 7-36, 1751-1760.

11. Wanas, A. K. 2019. New differential operator for holomorphic functions, *Earthline Journal of Mathematical Sciences* 2-2, 527-537.

12. Galib Atshan, W., Ibrahim Badawi, E., 2019. On Sandwich Theorems for Certain Univalent Functions Defined by a New operator. *Journal of Al-Qadisiyah for Computer Science and Mathematics*, *11*(2), math 72-80.

13. Atshan, W. G., Abbas, I. A., Yalcin, S., 2020. New Concept on Fourth-Order Differential Subordination and Superordination with Some Results for Multivalent Analytic Functions. *Journal of Al-Qadisiyah for Computer Science and Mathematics*, *12*(1), Math Page 96-107.

14. Frasin, B., & Murugusundaramoorthy, G., 2020. A subordination results for a class of analytic functions defined by q-differential operator, *Annales Universitatis Paedagogicae Cracoviensis. Studia Mathematica* (published online ahead of print 2020).

15. Çakmak, S., Yalçın, S., & Altınkaya, Ş., 2019. An application of the power series distribution for certain analytic univalent function classes. *Lıbertas Mathematıca (new series)*, *39*(1), 95-102.

16. Srivastava, H. M., & El-Deeb, S. M., 2020. A certain class of analytic functions of complex order connected with a q-analogue of integral operators. *Miskolc Mathematical Notes*, *21*(1), 417-433.

17. Khan, B., Srivastava, H. M., Khan, N., Darus, M., Tahir, M., & Ahmad, Q. Z., 2020. Coefficient estimates for a subclass of analytic functions associated with a certain leaf-like domain. *Mathematics*, 8(8), 1334.

18. Liu, G., Liu, Z., & Ponnusamy, S., 2020. Refined Bohr inequality for bounded analytic functions. *arXiv preprint arXiv:2006.08930*.

19. Porwal, S. 2020. Mapping Properties Of Certain Subclasses Of Analytic Functions Associated With Generalized Distribution Series. *Applied Mathematics E-Notes*, *20*, 39-45.

20. Atshan, W. G., Abbas, I. A., & Yalcin, S., 2020. New Concept on Fourth-Order Differential Subordination and Superordination with Some Results for Multivalent Analytic Functions. *Journal of Al-Qadisiyah for Computer Science and Mathematics*, *12*(1), Math Page 96-107.