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RESEARCH ARTICLE / ARAŞTIRMA MAKALESİ

PT-/non-PT-Symmetric and non-Hermitian q-deformed Trigonometric Scarf Potential via Path Integral Method

Path Integral Yöntemiyle PT-/ PT-Simetrik ve Hermityen Olmayan q-deformasyonlu Trigonometrik Scarf Potansiyeli

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Abstract
In this study, energy spectrum and corresponding wave function of Parity-Time (PT)-/non-PT- Symmetric and Non-Hermitian q-deformation Trigonometric Scarf Potential are obtained by using Path Integral method where P and T denotes parity and time operators, respectively. First, the kernel of this potential is derived in terms of the energy spectrum and the wave function by adopting parametric time. Then, the concomitant energy spectrum and the wave function are found by using the Green function stemming from the aforementioned kernel.
Keywords: Trigonometric Scarf Potential, Path integral, PT Symmetry, Non Hermitian potential, Green's Function

Öz
Bu çalışmada Parite-Zaman (PT)-/ PT-Simetrik ve Hermityen Olmayan q-deformasyonlu Trigonometrik Scarf Potansiyelinin enerji spektrumu ve karşılık gelen dalga fonksiyonu Path Integral yöntemi kullanılarak elde edilmiştir. P ve T burada parite ve zaman işlemcilerine karşılık gelmektedir. Öncelikle bu potansiyelin kerneli parametrik zaman kullanılarak enerji spektrumu ve dalga fonksiyonu cinsinden türetilmiştir. Daha sonra, bu kernelden elde edilen Green fonksiyonu ile enerji spektrumu ve dalga fonksiyonu bulunmuştur.
Anahtar Kelimeler Trigonometrik Scarf Potansiyeli, Path integral, PT Simetri, Non Hermityen potansiyel, Green fonksiyonu

I. INTRODUCTION

It is the usual practice in quantum mechanics that any measurement of a physical quantity is expressible as an eigenvalue of an operator which is bound to be Hermitian. This common practice holds when there is no interaction with the environment i.e. when the system under scrutiny is isolated. In the presence of an interaction, the Hamiltonian is not Hermitian. However, although PT-symmetric systems are not isolated, they act as Hermitian systems because they are in equilibrium and their energy levels are real [1,2]. An example of such an open quantum system is George Gamow's work on Alpha decay. In this study, it was shown that a particle can escape from the nucleus by tunneling at a rate that can be effectively defined by a complex energy eigenvalue. It has been found that the real and imaginary portions of these eigenvalues are related to the experimentally observed energy levels and widths of the respective nuclear resonances. Non-Hermitian concepts have been used in physics, nuclear and quantum, optical, microwave, electronic and mechanical systems in a number of sub-disciplines [3,4].

The exact solutions of the Schrödinger equation with potentials yielding real and complex eigenvalues are quite interesting. Analytical and numerical studies for various Hamiltonians with real and complex eigenvalues, which were first initiated by Bender and Boetcher, and then were continued by many authors [2-4, 16-18]. Following this line of research, we aim to obtain the energy spectrum and the wave functions of the PT-/Non-PT-Symmetric and non-Hermitian Scarf Potential using Feynman's Path integral method in this work. The point of departure of the path integral method involves the calculation of the quantum mechanical amplitude over all possible paths associated with classical action, namely, the kernel [7].

Trigonometric Scarf potential has been used in the construction of a periodic potential and employed in one-dimensional crystal models in solid state physics. In this study, we will discuss the trigonometric Scarf potential based on q deformation. The hyperbolic potentials with q-deformation are defined as follows [9,12]:

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$$sinh_q x = \frac{1}{2}(e^x - qe^{-x}), \quad cosh_q x = \frac{1}{2}(e^x + qe^{-x})$$
$$tanh_q x = \frac{sinh_q x}{cosh_q x}, \quad sech_q x = \frac{1}{cosh_q x}$$
$$coth_q x = \frac{cosh_q x}{sinh_q x}, \quad cosech_q x = \frac{1}{sinh_q x} \qquad (1)$$
$$\frac{d}{dx} cosh_q x = sinh_q x, \quad \frac{d}{dx} sinh_q x = cosh_q x$$

Firstly, q deformation hyperbolic functions, which were introduced by Arai to obtain complete solutions of supersymmetric potentials, were later used by various authors for various potentials. The q parameter is used as an additional parameter to define interatomic interactions.

This paper is organized as follows: In section 2.1, we derive the kernel of PT-Symmetric and q-deformed Trigonometric Scarf potentials. In section 2.2, we obtain the energy eigenvalues and the corresponding wave functions. In section 3, we find kernel, energy spectrum and wave functions of non-PT Symmetric and non- Hermitian q deformed Trigonometric Scarf potential.

II. MATERIAL AND METHOD

The kernel of a point particle moving in the potential V(x) in one dimension between the initial position x^' at time t^'=0 and final position x" at time t" has the following form [7]:

$$K(x'', t''; x', t') = \int \frac{DxDp}{2\pi} exp\left\{\frac{i}{\hbar}\int dt[p\dot{x} - \frac{p^2}{2m} - V(x)]\right\} \qquad (2)$$

This expression is the sum of the amplitudes that includes the contribution from all possible paths, which includes all the information about the system. The kernel, also called the propagator, is the Green function of the Schrodinger equation, and it is also anything that expresses a wave function that spreads over time. The kernel is defined as the energy integral of the Green function as

$$K(x'', x'; T) = \frac{1}{2\pi i\hbar}\int_{-\infty}^{\infty} e^{iET/\hbar}\, G(x'', x'; E) dE \qquad (3)$$

Where $T = t'' - t'$. The time interval is divided into N parts and the kernel is expressed as the limit of the discrete time as follows

$$K(x'', x', T) = \lim_{N\to\infty}\int \sum_{j=1}^{N} dx_j \prod_{j=1}^{N+1}\left(\frac{dp_j}{2\pi\hbar}\right) \exp\{\frac{i}{\hbar}\sum_{j=1}^{N+1}[p_j\, \Delta x_j - \frac{p_j^2}{2m} - V(x_j)]\epsilon\} \qquad (4)$$

Here $\Delta x_j = x_j - x_{j-1}$, $\epsilon = t_j - t_{j-1}$, $t' = t_0 = t_a$, $t'' = t_N = t_b$. By using this expression, the kernel of any potential is derived while the kernel of the system is derived by applying path integral method with the appropriate coordinate and momentum transformations. The energy spectrum and the Green's function of the system are found using Equation (3). In particular, the Fourier that analyzes the Propagator gives all the energy eigenvalues and each Fourier coefficient gives the wave functions of each energy. Propagator therefore contains all the dynamic information about a quantum system.

III. RESULTS

3.1. The Kernel of the PT-Symmetric and q-deformed Trigonometric Scarf potential

The q-deformed trigonometric Scarf potential in its most general form reads [14-16].

$$V(x) = -\frac{A}{sin_q^2 \alpha x} \qquad (5)$$

If A is real and $\alpha \to i\alpha$, PT-Symmetric and q-deformed Trigonometric Scarf potential given by Equation (1) becomes

$$V(x) = \frac{A}{sinh_q^2 \alpha x} \qquad (6)$$

Through Equation (4), the above potential yields the following kernel

$$K(x'', x', T) = \lim_{N\to\infty}\int \sum_{j=1}^{N} dx_j \prod_{j=1}^{N+1}\left(\frac{dp_j}{2\pi\hbar}\right) \exp\{\frac{i}{\hbar}\sum_{j=1}^{N+1}[p_j\, \Delta x_j - \frac{p_j^2}{2m} - \frac{4Ae^{-2\alpha x_j}}{(1+qe^{-2\alpha x_j})^2}]\epsilon\} \qquad (7)$$

By applying the position and momentum transformations below to the kernel above

$$x = \frac{1}{2\alpha} ln\frac{tanh^2 y}{q}, \quad p_x = \alpha sinhy coshy P_y \qquad (8)$$

Equation (7) now becomes

$$K(y_a, y_b; T) = \alpha sinhy_b coshy_b \int DyDp_y \times exp\left[i\int dt\left(p_y\dot{y} - \alpha^2 sinh^2 y cosh^2 y\frac{p_y^2}{2\mu} + 4Asinh^2 y cosh^2 y\right)\right] \qquad (9)$$

since position and momentum transformations yield the term $\alpha sinhy_b coshy_b$ due to the Jacobian.
A new time parameter is defined to eliminate the multiplier in front of the kinetic energy. To this aim, it is included into the kernel in the form of a Lagrange multiplier [9,10,19,20].

$$t = \frac{1}{\alpha^2}\int \frac{ds}{sinh^2y cosh^2y} \tag{10}$$

This new time parameter can also be expressed as the Fourier transform of the Delta function [7,8]

$$1 = \int dS \int \frac{dE}{2\pi}\frac{1}{\alpha^2 sinh^2y cosh^2y} \times exp\left[i\left(ET - \int \frac{dsE}{\alpha^2 sinh^2y cosh^2y}\right)\right] \tag{11}$$

Here S=s_b-s_a. Using Equation (10) and Equation (11), we can perform the calculation in Equation (9) as

$$K(y_b, y_a; T) = \frac{1}{i\,\alpha sinhy_b coshy_b}\int_{-\infty}^{\infty}\frac{dE}{2\pi}e^{iET}\int Ds \int DyDp_y e^{i\frac{4A}{\alpha^2}S} \times \exp[i\int ds\,(p_y\dot{y} - \frac{p_y^2}{2\mu} - \frac{\frac{E}{2\alpha^2}}{sinh^2y} - \frac{\frac{E}{2\alpha^2}}{cosh^2y})] \tag{12}$$

In order to make an equal contribution to the Jacobien at the beginning and end points, we need to symmetrize with respect to the points a and b. Having done this, we have

$$\frac{1}{sinhy_b coshy_b} = \frac{1}{\sqrt{sinh2y_a sinh2y_b}}\exp(i\int_0^S dsi\frac{cosh2y}{sinh2y}\dot{y}) \tag{13}$$

Thus we can write Equation (12) as

$$K(y_b, y_a; T) = \int_0^\infty \frac{dE}{2\pi}e^{iET} \times \int Ds e^{\frac{i}{2\mu}\left(1+\frac{4A}{\alpha^2}\right)S}\frac{1}{i\alpha\sqrt{sinh2y_a sinh2y_b}}K(y_b, y_a; S) \tag{14}$$

where

$$K(y_b, y_a; S) = \int DyDp_y exp\left\{i\int_0^S ds\left[p_y\dot{y} - \frac{p_y^2}{2\mu} - \frac{1}{2\mu}\left(\frac{\kappa(\kappa-1)}{sinh^2y} - \frac{\gamma(\gamma-1)}{cosh^2y} - \frac{ip_y cosh2y}{sinh2y}\right)\right]\right\} \tag{15}$$

The constants κ and λ are found equal and read

$$\kappa = \gamma = \frac{1}{2}\left[1 + \sqrt{1 + \frac{4\mu E}{\alpha^2}}\right] \tag{16}$$

We make use of the following relation [9,10,19,20] in Equation (10) as well as Equation (15)

$$\dot{y}_j \to \dot{y}_j \pm \frac{icosh2y}{2\mu sinh2y} \quad \text{or} \quad \frac{y_j - y_{j-1}}{\epsilon} \to \frac{y_j - y_{j-1}}{\epsilon} \pm \frac{ip_y cosh2y}{2\mu sinh2y} \tag{17}$$

However, note that the term above vanishes in the $\varepsilon \to 0$ limit so that Equation (15) becomes the kernel of the well-known Pöschl Teller potential [11,12]. Therefore, using the solutions of the Poschl Teller potential, we can directly write the solution of the PT-Symmetric and q-deformed Trigonometric Scarf potential as

$$K(y_b, y_a; S) = \sum_{n=0}^{\infty} exp\left[-i(\frac{S}{2\mu})(2n + \gamma - \kappa - 1)^2 \psi_n(y_a)\psi^*{}_n(y_b)\right] \tag{18}$$

Integrating over S to obtain the energy-dependent Green's function (see Equation (3)), we obtain

$$G(x_b, x_a; S) = \frac{1}{i\alpha\sqrt{sinh2y_a sinh2y_b}} \times \sum_{n=0}^{\infty}\int_{-\infty}^{\infty}\frac{dE}{2\pi}\frac{e^{iET}}{(\kappa + \gamma - 2n)^2 - 1}\psi_n(y_a)\psi_n^*(y_b) \tag{19}$$

Using Green's function above, the kernel finally reads

$$K(x_b, x_a; E) = \sum_{n=0}^{\infty} e^{-iE_nT}\psi_n(x_a)\psi_n^*(x_b) = \sum_{n=0}^{\infty} exp\left[-\frac{1}{8\mu\alpha q(n+1)^2}\right]\varphi_n(x_a)\varphi_n^*(x_b) \tag{20}$$

3.2. Energy Spectrum and Wave functions for PT-Symmetric q-deformed Trigonometric Scarf Potential

Integrating the kernel in Equation (20) over the action S and energy E, the energy eigenvalues are found as

$$E_n = -\frac{\alpha^2}{8\mu}\left(2n + 1 - \sqrt{\frac{8\mu A}{\alpha^2 q} + 1}\right)^2 \tag{21}$$

The normalized wave functions in terms of Jacobi polynoms [21] are given by

$$\varphi(x) = \frac{i}{2\sqrt{2}\sqrt{n+1}} \times \sqrt{4(n+1)^2 - (\gamma_n - \kappa_n)^2 \sqrt{\frac{\Gamma(n+1)\Gamma(-n-1)}{\Gamma\left(\kappa_n + n + \frac{1}{2}\right)\Gamma\left(\gamma_n + n + \frac{1}{2}\right)}}} \times \frac{\exp\left(\kappa_n - \frac{1}{2}\right)x/2\alpha}{\left(1 + e^{\frac{x}{\alpha}}\right)^{\left(\kappa_n + \gamma_n - \frac{1}{2}\right)}} P_n^{\left(\kappa_n - \frac{1}{2}, \gamma_n - \frac{1}{2}\right)}\left(\frac{1 - e^{\frac{x}{\alpha}}}{1 + e^{\frac{x}{\alpha}}}\right) \quad (22)$$

where

$$\kappa_n = \frac{1}{2} + \frac{1}{n+1}[(n+1)^2 - 2\mu\alpha^2 A], \quad \gamma_n = \frac{1}{2} - \frac{1}{n+1}[(n+1)^2 - 2\mu\alpha^2 A] \quad (23)$$

We see that PT symmetric q deformed Trigonometric Scarf potential has real energy eigenvalues as also verified in Ref. [12-15] through the direct calculation of the Schrödinger equation.

3.3. Non-PT-Symmetric and non-Hermitian q-deformed Trigonometric Scarf Potential

Non-PT-Symmetric and non-Hermitian q-deformed Trigonometric Scarf Potential can be determined by considering $A \to A + iB$ and $q \to iq$ and $\alpha \to i\,\alpha$. Following the similar steps in Sections (3.1) and (3.2), we obtain the energy eigenvalues and corresponding wave functions for the Non-PT-Symmetric and non-Hermitian q-deformed Trigonometric Scarf Potential. The discrete kernel for this potential reads

$$K(x'', x', T) = \lim_{N\to\infty} \int \sum_{j=1}^{N} dx_j \prod_{j=1}^{N+1} \left(\frac{dp_j}{2\pi\hbar}\right) \exp\{\frac{i}{\hbar}\sum_{j=1}^{N+1}[p_j\,\Delta x_j - \frac{p_j^2}{2m} - \frac{4(A+iB)e^{-2\alpha x_j}}{(1+iqe^{-2\alpha x_j})^2}]\epsilon\}. \quad (24)$$

By taking the appropriate coordinate and momentum transformations, we can rearrange the kernel as

$$K(y_a, y_b; T) = i\alpha sinhy_b coshy_b \int DyDp_y \times exp\left[i\int dt\left(p_y\dot{y} + \alpha^2 sinh^2 y cosh^2 y \frac{p_y^2}{2\mu} + 4(A+iB) sinh^2 y cosh^2 y\right)\right] \quad (25)$$

One now defines the parametric time as

$$t = \frac{1}{\alpha^2}\int \frac{ds}{sinh^2 y cosh^2 y} \quad (26)$$

in a similar manner as in section (3.2). If we follow the similar procedure in section (3.2), we obtain the energy eigenvalues of Non-PT Symmetric and non-Hermitian q-deformed Trigonometric Scarf Potential as

$$E_n = -\frac{\alpha^2}{8\mu}\left(2n + 1 - \sqrt{-\frac{8\mu(A+iB)}{\alpha^2 q} + 1}\right)^2 \quad (27)$$

The normalized wave functions are

$$\varphi(x) = \frac{i}{2\sqrt{2}\sqrt{n+1}} \times \sqrt{4(n+1)^2 - (\gamma_n - \kappa_n)^2 \sqrt{\frac{\Gamma(n+1)\Gamma(-n-1)}{\Gamma\left(\kappa_n + n + \frac{1}{2}\right)\Gamma\left(\gamma_n + n + \frac{1}{2}\right)}}} \times \frac{\exp\left(\kappa_n - \frac{1}{2}\right)ix/2\alpha}{\left(1 + e^{-i\frac{x}{\alpha}}\right)^{\left(\kappa_n + \gamma_n - \frac{1}{2}\right)}} P_n^{\left(\kappa_n - \frac{1}{2}, \gamma_n - \frac{1}{2}\right)}\left(\frac{1 - e^{-i\frac{x}{\alpha}}}{1 + e^{-i\frac{x}{\alpha}}}\right) \quad (28)$$

where κ_n and γ_n are given exactly as in Equation (22). Hence, the energy spectra is real only if $Re(A) = 0$. This expression is identical to that given in Ref. [16].

IV. CONCLUSIONS

In this study, we obtained the energy spectrum and the corresponding wave function of PT-/non-PT-Symmetric and non-Hermitian q-deformation Trigonometric Scarf Potential by using the Path Integral method. First of all, we derived the kernel of this potential using the method devised by Duru and Kleinert. We have defined parametric time in order to apply the path integral method and then derived the kernel in terms of the energy spectrum and wave function. We calculated the energy spectrum and wave function by using the Green function obtained from the kernel. In standard quantum mechanics, operators with real eigenvalues must be Hermitian. In this work, we provided an example where one can have real eigenvalues despite the underlying non-Hermiticity. The present study therefore sheds light on the structure of quantum theory and moreover can be used to understand the dynamics of the quantum mechanical systems interacting with the environment.

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