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Some Coupled Fixed Point Theorems for F-Contraction Mappings

Kübra ÖZKAN[1*]

[1] Manisa Celal Bayar University, Department of Mathematics, 45140, Manisa/TURKEY

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Abstract

In this article, some coupled fixed point theorems for F -contraction mappings in complete metric spaces are proved. In addition, some results related to these theorems are given.

Keywords: Fixed point theory, Metric space, F-contraction, Completeness.

F-Büzülme Dönüşümleri için Bazı İkili Sabit Nokta Teoremleri

Öz

Bu çalışmada, tam metrik uzaylarda F-büzülme dönüşümleri için bazı ikili sabit nokta teoremleri ispatlanmıştır. Ayrıca, bu teoremlerle ilgili bazı sonuçlar verilmiştir.

Anahtar Kelimeler: Metrik uzaylar, Sabit nokta teorisi, F-büzülme, Tamlık.

1. Introduction

The concept of coupled fixed point was introduced by Guo and Lakshmikantham (1987). And, Bhaskar and Lakshmikantham (2006) introduced coupled fixed point for partially ordered metric spaces. A lot of authors such as Mutlu et.al. (2017 and 2018); Sabetghadam et.al. (2009); Samet (2010); Van Luong and Thuan (2011), gave different generalization of these theorems.

Wardowski (2012) was introduced the concept of F-contraction and he gave a different generalization of Banach contraction principle. Afterwards, various researchers examined some fixed point theorems for such type contraction mappings and they got some interesting and useful results (see; Abbas et.al., 2013; Altun et.al., 2015; Batra and Vashistha, 2014; Cosentino and Vetro, 2014; Piri and Kumam 2014).

In this manuscript, we examine some coupled fixed point theorems for F -contraction mappings in complete metric spaces. In addition to this, we give some results related to these theorems.

*Corresponding Author:kubra.ozkan@hotmail.com

2. Preliminaries

Definition 2.1. (Wardowski 2012)

Let a mapping $F:\mathbb{R}^{+}\to\mathbb{R}$ satisfies the following conditions:

$(F1)$ F is strictly increasing,

$(F2)$ For each sequence $\{\alpha_n\}_{n\in\mathbb{N}}$ of positive numbers

$$\lim_{n\to\infty}\alpha_n=0\Leftrightarrow\lim_{n\to\infty}F(\alpha_n)=-\infty,$$

$(F3)$ There exists $k\in(0,1)$ such that $\lim_{\alpha\to 0^{+}}\alpha^{k}F(\alpha)=0$.

$\mathcal{F}$ is called as the family of all functions F which satisfy the conditions $(F1)-(F3)$.

Definition 2.2. (Wardowski 2012)

Let (X,d) be metric space. $T:X\to X$ is called an F -contraction if there exist $F\in\mathcal{F}$ and $\tau\in\mathbb{R}^{+}$ such that

$$d(Ta,Tb)>0\Rightarrow\tau+F(d(Ta,Tb))\leq F(d(a,b)) \tag{1}$$

for each $a,b\in X$.

Example 2.3. (Wardowski 2012)

Let $F:\mathbb{R}^{+}\to\mathbb{R}$ be denoted by $F(a)=\ln a$. It is obvious that, for any $k\in(0,1)$, the function F satisfies the conditions $(F1)-(F3)$. All self-mappings T on X , which satisfies (1) is an F -contraction such that

$$d(Ta,Tb)\leq e^{-\tau}d(a,b)$$

for all $a,b\in X$ such that $Ta\neq Tb$.

It is clear that the inequality

$$d(Ta,Tb)\leq e^{-\tau}d(a,b)$$

also holds for $a,b\in X$ such that $Ta=Tb$. Then T is a Banach contraction mapping.

Example 2.4. (Wardowski 2012)

Let $F:\mathbb{R}^{+}\to\mathbb{R}$ be denoted by $F(a)=\ln a+a$ $a\in(0,\infty)$. It is clear that, for any $k\in(0,1)$, the function F satisfies the conditions $(F1)-(F3)$. All self-mappings T on X , which satisfies (1) is an -contraction such that

$$\frac{d(Ta,Tb)}{d(a,b)}\leq e^{d(Ta,Tb)-d(a,b)}\leq e^{-\tau}$$

for all $a,b\in X$, $Ta\neq Tb$.

3. Main Results

Theorem 3.1.

Let (X,d) be a complete metric space and $S:X\times X\to X$ be a self-mapping on X . If there exist $F\in\mathcal{F}$ and $\tau\in(0,\infty)$ such that the following condition holds

$$\begin{aligned}&d(S(a,b),S(u,v))>0\\&\Rightarrow\tau+F(d(S(a,b),S(u,v)))\leq F(\alpha d(a,u)+\beta d(b,v))\end{aligned} \tag{2}$$

for all $a,b,u,v\in X$, $S(a,b)\neq S(u,v)$, where $\alpha+\beta<1$, then S has a unique coupled fixed point.

Proof:

We take $a_0,b_0\in X$ and set

$a_1=S(a_0,b_0),b_1=S(b_0,a_0),\ldots,$
$a_{n+1}=S(a_n,b_n),b_{n+1}=S(b_n,a_n).$

If $a_{n_0}=a_{n_0+1}$, $b_{n_0}=b_{n_0+1}$ for some $n_0\in\mathbb{N}$, then

$a_{n_0}=a_{n_0+1}=S(a_{n_0},b_{n_0}), b_{n_0}=b_{n_0+1}=S(b_{n_0},a_{n_0}).$

Thus, (a_{n_0},b_{n_0}) is a coupled fixed point for S. We examine the case of either $a_n\neq a_{n+1}=S(a_n,b_n)$ or $b_n\neq b_{n+1}=S(b_n,a_n)$ for all $n\in\mathbb{N}$. Then,

$$d(S(a_{n-1},b_{n-1}),S(a_n,b_n))=d(a_n,a_{n+1})>0 \text{ or}$$

$$d(S(b_{n-1},a_{n-1}),S(b_n,a_n))=d(b_n,b_{n+1})>0$$

for all $n\in\mathbb{N}$. Using (2), we have

$$\begin{aligned}\tau+F(d(a_{n+1},a_{n+2}))&=\tau+F(d(S(a_n,b_n),S(a_{n+1},b_{n+1})))\\&\leq F(\alpha d(a_n,a_{n+1})+\beta d(b_n,b_{n+1})).\end{aligned} \tag{3}$$

And, from (2), we also get

$$\begin{aligned}\tau+F(d(b_{n+1},b_{n+2}))&=\tau+F(d(S(b_n,a_n),S(b_{n+1},a_{n+1})))\\&\leq F(\alpha d(b_n,b_{n+1})+\beta d(a_n,a_{n+1})).\end{aligned} \tag{4}$$

Since F is strictly increasing, using (3) and (4), we obtain that

$$d(a_{n+1},a_{n+2})<\alpha d(a_n,a_{n+1})+\beta d(b_n,b_{n+1})$$

and

$$d(b_{n+1},b_{n+2})<\alpha d(b_n,b_{n+1})+\beta d(a_n,a_{n+1}).$$

Therefore, by letting

$$d_n=d(a_{n+1},a_{n+2})+d(b_{n+1},b_{n+2}),$$

we have

$$\begin{aligned}d_n&<(\alpha+\beta)(d(a_n,a_{n+1})+d(b_n,b_{n+1}))\\&=(\alpha+\beta)d_{n-1}\end{aligned}$$

for all $n\in\mathbb{N}$. Since $\alpha+\beta<1$, we get $d_n<d_{n-1}$ for all $n\in\mathbb{N}$. Consequently,

$\tau+F(d_n)\leq F(d_{n-1})$ for all $n\in\mathbb{N}$. We get

$$F(d_n)\leq F(d_{n-1})-\tau\leq\cdots\leq F(d_0)-n\tau \tag{5}$$

for all $n\in\mathbb{N}$. If we take limit as $n\to\infty$ in (5), we obtain

$$\lim_{n\to\infty}F(d_n)=-\infty.$$

From property $(F2)$, we have that $\lim_{n\to\infty}d_n=0$. Using property $(F3)$, we can say that there exists $k\in(0,\infty)$ such that $\lim_{n\to\infty}d_n^kF(d_n)=0$. Using the inequation (5), we get

$$\begin{aligned}d_n^kF(d_n)-d_n^kF(d_0)&\leq d_n^k(F(d_0)-n\tau)-d_n^kF(d_0)\\&=-n\tau d_n^k\leq 0.\end{aligned} \tag{6}$$

If we take limit as $n\to\infty$ in (6), we get $\lim_{n\to\infty}nd_n^k=0$. Then there exists $n_0\in\mathbb{N}$ such that $nd_n^k\leq 1$ for all $n\geq n_0$. Hence we get $d_n\leq\frac{1}{n^{\frac{1}{k}}}$ for all $n\geq n_0$. We consider $m,n\in\mathbb{N}$ such that $m>n>n_0$, we get

$$\begin{aligned}d(a_m,a_n)+d(b_m,b_n)&\leq d(a_m,a_{m-1})+d(b_m,b_{m-1})+\cdots\\&\quad+d(a_{n-1},a_n)+d(b_{n-1},b_n)\\&=d_{m-1}+d_{m-2}+\cdots+d_n\\&=\sum_{i=n}^{m-1}d_i\leq\sum_{i=1}^{\infty}d_i\leq\sum_{i=1}^{\infty}\frac{1}{i^{\frac{1}{k}}}.\end{aligned}$$

Since the series $\sum_{i=1}^{\infty}\frac{1}{i^{\frac{1}{k}}}$ are convergent, $\{a_n\}$ and $\{b_n\}$ are Cauchy sequences in X. From completeness of (X,d), we can say that there exist $a,b\in X$ such that $\lim_{n\to\infty} a_n = a$ and $\lim_{n\to\infty} b_n = b.$ From property of metric, we obtain

$$d(S(a,b),a)\le d(S(a,b),a_{n+1})+d(a_{n+1},a)$$

$$\Rightarrow d(S(a,b),a)-d(a_{n+1},a)\le d(S(a,b),S(a_n,b_n)). \quad (7)$$

In addition to this, from (2), we get

$$F(d(S(a,b),S(a_n,b_n)))<\tau+F(d(S(a,b),S(a_n,b_n)))$$
$$\le F(\alpha d(a,a_n)+\beta d(b,b_n)).$$

From property of $(F1)$, we have

$$d(S(a,b),S(a_n,b_n))<\alpha d(a,a_n)+\beta d(b,b_n). \quad (8)$$

From (7) and (8), we obtain

$$d(S(a,b),a)-d(a_{n+1},a)<\alpha d(a,a_n)+\beta d(b,b_n).$$

Letting $n\to\infty$, we get

$$d(S(a,b),a)\to 0\Rightarrow S(a,b)=a.$$

Similarly, we have also $S(b,a)=b$. Then (a,b) is a coupled fixed of S. On the other hand, we assume that (a',b') is another coupled fixed point of S such that $(a,b)\ne(a',b')$. From (2), we get

$$F(d(a,a'))=F(d(S(a,b),S(a',b')))$$
$$\le F(\alpha d(a,a')+\beta d(b,b'))-\tau. \quad (9)$$

and

$$F(d(b,b'))=F(d(S(b,a),S(b',a')))$$
$$\le F(\alpha d(b,b')+\beta d(a,a'))-\tau. \quad (10)$$

From property of $(F1)$, (9) and (10), we get

$$d(a,a')<\alpha d(a,a')+\beta d(b,b')$$

and

$$d(b,b')<\alpha d(b,b')+\beta d(a,a').$$

Then we have

$$d(a,a')+d(b,b')<(\alpha+\beta)(d(a,a')+d(b,b')).$$

Since $\alpha+\beta<1$, we get

$$d(a,a')+d(b,b')=0.$$

This implies that $(a,b)=(a',b')$, which is a contradiction. Then S has a unique fixed point (a,b).

If constants in Theorem 3.1. are taken equal, it is obtained the following corollary.

Corollary 3.2.

Let (X,d) be a complete metric space and $S:X\times X\to X$ be a self-mapping on X. If there exist $F\in\mathcal{F}$ and $\tau\in(0,\infty)$ such that the following condition holds

$$d(S(a,b),S(u,v))>0\Rightarrow \tau+F(d(S(a,b),S(u,v)))$$
$$\le F(\frac{\alpha}{2}(d(a,u)+d(b,v))) \quad (11)$$

for all $a,b,u,v\in X$, $S(a,b)\ne S(u,v)$, where $\alpha<1$, then S has a unique coupled fixed point.

Example 3.3.

Let $X=\mathbb{R}$ and $d(a,b)=|a-b|$ for all $a,b\in X$. We can easily say that $(\mathbb{R},d)$ is a complete metric space. We consider $F:(0,\infty)\to\mathbb{R}$ such that $F(a)=\ln a$ for $a>0$. And, we define the mapping $S:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ with $S(a,b)=\frac{a+b}{3e^{\tau}}$ for $\tau>0$.

Then for all $a,b,u,v \in X$, $S(a,b) \neq S(u,v)$ and $\alpha = \frac{3}{2}$, we get

$$\begin{aligned}\tau + F(d(S(a,b),S(u,v))) &= \tau + ln\left(\left|\frac{a+b}{3e^{\tau}} - \frac{u+v}{3e^{\tau}}\right|\right)\\ &\leq \tau + ln\left(\left|\frac{a-u}{3e^{\tau}}\right| + \left|\frac{b-v}{3e^{\tau}}\right|\right)\\ &= \tau + ln\left(\frac{1}{3}|a-u| + |b-v|\right) - lne^{\tau}\\ &= F(\frac{1}{3}(d(a,u)+d(b,v))).\end{aligned}$$

Then the expression (11) is satisfied. From Corollary 3.2., S has a unique coupled fixed point. This point is $(0,0) \in \mathbb{R} \times \mathbb{R}$.

Theorem 3.4.

Let (X,d) be a complete metric space and $S: X \times X \to X$ be a self-mapping on X. If there exist $F \in \mathcal{F}$ and $\tau \in (0,\infty)$ such that the following condition holds

$$\begin{aligned}&d(S(a,b),S(u,v)) > 0 \Rightarrow \tau + F(d(S(a,b),S(u,v)))\\ &\leq F(\alpha d(S(a,b),a) + \beta d(S(u,v),u))\end{aligned} \tag{12}$$

for all $a,b,u,v \in X$, $S(a,b) \neq S(u,v)$, where $\alpha + \beta < 1$, then S has a unique coupled fixed point.

Proof:

We take sequences $\{a_n\}$ and $\{b_n\}$ which have same properties in the proof of Theorem 3.1. such as $a_{n+1} = S(a_n,b_n)$ and $b_{n+1} = S(b_n,a_n)$. From (12), we get

$$\begin{aligned}\tau + F(d(a_n,a_{n+1})) &= \tau + F(d(S(a_{n-1},b_{n-1}),S(a_n,b_n)))\\ &\leq F(\alpha d(S(a_{n-1},b_{n-1}),x_{n-1}) + \beta d(S(a_n,b_n),a_n))\\ &= F(\alpha d(a_n,a_{n-1}) + \beta d(a_{n+1},a_n)).\end{aligned}$$

From property $(F1)$,

$$\begin{aligned}&d(a_n,a_{n+1}) < \alpha d(a_{n-1},a_n) + \beta d(a_n,a_{n+1})\\ &\Rightarrow d(a_n,a_{n+1}) < \frac{\alpha}{1-\beta} d(a_{n-1},a_n),\end{aligned}$$

where $0 < \frac{\alpha}{1-\beta} < 1$. Then we get

$$d(a_n,a_{n+1}) < d(a_{n-1},a_n)$$

for all $n \in \mathbb{N}$. We denote $\delta_n = d(a_n,a_{n+1})$. So, $\tau + F(\delta_n) \leq F(\delta_{n-1})$ for all $n \in \mathbb{N}$. The following holds

$$F(\delta_n) \leq F(\delta_{n-1}) - \tau \leq \cdots \leq F(\delta_0) - n\tau \tag{13}$$

for all $n \in \mathbb{N}$. If we take limit as $n \to \infty$ in (13), we get $\lim_{n\to\infty} F(\delta_n) = -\infty$. From property $(F2)$, we have that $\lim_{n\to\infty} \delta_n = 0$. Using $(F3)$, we can say that there exist $k \in (0,\infty)$ such that $\lim_{n\to\infty} \delta_n^k F(\delta_n) = 0$. From (13), we get

$$\begin{aligned}\delta_n^k F(\delta_n) - \delta_n^k F(\delta_0) &\leq \delta_n^k (F(\delta_0) - n\tau) - \delta_n^k F(\delta_0)\\ &= -n\tau\delta_n^k \leq 0.\end{aligned} \tag{14}$$

Taking limit as $n \to \infty$ in (14), we get $\lim_{n\to\infty} n\delta_n^k = 0$. There exist $n_1 \in \mathbb{N}$ such that $n\delta_n^k \leq 1$ for $n \geq n_1$. Then we get

$$\delta_n \leq \frac{1}{n^{\frac{1}{k}}}. \tag{15}$$

We consider $m,n \in \mathbb{N}$ such that $m > n \geq n_1$. From (15), we get

$$\begin{aligned} d(a_n,a_m) &\le d(a_n,a_{n+1})+\cdots+d(a_{m-1},a_m) \\ &= \delta_n+\delta_{n+1}+\cdots+\delta_{m-1} \\ &< \sum_{i=n}^{\infty}\delta_i \\ &\le \sum_{i=n}^{\infty}\frac{1}{i^{\frac{1}{k}}}. \end{aligned}$$

Then $\sum_{n=1}^{\infty}\frac{1}{i^{\frac{1}{k}}}$ is convergent. Thus $\{a_n\}$ is a Cauchy sequence in X. In a similar way, we can show that $\{b_n\}$ is a Cauchy sequence in X. From completeness of (X,d), there exist $a,b\in X$ such that $\lim_{n\to\infty}a_n=a$ and $\lim_{n\to\infty}b_n=b.$

$$\begin{aligned} d(S(a,b),a) &\le d(S(a,b),a_{n+1})+d(a_{n+1},a) \\ &= d(S(a,b),S(a_n,b_n))+d(a_{n+1},a) \end{aligned}$$

$$\Rightarrow d(S(a,b),a)-d(a_{n+1},a)\le d(S(a,b),S(a_n,b_n)). \tag{16}$$

On the other hand, from (12), we get

$$\begin{aligned} F(d(S(a,b),S(a_n,b_n))) &< \tau+F(d(S(a,b),S(a_n,b_n))) \\ &\le F(\alpha d(S(a,b),a)+\beta d(S(a_n,b_n),a_n)). \end{aligned}$$

From property of $(F1)$, we get

$$d(S(a,b),S(a_n,b_n))<\alpha d(S(a,b),a)+\beta d(S(a_n,b_n),b_n). \tag{17}$$

From (16) and (17), we obtain

$$\begin{aligned} d(S(a,b),a)-d(a_{n+1},a) &< \alpha d(S(a,b),a)+\beta d(S(a_n,b_n),a_n) \\ &= \alpha d(S(a,b),a)+\beta d(a_{n+1},a_n) \\ &\le \alpha d(S(a,b),a)+\beta(d(a_{n+1},a)+d(a,a_n)) \end{aligned}$$

$$d(S(a,b),a)\le\frac{1+\beta}{1-\alpha}d(a_{n+1},a)+\frac{\beta}{1-\alpha}d(a_n,a).$$

Letting $n\to\infty$, we get

$$d(S(a,b),a)\to 0\Rightarrow S(a,b)=a.$$

Similarly, we have also $S(b,a)=b$. Then (a,b) is a coupled fixed of S. On the other hand, we assume that (a',b') is another coupled fixed point of S such that $(a,b)\ne(a',b')$. From (12), we get

$$\begin{aligned} F(d(a,a')) &= F(d(S(a,b),S(a',b'))) \\ &\le F(\alpha d(S(a,b),a)+\beta d(S(a',b'),a'))-\tau. \end{aligned}$$

From property $(F1)$, we get $d(a,a')=0$. Similarly, we can show that $d(b,b')=0$ These imply that $(a,b)=(a',b')$, which is a contradiction. Then S has a unique fixed point (a,b).

Corollary 3.5.

Let (X,d) be a complete metric space and $S:X\times X\to X$ be a self-mapping on X. If there exist $F\in\mathcal{F}$ and $\tau\in(0,\infty)$ such that the following condition holds

$$\begin{aligned} d(S(a,b),S(u,v))>0 &\Rightarrow \tau+F(d(S(a,b),S(u,v))) \\ &\le F(\frac{\alpha}{2}(d(S(a,b),a)+d(S(u,v),u))) \end{aligned}$$

for all $a,b,u,v\in X$, $S(a,b)\ne S(u,v)$, where $\alpha<1$, then S has a unique coupled fixed point.

Example 3.6.

Let $X=[0,\infty)$. We define $d:[0,\infty)\times[0,\infty)\to\mathbb{R}^+$ with $d(a,b)=\max\{a,b\}$. $([0,\infty),d)$ is a complete metric space. We consider the mapping $S:[0,\infty)\times[0,\infty)\to[0,\infty)$ such that $S(a,b)=\frac{a}{12}$. And, we choose $F(a)=\ln(a)$, $a\in(0,\infty)$. Then it is clear that for all $a,b,u,v\in[0,\infty)$, $S(a,b)\ne S(u,v)$, $\tau=\ln 2$ and $\alpha=\frac{1}{3}$, the condition

$$\ln 2+F(d(S(a,b),S(u,v))) \leq F(\frac{1}{6}(d(S(a,b),a)+d(S(u,v),u)))$$

is satisfied. From the Corollary 3.5., S has a unique coupled fixed point.

Theorem 3.7.

Let (X,d) be a complete metric space and $S:X\times X\to X$ be a self-mapping on X. If there exist $F\in\mathcal{F}$ and $\tau\in(0,\infty)$ such that the following condition holds

$$d(S(a,b),S(u,v))>0\Rightarrow \tau+F(d(S(a,b),S(u,v))) \leq F(\alpha d(S(a,b),u)+\beta d(S(u,v),a)) \tag{18}$$

for all $a,b,u,v\in X$, $S(a,b)\neq S(u,v)$, where $\alpha+\beta<1$, then S has a unique coupled fixed point.

Proof:

We take $a_0,b_0\in X$ and set $a_1=S(a_0,b_0), y_1=S(b_0,a_0),\ldots, a_{n+1}=S(a_n,b_n), b_{n+1}=S(b_n,a_n)$.

From (18), we get

$$\begin{aligned}F(d(a_n,a_{n+1})) &= F(d(S(a_{n-1},b_{n-1}),S(a_n,b_n)))\\ &\leq F(\alpha d(S(a_{n-1},b_{n-1}),a_n)+\beta d(S(a_n,b_n),a_{n-1}))-\tau\\ &< F(\alpha d(a_n,a_n)+\beta d(a_{n+1},a_{n-1}))\\ &= F(\beta d(a_{n+1},a_{n-1}))\\ &\leq F(\beta d(a_{n+1},a_n)+\beta d(a_n,a_{n-1})).\end{aligned}$$

From property $(F1)$, we get

$$d(a_n,a_{n+1})<\frac{\beta}{1-\beta}d(a_{n-1},a_n)$$

Since $\alpha+\beta<1$, we get $\frac{1}{1-\beta}<1$. Then we get

$$d(a_n,a_{n+1})<d(a_{n-1},a_n)$$

for all $n\in\mathbb{N}$. If we denote $\delta_n=d(a_n,a_{n+1})$, then the proof similar to proof of Theorem 3.4. Thus, $\{a_n\}$ is a Cauchy sequence in X. In a similar way, we can show that $\{b_n\}$ is a Cauchy sequence in X. From completeness of (X,d), there exist $a,b\in X$ such that $\lim_{n\to\infty}a_n=a$ and $\lim_{n\to\infty}b_n=b$. As similar to proof of Theorem 3.4., we get

$$F(d(S(a,b),S(a_n,b_n)))<\tau+F(d(S(a,b),S(a_n,b_n))) \leq F(\alpha d(S(a,b),a_n)+\beta d(S(a_n,b_n),a)).$$

From property $(F1)$, we get

$$d(S(a,b),a)<\frac{1+\beta}{1-\alpha}d(a_{n+1},a)+\frac{\alpha}{1-\alpha}d(a_n,a).$$

Letting $n\to\infty$, we get

$$d(S(a,b),a)\to 0\Rightarrow S(a,b)=a.$$

Similarly, we have also $S(b,a)=b$. Then (a,b) is a coupled fixed of S. Now we show that the coupled fixed point is unique. We assume that (a',b') is another coupled fixed point of S such that $(a,b)\neq(a',b')$. From (18), we get

$$F(d(a,a'))=F(d(S(a,b),S(a',b'))) \leq F(\alpha d(S(a,b),a')+\beta d(S(a',b'),a))-\tau.$$

From property $(F1)$, we get

$$d(a,a')<\frac{\alpha}{1-\alpha-\beta}d(S(a,b),a)+\frac{\beta}{1-\alpha-\beta}d(S(a',b'),a')$$

We get $d(a,a')=0$. Similarly, we can show that $d(b,b')=0$ These imply that

$(a,b)=(a',b')$, which is a contradiction. Then S has a unique fixed point (a,b).

Corollary 3.8.

Let (X,d) be a complete metric space and $S: X\times X\to X$ be a self-mapping on X. If there exist $F\in\mathcal{F}$ and $\tau\in(0,\infty)$ such that the following condition holds

$$d(S(a,b),S(u,v))>0\Rightarrow \tau+F(d(S(a,b),S(u,v)))$$
$$\leq F(\frac{\alpha}{2}(d(S(a,b),u)+d(S(u,v),a)))$$

for all $a,b,u,v\in X$, $S(a,b)\neq S(u,v)$, where $\alpha<1$, then S has a unique coupled fixed point.

Example 3.9.

If we take as $\tau=\ln 3$ and $\alpha=\frac{1}{2}$ in Example 3.6., it is obvious that the condition

$$\ln 3+F(d(S(a,b),S(u,v)))$$
$$\leq F(\frac{1}{4}(d(S(a,b),u)+d(S(u,v),a)))$$

is also satisfied for all $a,b,u,v\in[0,\infty)$, $S(a,b)\neq S(u,v)$. From the Corollary 3.8., S has a unique coupled fixed point.

4. References

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